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On Proper Class Coprojectively Generated by Modules With Projective Socle

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Abstract — Let $\mathcal{E} : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of modules and module homomorphism. $\mathcal{E}$ is called gd-closed sequence if $\operatorname{Im} f$ is gd-closed in B. In this paper, the proper class $\mathcal{GD}-\mathcal{C}losed$, which is coprojectively generated by modules with projective socle, be studied and also its relations among $\mathcal{N}eat, \mathcal{C}losed, \mathcal{D}-\overline{\mathcal{C}losed}, \mathcal{S}-\overline{\mathcal{C}losed}$ be investigated. Additionally, we examine coprojective modules of this class.

Keywords — *$\mathcal{GD}-\mathcal{C}losed$, g-semartinian modules, G-Dickson torsion theory, gdc-flat modules*

1. Introduction

Throughout the paper, we assume that all rings are associative with identity and all modules are unitary right modules. As usual, $Mod-R$ denotes the category of all right R-modules over a ring R.

A submodule N of a module M is *essential* (or *large*) in M, denoted $N \trianglelefteq M$, if for every $0 \neq K \leq M$, we have $N \cap K \neq 0$; and N is said to be *closed* in M if N has no proper essential extension in M. We also say in this case that N is a closed submodule of M. Closed submodules are important in rings and modules, and relative homological algebra. See, for example, [1] for their properties. More recently, many authors have studied their generalizations, some of which are neat submodules, $\mathcal{S}$-closed submodules, $\mathcal{D}$-closed submodules (see, for example, [2, 3, 6, 11, 12, 15, 18]).

The singular submodule $Z(M)$ of a module M is the set of $m \in M$ such that, $mI = 0$ for some essential right ideal I of R. This takes the place of the torsion submodule in general setting. The module M is called nonsingular if $Z(M) = 0$, and singular if $M = Z(M)$, while the right singular ideal of R is $Z_r(R) = Z(R_R)$. The ring R is said to be right nonsingular if it is nonsingular as a right R-module. The Goldie torsion submodule $Z_2(M)$ of M is defined by the equality $Z_2(M)/Z(M) = Z(M/Z(M))$. A module M is called Goldie torsion if $Z_2(M) = M$. A module M is called semiartinian if every non-zero homomorphic image of M contains a simple submodule, that is, $\operatorname{Soc}(M/N) \neq 0$ for every submodule $N \leq M$. A module M is called socle-free if $Soc(M) = 0$. It is well known that a simple module is either singular or projective. A module M is called *g-semiartinian* if every non-zero homomorphic image of M contains a singular simple submodule. The class of g-semiartinian modules is a torsion class of G-Dickson torsion theory, which is generated by singular simple modules [9]. The class of modules with projective socle is the torsion-free class of the same torsion theory. Note that the class of all g-semiartinian modules is closed under submodules, homomorphic images, direct sums and extensions, while the class of all socle-projective (nonsingular, socle-free) modules is closed under submodules, direct products, extensions.

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The notion of s-closed submodules is a generalization of the notion of closed submodules and it was introduced in [16]. A submodule $N \leq M$ is called *s-closed* if M/N is nonsingular. Later, in [12], d-closed submodules were introduced. A submodule $N \leq M$ is called *d-closed* if M/N has zero socle. Inspired from these, a submodule N of a module M is called *gd-closed* if there is a submodule S in M such that $S \cap N = 0$ and $\mathrm{Soc}(M/(S \oplus N))$ is projective. Note that $\mathrm{Soc}(X)$ is projective for each nonsingular module X, and hence any s-closed submodule is gd-closed. Also, since zero module is projective, any d-closed submodule is gd-closed. A short exact sequence $0 \to A \xrightarrow{f} B \to C \to 0$ is called gd-closed (respectively, s-closed, d-closed) sequence if $\mathrm{Im}(f)$ is gd-closed (respectively, s-closed, d-closed) in B. The class of gd-closed sequences is a proper class [19, Theorem 2.1], which is projectively generated by g-semiartinian modules ([8, Proposition 3.5]). In general, the class of s-closed (respectively d-closed) exact sequences is not a proper class, ([11], [12]). But the class of gd-closed exact sequence is a proper class by [19, Theorem 2.1]. We will call a module M is gdc-flat if every exact sequence ending with M is gd-closed sequence. First of all, it is obvious that projective modules are gdc-flat. Furthermore, nonsingular modules and modules with zero socle are less obvious examples of gdc-flat modules.

After this introductory section, this paper is divided into three sections. In Section 2, we recall some torsion theoretic concepts and then give some properties of proper classes. In Section 3, we prove some inclusion relations among $\mathcal{GD}-\mathcal{C}losed$ and the well-known proper classes, such as $\mathcal{S}-\overline{\mathcal{C}losed}$, $\mathcal{D}-\overline{\mathcal{C}losed}$, $\mathcal{N}eat$. We show that R is a C-ring if and only if $\mathcal{GD}-\mathcal{C}losed = \mathcal{S}-\overline{\mathcal{C}losed}$. (Proposition 3.7). For any ring R, we show that $\mathcal{GD}-\mathcal{C}losed = \mathcal{N}eat$ if and only if each g-semiartinian module T can be represented as $T = S \oplus P$, where S is semisimple goldie torsion module and P is a projective module (Proposition 3.8).

In Section 4, we investigate modules M such that each short exact sequence ending with M belongs to $\mathcal{GD}-\mathcal{C}losed$. Such modules are called gdc-flat modules. We prove that if the torsion submodule $\tau_{gd}(E)$ of an injective module E is projective, then E is gdc-flat; and the converse is true for C-rings (Theorem 4.8). Finally, we prove that every $\mathcal{GD}-\mathcal{C}losed$ sequence is a $\mathcal{P}ure$ sequence if and only if every module with a projective socle is flat if and only if every gdc-flat module is flat if and only if every finitely generated module with a projective socle is flat (Proposition 4.12).

We use the notation $E(M)$, $Soc(M)$, $Sa(M)$, $Z(M)$, $Z_2(M)$ for the injective hull, socle, semi-artinian, singular, Goldie torsion submodule of a module M, respectively. The character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M will be denoted by M^+. For all other basic or background material, we refer the reader to [5, 14, 22].

2. Proper Classes and Torsion Theories

In this section, we denote by $\mathcal{P}$ a class of short exact sequences of modules and module homomorphisms. Let $\mathcal{E} : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence belonging to $\mathcal{P}$. We call f and g a $\mathcal{P}$-monomorphism and a $\mathcal{P}$-epimorphism, respectively. Note that the monomorphism f and the epimorphism g uniquely determine the exact sequence $\mathcal{E}$ up to isomorphism. The following definition is given in the sense of Buchsbaum [4].

Definition 2.1. The class $\mathcal{P}$ is called proper if it satisfies the following conditions (see for example [5, 19, 23]).

(P-1) If a short exact sequence $\mathcal{E}$ is in $\mathcal{P}$, then $\mathcal{P}$ contains every short exact sequence isomorphic to $\mathcal{E}$.

(P-2) $\mathcal{P}$ contains all the splitting exact sequences.

(P-3) The composite of two $\mathcal{P}$-monomorphisms (respectively $\mathcal{P}$-epimorphisms) is a $\mathcal{P}$-monomorphism (respectively $\mathcal{P}$-epimorphism) when the composite is defined.

(P-4) If g and f are monomorphisms and gf is a $\mathcal{P}$-monomorphism, then f is a $\mathcal{P}$-monomorphism. If g and f are epimorphisms and gf is a $\mathcal{P}$-epimorphism, then g is a $\mathcal{P}$-epimorphism.

Let $\mathcal{P}$ be a proper class. A module M is called $\mathcal{P}$-projective if it is projective with respect to all short exact sequences in $\mathcal{P}$, that is, $\mathrm{Hom}(M; \mathcal{E})$ is exact for every $\mathcal{E}$ in $\mathcal{P}$. A module M is called $\mathcal{P}$-coprojective if every short exact sequence which ends with M belongs to $\mathcal{P}$. For a given class $\mathcal{M}$ of modules, we denote the smallest proper class for which each $M \in \mathcal{M}$ is $\overline{k}(\mathcal{M})$-coprojective by $\overline{k}(\mathcal{M})$.

This proper class is called the proper class *coprojectively generated* by $\mathcal{M}$. The largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is $\mathcal{P}$-projective is called the proper class *projectively generated* by $\mathcal{M}$.

We now give some examples of proper classes. Let $\mathcal{E} : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of modules.

1. We denote the smallest proper class of short exact sequences by $\mathcal{S}plit$. This class consists of all splitting exact sequences of modules. Also, we denote the largest proper class of short exact sequence by $\mathcal{A}bs$. This class consists of all short exact sequences of modules.

2. A submodule N of a module M is called essential if every non-zero submodule of M has a non-zero intersection with N. A submodule N of a module M is called *closed* if it has no proper essential extension. $\mathcal{E}$ is called closed if $\operatorname{Im} f$ is closed in B. We denote the class of all closed short exact sequences by $\mathcal{C}losed$. It is known that $\mathcal{C}losed$ is a proper class (see [5, 10.5]) .

3. A submodule N of a module M is called pure if the sequence $0 \to N \otimes_R X \to M \otimes_R X$ is exact for every left R-module X. $\mathcal{E}$ is called a pure sequence if $\operatorname{Im} f$ is pure in B. The class of pure exact sequences is denoted by $\mathcal{P}ure$ and it is a proper class. (see, [10]).

4. A submodule N of a module M is called neat if the sequence $\operatorname{Hom}(S, M) \to \operatorname{Hom}(S, M/N) \to 0$ is exact for every simple module S. $\mathcal{E}$ is called a neat sequence if $\operatorname{Im} f$ is neat in B. The class of neat exact sequences is denoted by $\mathcal{N}eat$ and it is a proper class. It is projectively generated by simple modules (see [21]).

5. A submodule N of a module M is called extended s-closed if there is $S \subset M$ such that $S \cap N = 0$ and $M/(S \oplus N)$ is nonsingular. $\mathcal{E}$ is called extended s-closed sequence if $\operatorname{Im} f$ is extended s-closed in B. The class of extended s-closed exact sequences is denoted by $\mathcal{S} - \overline{\mathcal{C}losed}$ and it is a proper class. It is projectively generated by goldie torsion modules (see [11, Proposition 3.3, Proposition 3.4]).

6. A submodule N of a module M is called extended d-closed if there is $S \subset M$ such that $S \cap N = 0$ and $M/(S \oplus N)$ has zero socle. $\mathcal{E}$ is called extended d-closed sequence if $\operatorname{Im} f$ is extended d-closed in B. The class of extended d-closed exact sequences is denoted by $\mathcal{D} - \overline{\mathcal{C}losed}$ and it is a proper class. It is projectively generated by semiartinian modules (see [12, Proposition 5, Proposition 6]).

   In this paper, we study gd-closed submodules. A submodule N of a module M is called gd-closed if there is $S \subset M$ such that $S \cap N = 0$ and $M/(S \oplus N)$ has projective socle. $\mathcal{E}$ is called gd-closed sequence if $\operatorname{Im} f$ is gd-closed in B. G-semiartinian modules were introduced in [9]. A module M is g-semiartinian if every non-zero homomorphic image of M contains a simple singular module. The class of gd-closed exact sequences is denoted by $\mathcal{GD} - \mathcal{C}losed$ and it is a proper class. This class is projectively generated by g-semiartinian modules (see [19, Theorem 2.1]).

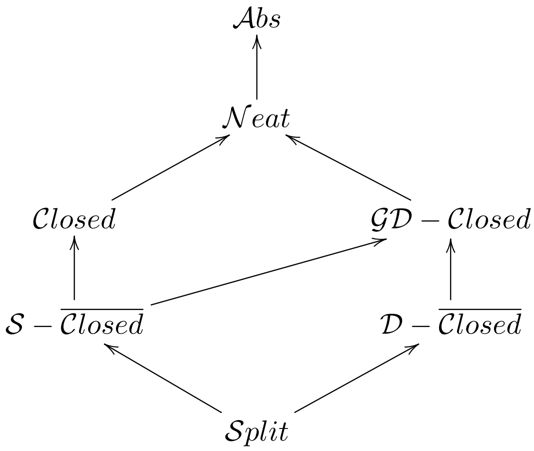
The following result can be obtained from [8, Proposition 3.5], so the proof is omitted.

Proposition 2.2. A submodule A of a module B is gd-closed in B if and only if $\operatorname{Hom}(T, B) \to \operatorname{Hom}(T, B/A) \to 0$ is exact for each g-semiartinian module T.

The class of all g-semiartinian modules is the torsion class of a torsion theory. This torsion theory is called G-dickson torsion theory and denoted by $\tau_{gd} = (\mathbb{T}_{GD}, \mathbb{F}_{GD})$ where $\mathbb{T}_{GD}$ and $\mathbb{F}_{GD}$ represent the torsion class and the torsion free class, respectively. Note that $\mathbb{F}_{GD}$ consists of the modules with projective socle. G-dickson torsion theory was introduced in [9]. Note that τ_{gd} is hereditary, that is, $\mathbb{T}_{GD}$ is closed under submodules. $\mathbb{T}_{GD}$ is closed under homomorphic images, direct sums and extensions as well. Furthermore, $\mathbb{F}_{GD}$ is closed under submodules, direct products, extensions and injective hulls. Any module M contains a unique maximal g-semiartinian submodule which is denoted by $\tau_{gd}(M)$. It is clear that $\tau_{gd}(M) = 0$ if and only if $\operatorname{Soc}(M)$ is projective. In addition $M/\tau_{gd}(M)$ has a projective socle. Note that $\tau_{gd}(M) = Z_2(M) \cap Sa(M)$.

3. Inclusion relations among the proper classes of modules

In this section, we prove some inclusion relations among the proper classes of modules that are considered in the present paper. These relations are given in the following diagram:



First of all, it is clear that $\mathcal{A}bs$ contains any proper class and $\mathcal{S}plit$ is contained in any proper class. It is clear that $\mathcal{S}-\overline{\mathcal{C}losed} \subset \mathcal{C}losed \subset \mathcal{N}eat$. Since every simple module is semiartinian, $\mathcal{D}-\overline{\mathcal{C}losed} \subset \mathcal{N}eat$. Also, $\mathcal{C}losed = \mathcal{N}eat$ exactly when R is a C-ring [17, Theorem 5]. In general, an extended s-closed submodule of a module is not extended d-closed submodule, and conversely, an extended d-closed submodule is not necessarily s-closed submodule. However, it was proven that $\mathcal{D}-\overline{\mathcal{C}losed} \subset \mathcal{S}-\overline{\mathcal{C}losed}$ if and only if R is C-ring [13, Proposition 3.5], and $\mathcal{D}-\overline{\mathcal{C}losed} \supset \mathcal{S}-\overline{\mathcal{C}losed}$ if and only if every simple module is singular [13, Proposition 3.6].

The following result can be obtained by [8, Proposition 3.7], so the proof is omitted.

Proposition 3.1. Let $\mathcal{K}$ be the class of all modules with projective socle. Then $\overline{k}(\mathcal{K}) = \mathcal{GD}-\mathcal{C}losed$.

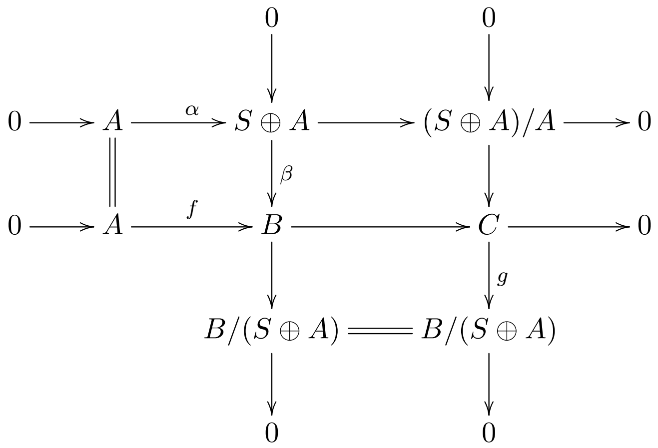
Proposition 3.2. The following statements are equivalent for a ring R:

1. Every module has projective socle.
2. $\mathcal{A}bs = \mathcal{GD}-\mathcal{C}losed$.
3. Every g-semiartinian module is projective.
4. R is semisimple.

Proof. (1) $\Rightarrow$ (2) If every module has projective socle, then $\mathcal{A}bs = \mathcal{GD}-\mathcal{C}losed$. (2) $\Rightarrow$ (3) This is an obvious consequence of Proposition 2.2. (3) $\Rightarrow$ (4) Every simple module is projective by assumption, that is R is semisimple ring. (4) $\Rightarrow$ (1) By assumption, every module is semisimple, and so every simple module is projective. Thus, every module has projective socle. □

Proposition 3.3. $\mathcal{S}plit = \mathcal{GD}-\mathcal{C}losed$ if and only if every module with projective socle is projective.

Proof. Let C be a a module with projective socle. We consider the short exact sequence $0 \to A \to P \to C \to 0$ with P projective. Since $C \cong P/A$, the sequence is a $\mathcal{GD}-\mathcal{C}losed$ exact sequence, and thus it splits. Since C is a direct summand of P, C is a projective module. For the converse implication, let $0 \to A \xrightarrow{f} B \to C \to 0$ be a $\mathcal{GD}-\mathcal{C}losed$ exact sequence. Then there is a submodule S of B such that $S \cap A = 0$ and $B/(S \oplus A)$ has projective socle. Now, consider the following diagram:



Since $B/(S \oplus A)$ is a projective module, β is $\mathcal{S}plit$-monomorphism and thus $f = \beta\alpha$ is also a $\mathcal{S}plit$-monomorphism. □

We know that $\mathcal{D} - \overline{\mathcal{C}losed}$ is projectively generated by semiartinian modules and $\mathcal{GD} - \mathcal{C}losed$ is projectively generated by g-semiartinian modules. Since each g-semiartinian module is semiartinian, it follows that $\mathcal{D} - \overline{\mathcal{C}losed} \subset \mathcal{GD} - \mathcal{C}losed$. See [9, Proposition 1] for the proof of the following proposition.

Proposition 3.4. Let R be a ring. Every semiartinian $R-$module is g-semiartinian if and only if every simple $R-$module is singular.

Proposition 3.5. Let R be a ring such that every semiartinian $R-$module is g-semiartinian. Then $\mathcal{GD} - \mathcal{C}losed = \mathcal{D} - \overline{\mathcal{C}losed}$.

Proof. $\mathcal{D} - \overline{\mathcal{C}losed} \subset \mathcal{GD} - \mathcal{C}losed$ because every g-semiartinian module is semiartinian. $\mathcal{GD} - \mathcal{C}losed \subset \mathcal{D} - \overline{\mathcal{C}losed}$ is clear by assumption. □

It is known that $\mathcal{S} - \overline{\mathcal{C}losed}$ is projectively generated by goldie torsion modules. Since each g-semiartinian is goldie torsion module, it follows that $\mathcal{S} - \overline{\mathcal{C}losed} \subset \mathcal{GD} - \mathcal{C}losed$. For the proof of the following Proposition, see [9, Theorem 4].

Proposition 3.6. R is a C-ring if and only if the class of g-semiartinian modules and the class of goldie torsion modules are the same.

Proposition 3.7. R is a C-ring if and only if $\mathcal{GD} - \mathcal{C}losed = \mathcal{S} - \overline{\mathcal{C}losed}$.

Proof. If R is a C-ring, then the class of g-semiartinian modules and the class of Goldie torsion modules are the same. This implies that $\mathcal{S} - \overline{\mathcal{C}losed}$ is projectively generated by g-semiartinian modules, hence $\mathcal{GD} - \mathcal{C}losed = \mathcal{S} - \overline{\mathcal{C}losed}$. Conversely, let M be a singular module. Then $M \cong F/A$ with F is projective and A is essential in F. Assume that M has a zero socle. Now consider the exact sequence $\mathcal{E} : 0 \to A \to F \to M \to 0$. since zero module is projective, $\mathcal{E}$ is gd-closed exact sequence. By assumption, there exists a submodule S of F such that $A \cap S = 0$ and $F/(A \oplus S)$ is nonsingular. But A is essential in F, which gives a contradiction. □

Since $\mathcal{N}eat$ is projectively generated by simple (singular) modules and simple (singular) modules are g-semiartinian, it follows that $\mathcal{GD} - \mathcal{C}losed \subset \mathcal{N}eat$.

Proposition 3.8. Let R be a ring. $\mathcal{GD} - \mathcal{C}losed = \mathcal{N}eat$ if and only if each g-semiartinian module T can be represented as $T = S \oplus P$, where S is semisimple goldie torsion module and P is a projective module.

Proof. Let $\mathcal{E} : 0 \to A \to B \to B/A \to 0$ be a neat exact sequence and T be a g-semiartinian module. By assumption, $\mathcal{E}$ is a gd-closed exact sequence. Therefore the sequence $\mathrm{Hom}(T, B) \to \mathrm{Hom}(T, B/A) \to 0$ is exact and this means that T is $\mathcal{N}eat$-projective. By [15, Theorem 2.6], T is a direct sum of a projective module and a semisimple module. Furthermore, since T is g-semiartinian, the semisimple component of T should be goldie torsion. Conversely, by assumption, we notice that every g-semiartinian module is $\mathcal{N}eat$-projective. It follows that every neat exact sequence is a gd-closed sequence.

□

Proposition 3.9. If R is Goldie torsion, then $\mathcal{GD}-\mathcal{C}losed = \mathcal{D}-\overline{\mathcal{C}losed}$.

PROOF. If R is Goldie torsion, then every right module is Goldie torsion. Since each semiartinian modules are g-semiartinian, it follows that $\mathcal{GD}-\mathcal{C}losed \subset \mathcal{D}-\overline{\mathcal{C}losed}$. The implication $\mathcal{D}-\overline{\mathcal{C}losed} \subset \mathcal{GD}-\mathcal{C}losed$ is clear because every g-semiartinian module is semiartinian. □

4. gdc-flat Modules

An R-module M is called *flat* if the functor $M \otimes_R -$ is exact. Note that M is flat if and only if every short exact sequence $0 \to A \to B \to M \to 0$ of R-modules is pure-exact ([22, Proposition 3.67]). This connection between flatness and purity give rise to a lot of studies on some classes of modules defined throught closed, neat submodules. That is, the modules M such that any short exact sequence ending with M is included in $\mathcal{C}losed$ ($\mathcal{N}eat$), which are called *weakly-flat* (respectively, *neat-flat*), have been studied in [24], (respectively [2]). Gdc-flat modules were introduced in [7] throught this relation. A module M is called gdc-flat if every exact sequence ending with M is a gd-closed sequence. By Poposition 2.2, it follows that M is gdc-flat if and only if for any epimorphism $Y \to M$, $\text{Hom}(T, Y) \to \text{Hom}(T, M) \to 0$ is exact for any g-semiartinian module T.

Remark 4.1. Projective modules are gdc-flat since any exact sequence ending with a projective module splits and splitting sequences are gd-closed sequences.

The following Proposition gives a characterization of gd-closed sequences. See [7, Proposition 2.2].

Proposition 4.2. The following are equivalent for a module M.

1. M is gdc-flat.
2. There exists a gd-closed exact sequence with $0 \to K \to F \to M \to 0$ with F is projective.
3. There exists a gd-closed exact sequence with $0 \to K \to F \to M \to 0$ with F is gdc-flat.

Remark 4.3.

1. A module with projective socle is gdc-flat.
2. R is a semisimple ring if and only if every right R-module is gdc-flat.
3. The class of gdc-flat modules is closed under finite direct sums and direct summands.
4. Nonsingular modules are gdc-flat.
5. The class of gdc-flat modules is closed under extensions.
6. A g-semiartinian module is gdc-flat if and only it is projective by Proposition 2.2.

Definition 4.4. R is called right PS-ring if every simple right ideal of R is projective, that is $\text{Soc}(R_R)$ is projective ([20]).

Proposition 4.5. [7, Proposition 3.1] The following are equivalent.

1. R is a PS-ring.
2. Gdc-flat modules are exactly the modules with projective socle.
3. Every submodule of a gdc-flat module is gdc-flat.
4. Every right ideal of R is a gdc-flat module.

Proposition 4.6. Let $f : N \to M$ be an epimorphism of modules. If M is a gdc-flat module, then any g-semiartinian submodule of M is isomorphic to a g-semiartinian submodule of N. In particular, the torsion submodule $\tau_{gd}(M)$ of M embeds in a projective module.

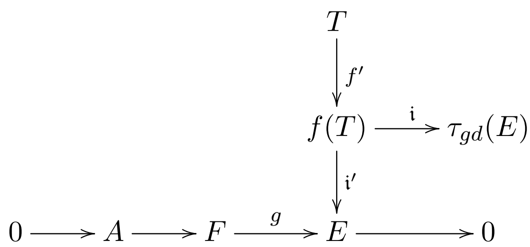
Proof. Take a g-semiartinian submodule T of M and consider the inclusion map $\mathfrak{i} : T \to M$. Since M is gdc-flat, the sequence $\mathrm{Hom}(T, N) \to \mathrm{Hom}(T, M) \to 0$ is exact. It follows that there exists $g : T \to N$ such that $fg = \mathfrak{i}$. Clearly g is a monomorphism and $g(T)$ is a g-semiartinian submodule of N since g-semiartinian modules are closed under homomorphic images. For the particular case, take an epimorphism $f : P \to M$ with P is projective and let $T = \tau_{gd}(M)$. □

Definition 4.7. A ring R is said to be a C-ring if for every R-module of B and every essential proper submodule A of B, $\mathrm{Soc}(B/A) \neq 0$.

The notion of a right C-ring has been introduced in [21]. It is known that R is a C-ring if and only if $\mathrm{Soc}(R/I) \neq 0$ for every essential ideal I of R .

Theorem 4.8. Let E be an injective module. If the torsion submodule $\tau_{gd}(E)$ of E is projective, then E is gdc-flat module. Furthermore, the converse statement holds for right C-rings.

Proof. Assume that $\tau_{gd}(E)$ is a projective module and $0 \to A \to F \xrightarrow{g} E \to 0$ be a short exact sequence with F is projective. Let T is a g-semiartinian module and $f : T \to E$ be a homomorphism. We seek a homomorphism $h : T \to F$ such that $gh = f$. As T is a g-semiartinian module, $f(T)$ is also g-semiartinian. It follows that there is an inclusion map $\mathfrak{i} : f(T) \to \tau_{gd}(E)$. Let $\mathfrak{i}' : f(T) \to E$ be the other inclusion map. The following commutative diagram is obtained by combining these maps:



Injectivity of E implies existence of a homomorphism $v : \tau_{gd}(E) \to E$ such that $v\mathfrak{i} = \mathfrak{i}'$. Projectivity of $\tau_{gd}(E)$ implies that there is a homomorphism $u : \tau_{gd}(E) \to F$ such that $gu = v$. Setting $h = u\mathfrak{i}f'$, we get that $gh = f$. Therefore E is gdc-flat by Proposition 4.2.

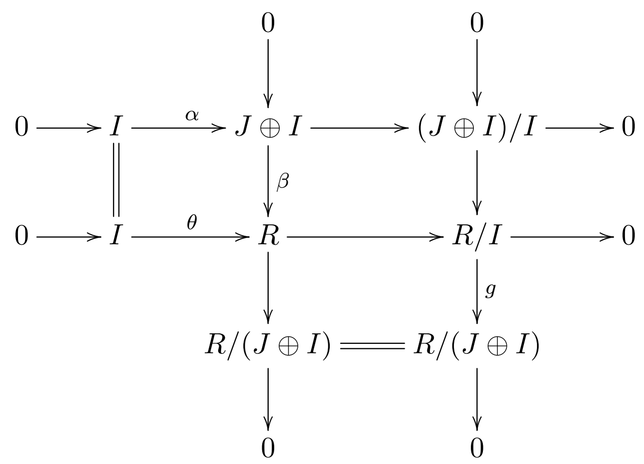
For the converse, suppose that E is gdc-flat. Proposition 4.6 implies that $\tau_{gd}(E)$ embeds in a projective module P. Since $\mathrm{Soc}(E/\tau_{gd}(E))$ is projective, $\tau_{gd}(E)$ is a gd-closed submodule of E and this means that it is a neat submodule of E. Since neat submodules and closed submodules coincide for a right C-ring, it follows that $\tau_{gd}(E)$ is closed in E. Hence $\tau_{gd}(E)$ is a direct summand of E because E is injective. It follows that $\tau_{gd}(E)$ is also injective. Therefore $\tau_{gd}(E)$ is a direct summand of P and it is projective. □

Definition 4.9. A ring R is called a right Kasch ring if each simple R-module embeds in R_R.

For a C-ring, it is true that the injective hull of a g-semiartinian module is also g-semiartinian. We use this fact in the following proof.

Proposition 4.10. Every cyclic module with projective socle is projective if and only if every cyclic gdc-flat module is projective.

Proof. ($\Rightarrow$) Let M be a cyclic gdc-flat module. Then there is a right ideal I of R such that $M \cong R/I$. Since I is gd-closed ideal of R, there is a right ideal J of R such that $J \cap I = 0$ and $\mathrm{Soc}(R/(J \oplus I))$ is projective. Consider the following commutative diagram:



$R/(J \oplus I)$ is projective since $\mathrm{Soc}(R/(J \oplus I))$ is projective. It follows that β is $\mathcal{S}plit$-monomorphism. Moreover, α is a $\mathcal{S}plit$-monomorphism. Therefore $\theta = \beta\alpha$ is a $\mathcal{S}plit$-monomorphism. Hence $M \cong R/I$ is projective since it is a direct summand of R.
($\Leftarrow$) Let M be a cyclic module with projective socle. Then it is gdc-flat by Remark 4.3 (1). Therefore M is projective. □

Proposition 4.11. The following statements are equivalent.

1. Every module with projective socle is projective.
2. Every gdc-flat module is projective.
3. Every gd-closed submodule of a projective module is a direct summand.

Proof. (1)$\Rightarrow$ (2) Let C be a gdc-flat module. Then there is a projective module B and gd-closed sequence $0 \to A \xrightarrow{f} B \to C \to 0$. It means that there is a submodule $S \subset B$ such that $S \cap A = 0$ and $\mathrm{Soc}(B/S \oplus A)$ is projective. Consider the diagram in the proof of Proposition 3.3. β is a $\mathcal{S}plit$-monomorphism since $B/(S \oplus A)$ is projective by assumption. It follows that $f = \beta\alpha$ is a $\mathcal{S}plit$-monomorphism as well. Since C is a direct summand of B, it is projective.
(2) $\Rightarrow$ (3) Let B be a projective module and A be a gd-closed submodule of B. Proposition 4.2 implies that C is gdc-flat since $\mathcal{E} : 0 \to A \to B \to C \to 0$ is gd-closed with B is projective. By assumption, C is projective, hence $\mathcal{E}$ splits. It follows that A is a direct summand of B.
(3)$\Rightarrow$ (1) Let C be a module and assume that it has a projective socle. Then there exists a projective module B and a gd-closed exact sequence $0 \to A \to B \to C \to 0$. Hence, A is a direct summand of B by assumption. Indeed, $B \cong A \oplus C$. Thus C is projective. □

Proposition 4.12. The following are equivalent.

1. Every gd-closed sequence is pure.
2. Every gdc-flat is flat.
3. Every module with projective socle is flat.
4. Every finitely generated module with projective socle is flat.

Proof. The implications (1)$\Rightarrow$(2), (2)$\Rightarrow$(3), (3)$\Rightarrow$(4) are obvious. (4)$\Rightarrow$(1) By [22, Corollary 3.49], every module having projective socle is flat. Let $\mathcal{E} : 0 \to A \xrightarrow{f} B \to C \to 0$ be a gd-closed exact sequence, It means that there exists a submodule $S \subset B$ such that $S \cap A = 0$ and $\mathrm{Soc}(B/(S \oplus A))$ is projective. Now consider the commutative diagram in the proof of Proposition 4.11. β is a $\mathcal{P}ure$-monomorphism since $B/(S \oplus A)$ is flat. Hence $f = \beta\alpha$ is a $\mathcal{P}ure$-monomorphism. □

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