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*Research Article*

# Abstract generalized fractional Landau inequalities over $\mathbb{R}$

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ABSTRACT. We present uniform and $L_p$ mixed Caputo-Bochner abstract generalized fractional Landau inequalities over $\mathbb{R}$ of fractional orders $2 < \alpha \leq 3$. These estimate the size of first and second derivatives of a composition with a Banach space valued function over $\mathbb{R}$. We give applications when $\alpha = 2.5$.

**Keywords:** Abstract generalized fractional Landau inequality, right and left Caputo abstract generalized fractional derivatives.

**2020 Mathematics Subject Classification:** 26A33, 26D10, 26D15.

*Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.*

## 1. INTRODUCTION

Let $p \in [1, \infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$ and $f : I \to \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of $f$, such that

$$\|f'\|_{p,I} \leq C_p(I) \|f\|_{p,I}^{\frac{1}{2}} \|f''\|_{p,I}^{\frac{1}{2}}, \tag{1}$$

where $\|\cdot\|_{p,I}$ is the $p$-norm on the interval $I$, see [1], [5]. The research on these inequalities started by E. Landau [10] in 1913. For the case of $p = \infty$, he proved that

$$C_\infty(\mathbb{R}_+) = 2 \text{ and } C_\infty(\mathbb{R}) = \sqrt{2} \tag{2}$$

are the best constants in (1). In 1932, G. H. Hardy and J. E. Littlewood [7] proved (1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2} \text{ and } C_2(\mathbb{R}) = 1. \tag{3}$$

In 1935, G. H. Hardy, E. Landau and J. E. Littlewood [8] showed that the best constants $C_p(\mathbb{R}_+)$ in (1) satisfies the estimate

$$C_p(\mathbb{R}_+) \leq 2, \text{ for } p \in [1, \infty), \tag{4}$$

which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$.

In fact, in [6] and [9] was shown that $C_p(\mathbb{R}) \leq \sqrt{2}$. We need the following concepts from abstract generalized fractional calculus. Our integrals next are of Bochner type [11]. We need the following definition.

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**Definition 1.1.** *([4], p. 104) Let $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ a Banach space, $g \in C^1([a,b])$ and increasing, $f \in C([a,b], X)$, $\nu > 0$. We define the left Riemann-Liouville generalized fractional Bochner integral operator*

(5)
$$\left(J_{a;g}^{\nu} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_a^x (g(x) - g(z))^{\nu-1} g'(z) f(z) \, dz,$$

*$\forall\, x \in [a,b]$, where $\Gamma$ is the gamma function. The last integral is of Bochner type. Since $f \in C([a,b], X)$, then $f \in L_\infty([a,b], X)$. By Theorem 4.10, p. 98, [4], we get that $\left(J_{a;g}^{\nu} f\right) \in C([a,b], X)$. Above we set $J_{a;g}^0 f := f$ and see that $\left(J_{a;g}^{\nu} f\right)(a) = 0$.*

We need the following definition.

**Definition 1.2.** *([4], p. 105) Let $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ a Banach space, $g \in C^1([a,b])$ and increasing, $f \in C([a,b], X)$, $\nu > 0$. We define the right Riemann-Liouville generalized fractional Bochner integral operator*

(6)
$$\left(J_{b-;g}^{\nu} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_x^b (g(z) - g(x))^{\nu-1} g'(z) f(z) \, dz,$$

*$\forall\, x \in [a,b]$, where $\Gamma$ is the gamma function. The last integral is of Bochner type. Since $f \in C([a,b], X)$, then $f \in L_\infty([a,b], X)$. By Theorem 4.11, p. 101, [4], we get that $\left(J_{b-;g}^{\nu} f\right) \in C([a,b], X)$. Above we set $J_{b-;g}^0 f := f$ and see that $\left(J_{b-;g}^{\nu} f\right)(b) = 0$.*

We also need the following definition.

**Definition 1.3.** *([4], p. 106) Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. We define the left generalized g-fractional derivative $X$-valued of $f$ of order $\alpha$ as follows:*

(7)
$$\left(D_{a+;g}^{\alpha} f\right)(x) := \frac{1}{\Gamma(n-\alpha)} \int_a^x (g(x) - g(t))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t)) \, dt,$$

*$\forall\, x \in [a,b]$. The last integral is of Bochner type. Ordinary vector valued derivative is as in [12], similar to numerical one. If $\alpha \notin \mathbb{N}$, by Theorem 4.10, p. 98, [4], we have that $\left(D_{a+;g}^{\alpha} f\right) \in C([a,b], X)$. We see that*

(8)
$$\left(J_{a;g}^{n-\alpha} \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)\right)(x) = \left(D_{a+;g}^{\alpha} f\right)(x), \;\; \forall\, x \in [a,b].$$

*We set*

(9)
$$D_{a+;g}^{n} f(x) := \left(\left(f \circ g^{-1}\right)^{n} \circ g\right)(x) \in C([a,b], X), \;\; n \in \mathbb{N},$$

$$D_{a+;g}^{0} f(x) = f(x), \;\; \forall\, x \in [a,b].$$

*When $g = id$, then*

(10)
$$D_{a+;g}^{\alpha} f = D_{a+;id}^{\alpha} f = D_{*a}^{\alpha} f,$$

*the usual left $X$-valued Caputo fractional derivative, see* [4, Chapter 1].

We mention the following definition.

**Definition 1.4.** *([4], p. 107) Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. We define the right generalized g-fractional derivative X-valued of f of order $\alpha$ as follows:*

$$(11) \qquad \left(D^{\alpha}_{b-;g} f\right)(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (g(t) - g(x))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt,$$

*$\forall\, x \in [a,b]$. The last integral is of Bochner type. If $\alpha \notin \mathbb{N}$, by Theorem 4.11, p. 101, [4], we have that $\left(D^{\alpha}_{b-;g} f\right) \in C([a,b], X)$. We see that*

$$(12) \qquad J^{n-\alpha}_{b-;g} \left( (-1)^n \left(f \circ g^{-1}\right)^{(n)} \circ g \right)(x) = \left(D^{\alpha}_{b-;g} f\right)(x), \quad a \le x \le b.$$

*We set*

$$(13) \qquad D^{n}_{b-;g} f(x) := (-1)^n \left( \left(f \circ g^{-1}\right)^{n} \circ g \right)(x) \in C([a,b], X), \ n \in \mathbb{N},$$

$$D^{0}_{b-;g} f(x) := f(x), \quad \forall\, x \in [a,b].$$

*When $g = id$, then*

$$(14) \qquad D^{\alpha}_{b-;g} f(x) = D^{\alpha}_{b-;id} f(x) = D^{\alpha}_{b-} f,$$

*the usual right X-valued Caputo fractional derivative, see* [4, Chapter 2].

We mention the generalized left fractional Taylor formula:

**Theorem 1.1.** *([4], p. 107) Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$, $a \le x \le b$. Then,*

$$\begin{aligned} f(x) = {} & f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) \\ & + \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \left(D^{\alpha}_{a+;g} f\right)(t)\, dt \\ = {} & f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) \\ (15) \qquad & + \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha-1} \left( \left(D^{\alpha}_{a+;g} f\right) \circ g^{-1} \right)(z)\, dz. \end{aligned}$$

We also mention the generalized right fractional Taylor formula:

**Theorem 1.2.** *([4], p. 108) Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$,*

*$a \leq x \leq b$. Then,*

$$f(x) = f(b) + \sum_{i=1}^{n-1} \frac{(g(x) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b)) + \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \left(D_{b-;g}^{\alpha} f\right)(t)\, dt$$

$$= f(b) + \sum_{i=1}^{n-1} \frac{(g(x) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b)) + \frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{\alpha-1} \left(\left(D_{b-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz. \tag{16}$$

*By convention, we suppose that*

$$\left(D_{x_0+;g}^{\alpha} f\right)(x) = 0, \text{ for } x < x_0,$$
$$\left(D_{x_0-;g}^{\alpha} f\right)(x) = 0, \text{ for } x > x_0, \tag{17}$$

*for any $x, x_0 \in [a, b]$.*

The author has already done an extensive amount of work on fractional Landau inequalities, see [3], and on abstract fractional Landau inequalities, see [4]. However, there the proving methods came out of applications of fractional Ostrowski inequalities ([2], [4]) and the derived inequalities were for small fractional orders, i.e. $\alpha \in (0, 1)$. Usually there the domains where $[A, +\infty)$ or $(-\infty, B]$, with $A, B \in \mathbb{R}$ and in one mixed case the domain was all of $\mathbb{R}$.

In this work with less assumptions, we establish uniform and $L_p$ type mixed Caputo-Bochner abstract generalized fractional Landau inequalities over $\mathbb{R}$ for fractional orders $2 < \alpha \leq 3$. The method of proving is based on left and right Caputo-Bochner generalized fractional Taylor's formulae with integral remainder, see Theorems 1.1,1.2. We give also applications for $\alpha = 2.5$. Certainly, we are also inspired by [3], [4].

## 2. Main Results

We give the following abstract mixed generalized fractional Landau inequalities over $\mathbb{R}$.

**Theorem 2.3.** *Let $2 < \alpha \leq 3$ and $f \in C^3(\mathbb{R}, X)$, where $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1(\mathbb{R})$, strictly increasing, such that $g^{-1} \in C^3(g(\mathbb{R}))$. We assume that $\|\|f\|\|_{\infty,\mathbb{R}} < \infty$ and that*

$$K := \max\left\{ \left\|\left\|\left(\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\right\|\right\|_{\infty,\mathbb{R}\times g(\mathbb{R})}, \left\|\left\|\left(\left(D_{a-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\right\|\right\|_{\infty,\mathbb{R}\times g(\mathbb{R})} \right\} < \infty, \tag{18}$$

*where $(a, z) \in \mathbb{R} \times g(\mathbb{R})$. Then,*

$$\left\|\left\|\left(f \circ g^{-1}\right)' \circ g\right\|\right\|_{\infty,\mathbb{R}} \leq \alpha \left(\frac{K}{\Gamma(\alpha+1)}\right)^{\frac{1}{\alpha}} \left(\frac{\|\|f\|\|_{\infty,\mathbb{R}}}{\alpha - 1}\right)^{\frac{\alpha-1}{\alpha}} \tag{19}$$

*and*

$$\left\|\left\|\left(f \circ g^{-1}\right)'' \circ g\right\|\right\|_{\infty,\mathbb{R}} \leq \alpha \left(\frac{K}{\Gamma(\alpha+1)}\right)^{\frac{2}{\alpha}} \left(\frac{4\|\|f\|\|_{\infty,\mathbb{R}}}{\alpha - 2}\right)^{\frac{\alpha-2}{\alpha}}. \tag{20}$$

*That is,*

$$\left\|\left\|\left(f \circ g^{-1}\right)' \circ g\right\|\right\|_{\infty,\mathbb{R}}, \left\|\left\|\left(f \circ g^{-1}\right)'' \circ g\right\|\right\|_{\infty,\mathbb{R}} < \infty.$$

*Proof.* Here $2 < \alpha \leq 3$, i.e. $\lceil \alpha \rceil = 3$. Let $f \in C^3(\mathbb{R}, X)$, where $(X, \|\cdot\|)$ is a Banach space, $a \in \mathbb{R}$ is fixed momentarily. We need the following abstract generalized fractional Taylor formulae for $n = 3$. By Theorem 1.1, we get

$$f(x) - f(a) = (g(x) - g(a)) \left(f \circ g^{-1}\right)'(g(a)) + \frac{(g(x) - g(a))^2}{2} \left(f \circ g^{-1}\right)''(g(a)) + \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha-1} \left(\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz, \quad \forall\, x \geq a. \tag{21}$$

And by Theorem 1.2, we get

$$f(x) - f(a) = (g(x) - g(a)) \left(f \circ g^{-1}\right)'(g(a)) + \frac{(g(x) - g(a))^2}{2} \left(f \circ g^{-1}\right)''(g(a)) + \frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(a)} (z - g(x))^{\alpha-1} \left(\left(D_{a-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz, \quad \forall\, x \leq a. \tag{22}$$

Let $x_1 > a$, then

$$(g(x_1) - g(a)) \left(f \circ g^{-1}\right)'(g(a)) + \frac{(g(x_1) - g(a))^2}{2} \left(f \circ g^{-1}\right)''(g(a)) = (f(x_1) - f(a)) - \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x_1)} (g(x_1) - z)^{\alpha-1} \left(\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz =: A, \tag{23}$$

and let $x_2 < a$, then

$$(g(x_2) - g(a)) \left(f \circ g^{-1}\right)'(g(a)) + \frac{(g(x_2) - g(a))^2}{2} \left(f \circ g^{-1}\right)''(g(a)) = (f(x_2) - f(a)) - \frac{1}{\Gamma(\alpha)} \int_{g(x_2)}^{g(a)} (z - g(x_2))^{\alpha-1} \left(\left(D_{a-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz =: B. \tag{24}$$

Let $h > 0$, we can choose $x_1$ such that $g(x_1) - g(a) = h$ and we can choose $x_2$ such that $g(a) - g(x_2) = h$. That is $g(x_1) = g(a) + h$ and $g(x_2) = g(a) - h$, and $g(x_2) - g(a) = -h$. Furthermore, it holds $g(x_2) - g(x_1) = -2h$. We can rewrite (23) as

$$h \left(f \circ g^{-1}\right)'(g(a)) + \frac{h^2}{2} \left(f \circ g^{-1}\right)''(g(a)) = A, \tag{25}$$

and we can rewrite (24) as

$$-h \left(f \circ g^{-1}\right)'(g(a)) + \frac{h^2}{2} \left(f \circ g^{-1}\right)''(g(a)) = B. \tag{26}$$

Solving the system of (25) and (26), we find

$$\left(f \circ g^{-1}\right)'(g(a)) = \frac{A - B}{2h} \tag{27}$$

and

$$\left(f \circ g^{-1}\right)''(g(a)) = \frac{A + B}{h^2}.$$

We assumed that

$$\left\|\left\|\left(\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\right\|\right\|_{\infty, \mathbb{R} \times g(\mathbb{R})}, \left\|\left\|\left(\left(D_{a-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\right\|\right\|_{\infty, \mathbb{R} \times g(\mathbb{R})} < \infty.$$

We obtain,

$$\left\|\left(f \circ g^{-1}\right)'(g(a))\right\| = \frac{1}{2h} \|A - B\|$$

and

$$\left\| \left(f \circ g^{-1}\right)'' (g(a)) \right\| \leq \frac{1}{h^2} \left( \|A\| + \|B\| \right). \tag{28}$$

We get

$$\begin{aligned} \|A\| &= \left\| (f(x_1) - f(a)) - \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x_1)} (g(x_1) - z)^{\alpha-1} \left( \left( D_{a+;g}^{\alpha} f \right) \circ g^{-1} \right)(z)\, dz \right\| \\ &\leq 2 \|\|f\|\|_{\infty,\mathbb{R}} + \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x_1)} (g(x_1) - z)^{\alpha-1} \left\| \left( \left( D_{a+;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\| dz \\ &\leq 2 \|\|f\|\|_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha)} \left( \int_{g(a)}^{g(x_1)} (g(x_1) - z)^{\alpha-1} dz \right) \\ &= 2 \|\|f\|\|_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha+1)} (g(x_1) - g(a))^{\alpha} = 2 \|\|f\|\|_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha+1)} h^{\alpha}. \end{aligned} \tag{29}$$

That is,

$$\|A\| \leq 2 \|\|f\|\|_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha+1)} h^{\alpha}, \quad h > 0. \tag{30}$$

Similarly, it holds

$$\begin{aligned} \|B\| &= \left\| (f(x_2) - f(a)) - \frac{1}{\Gamma(\alpha)} \int_{g(x_2)}^{g(a)} (z - g(x_2))^{\alpha-1} \left( \left( D_{a-;g}^{\alpha} f \right) \circ g^{-1} \right)(z)\, dz \right\| \\ &\leq 2 \|\|f\|\|_{\infty,\mathbb{R}} + \frac{1}{\Gamma(\alpha)} \int_{g(x_2)}^{g(a)} (z - g(x_2))^{\alpha-1} \left\| \left( \left( D_{a-;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\| dz \\ &\leq 2 \|\|f\|\|_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha)} \left( \int_{g(x_2)}^{g(a)} (z - g(x_2))^{\alpha-1} dz \right) \\ &= 2 \|\|f\|\|_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha+1)} (g(a) - g(x_2))^{\alpha} = 2 \|\|f\|\|_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha+1)} h^{\alpha}. \end{aligned} \tag{31}$$

That is,

$$\|B\| \leq 2 \|\|f\|\|_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha+1)} h^{\alpha}, \quad h > 0. \tag{32}$$

Furthermore, we have

$$\|A\| + \|B\| \leq 4 \|\|f\|\|_{\infty,\mathbb{R}} + \frac{2K}{\Gamma(\alpha+1)} h^{\alpha}, \quad h > 0. \tag{33}$$

We also notice that

$$\|A-B\| = \left\| f(x_1) - f(a) - \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x_1)} (g(x_1)-z)^{\alpha-1} \left( \left( D_{a+;g}^{\alpha} f \right) \circ g^{-1} \right)(z)\, dz \right.$$

$$\left. - f(x_2) + f(a) + \frac{1}{\Gamma(\alpha)} \int_{g(x_2)}^{g(a)} (z-g(x_2))^{\alpha-1} \left( \left( D_{a-;g}^{\alpha} f \right) \circ g^{-1} \right)(z)\, dz \right\|$$

$$\leq \|f(x_1)-f(x_2)\| + \frac{1}{\Gamma(\alpha)} \left[ \int_{g(a)}^{g(x_1)} (g(x_1)-z)^{\alpha-1} \left\| \left( \left( D_{a+;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\| dz \right. \tag{34}$$

$$\left. + \int_{g(x_2)}^{g(a)} (z-g(x_2))^{\alpha-1} \left\| \left( \left( D_{a-;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\| dz \right]$$

$$\leq 2\, \| \|f\| \|_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha)} \left[ \int_{g(a)}^{g(x_1)} (g(x_1)-z)^{\alpha-1} dz + \int_{g(x_2)}^{g(a)} (z-g(x_2))^{\alpha-1} dz \right]$$

$$= 2\, \| \|f\| \|_{\infty,\mathbb{R}} + \frac{K}{\Gamma(\alpha+1)} \left[ (g(x_1)-g(a))^{\alpha} + (g(a)-g(x_2))^{\alpha} \right]$$

$$= 2\, \| \|f\| \|_{\infty,\mathbb{R}} + \frac{2Kh^{\alpha}}{\Gamma(\alpha+1)}.$$

That is,

$$\frac{\|A-B\|}{2} \leq \| \|f\| \|_{\infty,\mathbb{R}} + \frac{Kh^{\alpha}}{\Gamma(\alpha+1)}, \quad h>0. \tag{35}$$

Consequently, we obtain

$$\left\| \left( f \circ g^{-1} \right)' (g(a)) \right\| \overset{((28),(35))}{\leq} \frac{\| \|f\| \|_{\infty,\mathbb{R}}}{h} + \frac{Kh^{\alpha-1}}{\Gamma(\alpha+1)}$$

and

$$\left\| \left( f \circ g^{-1} \right)'' (g(a)) \right\| \overset{((28),(33))}{\leq} \frac{4\, \| \|f\| \|_{\infty,\mathbb{R}}}{h^2} + \frac{2Kh^{\alpha-2}}{\Gamma(\alpha+1)}, \tag{36}$$

$h>0$, for any $a \in \mathbb{R}$. Hence,

$$\left\| \left\| \left( f \circ g^{-1} \right)' \circ g \right\| \right\|_{\infty,\mathbb{R}} \leq \frac{\| \|f\| \|_{\infty,\mathbb{R}}}{h} + \frac{Kh^{\alpha-1}}{\Gamma(\alpha+1)}$$

and

$$\left\| \left\| \left( f \circ g^{-1} \right)'' \circ g \right\| \right\|_{\infty,\mathbb{R}} \leq \frac{4\, \| \|f\| \|_{\infty,\mathbb{R}}}{h^2} + \frac{2Kh^{\alpha-2}}{\Gamma(\alpha+1)}, \tag{37}$$

true $\forall\, h>0$, $2<\alpha\leq 3$. Call

$$\mu := \| \|f\| \|_{\infty,\mathbb{R}}, \quad \theta = \tfrac{K}{\Gamma(\alpha+1)}, \tag{38}$$

both are greater than zero. Set also $\rho := \alpha - 1 > 1$. We consider the function

$$y(h) := \frac{\mu}{h} + \theta h^{\rho}, \quad \forall\, h>0. \tag{39}$$

We have

$$y'(h) = -\frac{\mu}{h^2} + \rho\theta h^{\rho-1} = 0,$$

then

$$\rho\theta h^{\rho+1} = \mu,$$

with a unique solution

$$h_0 := h_{crit.no} = \left(\frac{\mu}{\rho\theta}\right)^{\frac{1}{\rho+1}}. \tag{40}$$

We have that

$$y''(h) = 2\mu h^{-3} + \rho(\rho-1)\theta h^{\rho-2}. \tag{41}$$

We observe that

$$\begin{aligned} y''(h_0) &= 2\mu\left(\frac{\mu}{\rho\theta}\right)^{-\frac{3}{\rho+1}} + \rho(\rho-1)\theta\left(\frac{\mu}{\rho\theta}\right)^{\frac{((\rho+1)-3)}{\rho+1}} \\ &= \left(\frac{\mu}{\rho\theta}\right)^{-\frac{3}{\rho+1}}[2\mu + \mu(\rho-1)] = \mu\left(\frac{\mu}{\rho\theta}\right)^{-\frac{3}{\rho+1}}(\rho+1) > 0. \end{aligned} \tag{42}$$

Therefore, $y$ has a global minimum at $h_0 = \left(\frac{\mu}{\rho\theta}\right)^{\frac{1}{\rho+1}}$, which is

$$\begin{aligned} y(h_0) &= \frac{\mu}{\left(\frac{\mu}{\rho\theta}\right)^{\frac{1}{\rho+1}}} + \theta\left(\frac{\mu}{\rho\theta}\right)^{\frac{\rho}{\rho+1}} \\ &= (\rho\theta)^{\frac{1}{\rho+1}}\frac{\mu}{\mu^{\frac{1}{\rho+1}}} + \frac{\theta\mu^{\frac{\rho}{\rho+1}}}{\rho^{\frac{\rho}{\rho+1}}\theta^{\frac{\rho}{\rho+1}}} \\ &= (\theta\mu^\rho)^{\frac{1}{\rho+1}}\left(\rho^{\frac{1}{\rho+1}} + \frac{1}{\rho^{\frac{\rho}{\rho+1}}}\right) = (\theta\mu^\rho)^{\frac{1}{\rho+1}}\left(\frac{\rho+1}{\rho^{\frac{\rho}{\rho+1}}}\right) \\ &= (\theta\mu^\rho)^{\frac{1}{\rho+1}}(\rho+1)\rho^{-\frac{\rho}{\rho+1}}. \end{aligned} \tag{43}$$

That is,

$$y(h_0) = (\theta\mu^\rho)^{\frac{1}{\rho+1}}(\rho+1)\rho^{-\frac{\rho}{\rho+1}}. \tag{44}$$

Consequently,

$$y(h_0) = \left(\frac{K}{\Gamma(\alpha+1)}\||\|f\||\|_{\infty,\mathbb{R}}^{\alpha-1}\right)^{\frac{1}{\alpha}}\alpha(\alpha-1)^{-\left(\frac{\alpha-1}{\alpha}\right)}. \tag{45}$$

We have proved that

$$\left\|\left|\left\|\left(f\circ g^{-1}\right)'\circ g\right\|\right|\right\|_{\infty,\mathbb{R}} \leq \left(\frac{K}{\Gamma(\alpha+1)}\||\|f\||\|_{\infty,\mathbb{R}}^{\alpha-1}\right)^{\frac{1}{\alpha}}\alpha(\alpha-1)^{-\left(\frac{\alpha-1}{\alpha}\right)}. \tag{46}$$

Next call

$$\xi := 4\||\|f\||\|_{\infty,\mathbb{R}}, \quad \psi = \frac{2K}{\Gamma(\alpha+1)}, \tag{47}$$

both are greater than zero. Set also $\varphi := \alpha - 2 > 0$. We consider the function

$$\gamma(h) := \frac{\xi}{h^2} + \psi h^\varphi = \xi h^{-2} + \psi h^\varphi, \ \ \forall\, h > 0. \tag{48}$$

We have

$$\gamma'(h) = -2\xi h^{-3} + \varphi\psi h^{\varphi-1} = 0,$$

then

$$\varphi\psi h^{\varphi+2}=2\xi,$$

with a unique solution

$$h_0:=h_{crit.no}=\left(\frac{2\xi}{\varphi\psi}\right)^{\frac{1}{\varphi+2}}. \tag{49}$$

We have that

$$\gamma''(h)=6\xi h^{-4}+\varphi(\varphi-1)\psi h^{\varphi-2}. \tag{50}$$

We see that

$$\begin{aligned}\gamma''(h_0)&=6\xi\left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{4}{\varphi+2}}+\varphi(\varphi-1)\psi\left(\frac{2\xi}{\varphi\psi}\right)^{\frac{(\varphi+2)-4}{\varphi+2}}\\&=\left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{4}{\varphi+2}}[6\xi+(\varphi-1)2\xi]=2\xi\left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{4}{\varphi+2}}(\varphi+2)>0.\end{aligned} \tag{51}$$

Therefore, $\gamma$ has a global minimum at $h_0=\left(\frac{2\xi}{\varphi\psi}\right)^{\frac{1}{\varphi+2}}$, which is

$$\begin{aligned}\gamma(h_0)&=\xi\left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{2}{\varphi+2}}+\psi\left(\frac{2\xi}{\varphi\psi}\right)^{\frac{\varphi+2-2}{\varphi+2}}\\&=\left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{2}{\varphi+2}}\left[\xi+\psi\frac{2\xi}{\varphi\psi}\right]=\frac{\xi}{\varphi}\left(\frac{\xi}{\varphi}\right)^{-\frac{2}{\varphi+2}}\left(\frac{2}{\psi}\right)^{-\frac{2}{\varphi+2}}(\varphi+2)\\&=\left(\frac{\xi}{\varphi}\right)^{\frac{\varphi}{\varphi+2}}\left(\frac{\psi}{2}\right)^{\frac{2}{\varphi+2}}(\varphi+2).\end{aligned} \tag{52}$$

That is,

$$\gamma(h_0)=\left(\frac{\xi}{\varphi}\right)^{\frac{\varphi}{\varphi+2}}\left(\frac{\psi}{2}\right)^{\frac{2}{\varphi+2}}(\varphi+2). \tag{53}$$

Consequently,

$$\gamma(h_0)=\left(\frac{4\,\|\|\|f\|\|\|_{\infty,\mathbb{R}}}{\alpha-2}\right)^{\frac{\alpha-2}{\alpha}}\left(\frac{K}{\Gamma(\alpha+1)}\right)^{\frac{2}{\alpha}}\alpha. \tag{54}$$

We have proved that

$$\left\|\left\|\left\|\left(f\circ g^{-1}\right)''\circ g\right\|\right\|\right\|_{\infty,\mathbb{R}}\leq\left(\frac{4\,\|\|\|f\|\|\|_{\infty,\mathbb{R}}}{\alpha-2}\right)^{\frac{\alpha-2}{\alpha}}\left(\frac{K}{\Gamma(\alpha+1)}\right)^{\frac{2}{\alpha}}\alpha. \tag{55}$$

The theorem is established. □

We also give an $L_p$ analog of a generalized fractional Landau inequality

**Theorem 2.4.** *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $2<\alpha\leq 3$ and $f\in C^3(\mathbb{R},X)$, where $(X,\|\cdot\|)$ is a Banach space. Let $g\in C^1(\mathbb{R})$, strictly increasing, such that $g^{-1}\in C^3(g(\mathbb{R}))$. We assume that*

$\||f\||_{\infty,\mathbb{R}} < \infty$, *and that*

$$M := \max\left\{ \sup_{a\in\mathbb{R}} \left\|\left\|\left( \left( D^{\alpha}_{a+;g} f\right) \circ g^{-1}\right)(z)\right\|\right\|_{p,g(\mathbb{R})}, \right.$$

$$\left. \sup_{a\in\mathbb{R}} \left\|\left\|\left( \left( D^{\alpha}_{a-;g} f\right) \circ g^{-1}\right)(z)\right\|\right\|_{p,g(\mathbb{R})} \right\} < \infty. \tag{56}$$

*Then,*
*1)*

$$\left\|\left\|\left(f\circ g^{-1}\right)' \circ g\right\|\right\|_{\infty,\mathbb{R}}$$

$$\leq \left(\alpha - \frac{1}{p}\right) \left( \frac{M}{\Gamma(\alpha)\left(q(\alpha-1)+1\right)^{\frac{1}{q}}} \right)^{\left(\frac{1}{\alpha-\frac{1}{p}}\right)} \left( \frac{\||f\||_{\infty,\mathbb{R}}}{\alpha - 1 - \frac{1}{p}} \right)^{\left(\frac{\alpha-1-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)} \tag{57}$$

*2) under the additional assumption* $2+\frac{1}{p} < \alpha \leq 3$, *we have*

$$\left\|\left\|\left(f\circ g^{-1}\right)'' \circ g\right\|\right\|_{\infty,\mathbb{R}}$$

$$\leq \left(\alpha - \frac{1}{p}\right) \left( \frac{M}{\Gamma(\alpha)\left(q(\alpha-1)+1\right)^{\frac{1}{q}}} \right)^{\left(\frac{2}{\alpha-\frac{1}{p}}\right)} \left( \frac{4\||f\||_{\infty,\mathbb{R}}}{\alpha - 2 - \frac{1}{p}} \right)^{\left(\frac{\alpha-2-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)}. \tag{58}$$

*That is,*

$$\left\|\left\|\left(f\circ g^{-1}\right)' \circ g\right\|\right\|_{\infty,\mathbb{R}}, \left\|\left\|\left(f\circ g^{-1}\right)'' \circ g\right\|\right\|_{\infty,\mathbb{R}} < \infty.$$

*Proof.* We continue with the proof of Theorem 2.3. By (23), we have

$$\|A\| = \left\| (f(x_1) - f(a)) - \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x_1)} (g(x_1)-z)^{\alpha-1} \left( \left( D^{\alpha}_{a+;g} f\right)\circ g^{-1}\right)(z)\,dz \right\|$$

$$\leq 2\||f\||_{\infty,\mathbb{R}} + \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x_1)} (g(x_1)-z)^{\alpha-1} \left\|\left( \left( D^{\alpha}_{a+;g} f\right)\circ g^{-1}\right)(z)\right\| dz$$

$$\leq 2\||f\||_{\infty,\mathbb{R}} + \frac{1}{\Gamma(\alpha)} \left( \int_{g(a)}^{g(x_1)} (g(x_1)-z)^{q(\alpha-1)} dz \right)^{\frac{1}{q}}$$

$$\times \left( \int_{g(a)}^{g(x_1)} \left\|\left( \left( D^{\alpha}_{a+;g} f\right)\circ g^{-1}\right)(z)\right\|^{p} dz \right)^{\frac{1}{p}} \tag{59}$$

$$\leq 2\||f\||_{\infty,\mathbb{R}} + \frac{1}{\Gamma(\alpha)} \frac{(g(x_1)-g(a))^{\frac{(q(\alpha-1)+1)}{q}}}{(q(\alpha-1)+1)^{\frac{1}{q}}} \left\|\left\|\left( \left( D^{\alpha}_{a+;g} f\right)\circ g^{-1}\right)(z)\right\|\right\|_{p,g(\mathbb{R})}$$

$$\leq 2\||f\||_{\infty,\mathbb{R}} + \frac{1}{\Gamma(\alpha)} \frac{h^{\alpha-\frac{1}{p}}}{(q(\alpha-1)+1)^{\frac{1}{q}}} \left( \sup_{a\in\mathbb{R}} \left\|\left\|\left( \left( D^{\alpha}_{a+;g} f\right)\circ g^{-1}\right)(z)\right\|\right\|_{p,g(\mathbb{R})} \right)$$

$$\leq 2\||f\||_{\infty,\mathbb{R}} + \frac{h^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(q(\alpha-1)+1)^{\frac{1}{q}}} M.$$

That is,

$$\|A\| \leq 2\,\|\|f\|\|_{\infty,\mathbb{R}} + \frac{M}{\Gamma(\alpha)\,(q(\alpha-1)+1)^{\frac{1}{q}}} h^{\alpha-\frac{1}{p}}, \quad h>0. \tag{60}$$

Similarly, from (24), we get

$$\|B\| = \left\| (f(x_2) - f(a)) - \frac{1}{\Gamma(\alpha)} \int_{g(x_2)}^{g(a)} (z - g(x_2))^{\alpha-1} \left( \left( D_{a-;g}^{\alpha} f \right) \circ g^{-1} \right)(z)\, dz \right\|$$

$$\leq 2\,\|\|f\|\|_{\infty,\mathbb{R}} + \frac{1}{\Gamma(\alpha)} \int_{g(x_2)}^{g(a)} (z - g(x_2))^{\alpha-1} \left\| \left( \left( D_{a-;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\| dz$$

$$\leq 2\,\|\|f\|\|_{\infty,\mathbb{R}} + \frac{1}{\Gamma(\alpha)} \left( \int_{g(x_2)}^{g(a)} (z - g(x_2))^{q(\alpha-1)} dz \right)^{\frac{1}{q}}$$

$$\times \left( \int_{g(x_2)}^{g(a)} \left\| \left( \left( D_{a-;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\|^{p} dz \right)^{\frac{1}{p}}$$

$$\leq 2\,\|\|f\|\|_{\infty,\mathbb{R}} + \frac{1}{\Gamma(\alpha)} \frac{(g(a) - g(x_2))^{\alpha-\frac{1}{p}}}{(q(\alpha-1)+1)^{\frac{1}{q}}} \left( \sup_{a\in\mathbb{R}} \left\|\left\| \left( \left( D_{a-;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\|\right\|_{p,g(\mathbb{R})} \right) \tag{61}$$

$$\leq 2\,\|\|f\|\|_{\infty,\mathbb{R}} + \frac{h^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\,(q(\alpha-1)+1)^{\frac{1}{q}}} M.$$

That is,

$$\|B\| \leq 2\,\|\|f\|\|_{\infty,\mathbb{R}} + \frac{M}{\Gamma(\alpha)\,(q(\alpha-1)+1)^{\frac{1}{q}}} h^{\alpha-\frac{1}{p}}, \quad h>0. \tag{62}$$

Hence, it holds

$$\|A+B\| \leq \|A\ \| + \|B\|$$

$$\overset{\text{(by (60), (62))}}{\leq} 4\,\|\|f\|\|_{\infty,\mathbb{R}} + \frac{2M}{\Gamma(\alpha)\,(q(\alpha-1)+1)^{\frac{1}{q}}} h^{\alpha-\frac{1}{p}}, \quad h>0. \tag{63}$$

Furthermore, we have

$$\|A-B\| \overset{(34)}{\leq} 2\,\|\|f\|\|_{\infty,\mathbb{R}}$$

$$+ \frac{1}{\Gamma(\alpha)} \left[ \left( \int_{g(a)}^{g(x_1)} (g(x_1) - z)^{q(\alpha-1)} dz \right)^{\frac{1}{q}} \left\|\left\| \left( \left( D_{a+;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\|\right\|_{p,g(\mathbb{R})} \right.$$

$$\left. + \left( \int_{g(x_2)}^{g(a)} (z - g(x_2))^{q(\alpha-1)} dz \right)^{\frac{1}{q}} \left\|\left\| \left( \left( D_{a-;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\|\right\|_{p,g(\mathbb{R})} \right]$$

$$\leq 2\,\|\|f\|\|_{\infty,\mathbb{R}} + \frac{M}{\Gamma(\alpha)} \left[ \frac{2h^{\alpha-\frac{1}{p}}}{(q(\alpha-1)+1)^{\frac{1}{q}}} \right]. \tag{64}$$

We have proved that

$$\frac{\|A-B\|}{2} \leq \|\|f\|\|_{\infty,\mathbb{R}} + \frac{M}{\Gamma(\alpha)\,(q(\alpha-1)+1)^{\frac{1}{q}}} h^{\alpha-\frac{1}{p}}, \quad h>0. \tag{65}$$

From (27), (65), we have

$$\left\| \left(f \circ g^{-1}\right)' \left(g\left(a\right)\right) \right\| \leq \frac{\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}}}{h} + \frac{M}{\Gamma\left(\alpha\right)\left(q\left(\alpha-1\right)+1\right)^{\frac{1}{q}}} h^{\alpha-\frac{1}{p}-1}, \tag{66}$$

$h > 0$, any $a \in \mathbb{R}$. And from (27), (63), we get that

$$\left\| \left(f \circ g^{-1}\right)'' \left(g\left(a\right)\right) \right\| \leq \frac{4\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}}}{h^2} + \frac{2M}{\Gamma\left(\alpha\right)\left(q\left(\alpha-1\right)+1\right)^{\frac{1}{q}}} h^{\alpha-\frac{1}{p}-2}, \tag{67}$$

$h > 0$, any $a \in \mathbb{R}$. Hence,

$$\left\|\left\| \left(f \circ g^{-1}\right)' \circ g \right\|\right\|_{\infty,\mathbb{R}} \leq \frac{\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}}}{h} + \left(\frac{M}{\Gamma\left(\alpha\right)\left(q\left(\alpha-1\right)+1\right)^{\frac{1}{q}}}\right) h^{\left(\alpha-\frac{1}{p}-1\right)} \tag{68}$$

and

$$\left\|\left\| \left(f \circ g^{-1}\right)'' \circ g \right\|\right\|_{\infty,\mathbb{R}} \leq \frac{4\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}}}{h^2} + \left(\frac{2M}{\Gamma\left(\alpha\right)\left(q\left(\alpha-1\right)+1\right)^{\frac{1}{q}}}\right) h^{\left(\alpha-\frac{1}{p}-2\right)}, \tag{69}$$

true $\forall\, h > 0$, $2 < \alpha \leq 3$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. In (69), we restrict ourselves to $2 + \frac{1}{p} < \alpha \leq 3$. Call

$$\mu := \left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}}, \quad \theta = \frac{M}{\Gamma(\alpha)(q(\alpha-1)+1)^{\frac{1}{q}}}, \tag{70}$$

both are greater than zero. Set also $\rho := \alpha - 1 - \frac{1}{p} > \frac{1}{q} > 0$. We consider the function

$$y\left(h\right) := \frac{\mu}{h} + \theta h^{\rho}, \quad \forall\, h > 0. \tag{71}$$

As in the proof of Theorem 2.3, it has only one critical number

$$h_0 := h_{crit.no} = \left(\frac{\mu}{\rho\theta}\right)^{\frac{1}{\rho+1}} \tag{72}$$

and a global minimum

$$y\left(h_0\right) = \theta^{\frac{1}{\rho+1}} \mu^{\frac{\rho}{\rho+1}} \left(\rho+1\right) \rho^{-\frac{\rho}{\rho+1}}. \tag{73}$$

Consequently,

$$y\left(h_0\right) = \left(\frac{M}{\Gamma\left(\alpha\right)\left(q\left(\alpha-1\right)+1\right)^{\frac{1}{q}}}\right)^{\frac{1}{\left(\alpha-\frac{1}{p}\right)}} \left(\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}}\right)^{\left(\frac{\alpha-1-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)}$$

$$\left(\alpha-\frac{1}{p}\right)\left(\alpha-1-\frac{1}{p}\right)^{-\left(\frac{\alpha-1-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)}. \tag{74}$$

We have proved that (see (68))

$$\left\|\left\| \left(f \circ g^{-1}\right)' \circ g \right\|\right\|_{\infty,\mathbb{R}}$$

$$\leq \left(\frac{M}{\Gamma\left(\alpha\right)\left(q\left(\alpha-1\right)+1\right)^{\frac{1}{q}}}\right)^{\frac{1}{\left(\alpha-\frac{1}{p}\right)}} \left(\frac{\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}}}{\alpha-1-\frac{1}{p}}\right)^{\left(\frac{\alpha-1-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)} \left(\alpha-\frac{1}{p}\right). \tag{75}$$

We also call

$$\xi := 4\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}}, \quad \psi = \frac{2M}{\Gamma(\alpha)(q(\alpha-1)+1)^{\frac{1}{q}}}, \tag{76}$$

both are greater than zero. Set also $\varphi := \alpha - 2 - \frac{1}{p} > 0$. We consider the function

$$\gamma(h) := \frac{\xi}{h^2} + \psi h^{\varphi}, \quad \forall\, h > 0. \tag{77}$$

As in the proof of Theorem 2.3, $\gamma$ has a global minimum at

$$h_0 = \left(\frac{2\xi}{\varphi\psi}\right)^{\frac{1}{\varphi+2}}, \tag{78}$$

which is

$$\gamma(h_0) = \left(\frac{\xi}{\varphi}\right)^{\frac{\varphi}{\varphi+2}} \left(\frac{\psi}{2}\right)^{\frac{2}{\varphi+2}} (\varphi + 2). \tag{79}$$

Consequently,

$$\gamma(h_0) = \left(\frac{4\,\|\|f\|\|_{\infty,\mathbb{R}}}{\alpha - 2 - \frac{1}{p}}\right)^{\left(\frac{\alpha-2-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)} \left(\frac{M}{\Gamma(\alpha)\left(q(\alpha-1)+1\right)^{\frac{1}{q}}}\right)^{\left(\frac{2}{\alpha-\frac{1}{p}}\right)} \left(\alpha - \frac{1}{p}\right). \tag{80}$$

We have proved that (see (69))

$$\begin{aligned} &\left\|\left\|\left(f \circ g^{-1}\right)'' \circ g\right\|\right\|_{\infty,\mathbb{R}} \\ &\leq \left(\frac{4\,\|\|f\|\|_{\infty,\mathbb{R}}}{\alpha - 2 - \frac{1}{p}}\right)^{\left(\frac{\alpha-2-\frac{1}{p}}{\alpha-\frac{1}{p}}\right)} \left(\frac{M}{\Gamma(\alpha)\left(q(\alpha-1)+1\right)^{\frac{1}{q}}}\right)^{\left(\frac{2}{\alpha-\frac{1}{p}}\right)} \left(\alpha - \frac{1}{p}\right). \end{aligned} \tag{81}$$

The theorem is established. □

Next, we apply Theorems 2.3, 2.4 for $g(t) = e^t$, $t \in \mathbb{R}$ and $\alpha = 2.5$.

**Corollary 2.1.** *Let $f \in C^3(\mathbb{R}, X)$, where $(X, \|\cdot\|)$ is a Banach space. We assume that $\|\|f\|\|_{\infty,\mathbb{R}} < \infty$, and that*

$$\begin{aligned} K_{2.5} := \max \Big\{ &\left\|\left\|\left(\left(D^{2.5}_{a+;e^t} f\right) \circ \ln\right)(z)\right\|\right\|_{\infty,\mathbb{R}\times(0,\infty)}, \\ &\left\|\left\|\left(\left(D^{2.5}_{a-;e^t} f\right) \circ \ln\right)(z)\right\|\right\|_{\infty,\mathbb{R}\times(0,\infty)} \Big\} < \infty, \end{aligned} \tag{82}$$

*where $(a, z) \in \mathbb{R} \times (0, \infty)$. Then,*

$$\left\|\left\|(f \circ \ln)' \circ e^t\right\|\right\|_{\infty,\mathbb{R}} \leq 1.21136\,(K_{2.5})^{0.4} \left(\|\|f\|\|_{\infty,\mathbb{R}}\right)^{0.6} \tag{83}$$

*and*

$$\left\|\left\|(f \circ \ln)'' \circ e^t\right\|\right\|_{\infty,\mathbb{R}} \leq 1.44713\,(K_{2.5})^{0.8} \left(\|\|f\|\|_{\infty,\mathbb{R}}\right)^{0.2}. \tag{84}$$

*That is,*

$$\left\|\left\|(f \circ \ln)' \circ e^t\right\|\right\|_{\infty,\mathbb{R}}, \left\|\left\|(f \circ \ln)'' \circ e^t\right\|\right\|_{\infty,\mathbb{R}} < \infty.$$

*Proof.* By Theorem 2.3. □

We finish with the following result.

**Corollary 2.2.** *(case of $g(t) = e^t$, $\alpha = 2.5$, $p = q = 2$) Let $f \in C^3(\mathbb{R}, X)$, where $(X, \|\cdot\|)$ is a Banach space. We assume that $\||f|\|_{\infty,\mathbb{R}} < \infty$, and that*

$$M_{2.5} := \max\left\{\sup_{a\in\mathbb{R}} \left\|\left\|\left(\left(D^{2.5}_{a+;e^t} f\right)\circ \ln\right)(z)\right\|\right\|_{2,(0,\infty)},\right.$$

$$\left.\sup_{a\in\mathbb{R}} \left\|\left\|\left(\left(D^{2.5}_{a-;e^t} f\right)\circ \ln\right)(z)\right\|\right\|_{2,(0,\infty)}\right\} < \infty. \tag{85}$$

*Then,*

$$\left\|\left\|(f\circ \ln)'\circ e^t\right\|\right\|_{\infty,\mathbb{R}} \leq 1.226583057\sqrt{M_{2.5}\, \||f|\|_{\infty,\mathbb{R}}} < \infty. \tag{86}$$

*Proof.* By Theorem 2.4, (57). □

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