Mean value theorem and semigroups of operators for interval-valued functions on time scales

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Abstract

In this paper, a new version of mean value theorem for interval-valued functions on time scales is established. Meantime, some basic concepts and results associated with semigroups of operators for interval-valued functions on time scales are presented. As an application of semigroups of operators, under certain conditions, we consider the initial value problem for interval-valued differential equations on time scales. Finally, two issues worthy of further discussion are presented.

1. Introduction

In 1988, the notion of time scale was introduced by Hilger [6] to unify continuous and discrete analysis. There is no doubt that the time scale calculus provide a unified framework for the study of differential equations and difference equations. In practice, many problems involve various types of uncertainty. Usually, the knowledge about the parameters of a real world system is imprecise or uncertain because, generally, it is difficult to accurately observe or measure the true value of these parameters. In these cases, the value of a parameter cannot be characterized by an ordinary real number. Accordingly, interval numbers and fuzzy numbers are two important tools to deal with these problems. In fact, interval numbers can be regarded as a special case of fuzzy numbers. Taking into account the shortcoming of the difference of fuzzy numbers, it is necessary to carry out the study of interval analysis. More importantly, interval analysis can provide important methodologies and foundations for fuzzy analysis. In 1993, Markov [8] first studied the differentiability and integrability of interval-valued functions. Later, Stefanini and Bede [11] together with Chalco-Cano et al. [3] further extended the theory of calculus of interval-valued functions. In 2013, Lupulescu [7] introduced the differentiability and integrability for interval-valued functions on time scales by using the generalized Hukuhara differentiability.

The mean value theorem for real-valued functions has important and extensive application in the classical calculus. In [8], the mean value theorem for interval-valued functions was established. Afterwards, the work was extended to the interval-valued functions on time scales by Lupulescu [7]. One purpose of this paper is to give a new version of the

mean value theorem for interval-valued functions on time scales. In addition, semigroups of operators are very important in the study of differential equations. In 2005, the semigroups of operators on spaces of fuzzy-number-valued functions were proposed by Gal and Gal [4] and were applied to study fuzzy differential equations. Recently, Hamza and Oraby [5] developed the theory of semigroups of operators on time scales. Motivated by these works, the other purpose of the present paper is to present some basic concepts and results related to semigroups of operators for interval-valued functions on time scales.

2. Preliminaries

Let $\mathbb{Z}_0^+$, $\mathbb{R}_0^+$ and $\mathbb{R}$ denote the set of all nonnegative integers, nonnegative real numbers and real numbers, respectively. Denote by $\mathcal{K}$ the set of all nonempty compact convex subsets (i.e., bounded and closed intervals) of the real line $\mathbb{R}$. For $A = [a^-, a^+], B = [b^-, b^+] \in \mathcal{K}, \lambda \in \mathbb{R}$, the Minkowski addition $A + B$ and scalar multiplication $\lambda \cdot A$ (denoted by λA) can be defined by

$$A + B = [a^-, a^+] + [b^-, b^+] = [a^- + b^-, a^+ + b^+]$$

and

$$\lambda \cdot A = \lambda A = \lambda[a^-, a^+] = [\min\{\lambda a^-, \lambda a^+\}, \max\{\lambda a^-, \lambda a^+\}].$$

It is well know that the addition is associative and commutative and with the neutral element $\{0\}$. Especially, if $\lambda = -1$, then the scalar multiplication gives the opposite $-A = (-1)A = [-a^+, -a^-]$. However, in general, $A + (-A) \neq \{0\}$. That is to say, the opposite of A is not the inverse of A with respect to the Minkowski addition, unless A is a singleton.

Let $A, B \in \mathcal{K}$. If there exists $C \in \mathcal{K}$ such that $A = B + C$, then C is called the Hukuhara difference (or H-difference) of A and B, and it is denoted by $C := A \ominus B$. Note that the H-difference is unique, but it does not always exist for any two intervals. Given two intervals $A, B \in \mathcal{K}$, it is easy to know that the H-difference $A \ominus B$ exists if and only if $len(A) \geq len(B)$, where $len(\cdot)$ denotes the length of the interval, i.e., $len(A) = a^+ - a^-$. In order to overcome this shortcoming, the generalized difference is introduced as follows.

Definition 2.1 (Markov [8], Stefanini [10]). Let $A, B \in \mathcal{K}$. The generalized Hukuhara difference (gH-difference for short) is defined as

$$A \ominus_g B = C \Leftrightarrow \begin{cases} (i) \ A = B + C \Leftrightarrow A \ominus B = C, \\ or \ (ii) \ B = A + (-C) \Leftrightarrow B \ominus A = -C. \end{cases}$$

According to Def. 2.1, if $A = [a^-, a^+], B = [b^-, b^+] \in \mathcal{K}$, then we have

$$\begin{aligned} A \ominus_g B &= [a^-, a^+] \ominus_g [b^-, b^+] \\ &= [\min\{a^- - b^-, a^+ - b^+\}, \max\{a^- - b^-, a^+ - b^+\}] \\ &= \begin{cases} [a^- - b^-, a^+ - b^+], & len(A) \geq len(B), \\ [a^+ - b^+, a^- - b^-], & len(A) < len(B). \end{cases} \end{aligned}$$

From [8,10,12], some basic properties of gH-difference can be summarized as follows.

(i) $A \ominus_g A = \{0\}$, $A \ominus_g \{0\} = A$, $\{0\} \ominus_g A = -A$;
(ii) $A \ominus_g B = (-B) \ominus_g (-A) = -(B \ominus_g A)$;
(iii) $A \ominus_g (-B) = B \ominus_g (-A)$, $(-A) \ominus_g B = (-B) \ominus_g A$;
(iv) $(A + B) \ominus_g B = A$, $A \ominus_g (A + B) = -B$;
(v) $(A \ominus_g B) + B = A$ if $len(A) \geq len(B)$, $A + (-1)(A \ominus_g B) = B$ if $len(A) < len(B)$;
(vi) $\lambda(A \ominus_g B) = \lambda A \ominus_g \lambda B$, $\lambda \in \mathbb{R}$;
(vii) $(\lambda + \mu)A = \lambda A + \mu A$ if $\lambda\mu \geq 0$, $(\lambda + \mu)A = \lambda A \ominus_g (-\mu A)$ if $\lambda\mu < 0$.

Lemma 2.2. *Let* $A = [a^-, a^+], B = [b^-, b^+]$ *and* $C = [c^-, c^+]$ *belong to* $\mathcal{K}$. *Then:*

   (i) *If* $len(A) \geq len(C)$, *then* $(A + B) \ominus_g C = (A \ominus_g C) + B$;

   (ii) *If* $len(A) < len(C)$, *then* $(A + B) \ominus_g C = (A \ominus_g C) \ominus_g (-B)$.

Proof. For simplicity, we write $(A + B) \ominus_g C = D$, where $D = [d^-, d^+]$.

(i) If $len(A) \geq len(C)$, then $(A + B) \ominus_g C = (A + B) \ominus C$. Using the representation of endpoints, we have

$$
\begin{aligned}
(A + B) \ominus_g C &= (A + B) \ominus C \\
&= [a^- + b^- - c^-, a^+ + b^+ - c^+] \\
&= [a^- - c^-, a^+ - c^+] + [b^-, b^+] \\
&= (A \ominus C) + B \\
&= (A \ominus_g C) + B.
\end{aligned}
$$

(ii) If $len(A) < len(C)$, then $A \ominus_g C = -(C \ominus A)$. Therefore, we can infer from Definition 2.1 that

$$
\begin{aligned}
&(A \ominus_g C) \ominus_g (-B) \\
&= [a^+ - c^+, a^- - c^-] \ominus_g [-b^+, -b^-] \\
&= [\min\{a^+ - c^+ + b^+, a^- - c^- + b^-\}, \max\{a^+ - c^+ + b^+, a^- - c^- + b^-\}] \\
&= (A + B) \ominus_g C.
\end{aligned}
$$

$\square$

Now we define a functional $\|\cdot\| : \mathcal{K} \to [0, \infty)$ by $\|A\| = \max\{|a^-|, |a^+|\}$ for every $A = [a^-, a^+] \in \mathcal{K}$. It can easily be shown that $\|\cdot\|$ is a norm on $\mathcal{K}$, and thus the quadruple $(\mathcal{K}, +, \cdot, \|\cdot\|)$ is a normed quasilinear space [9].

Given two intervals $A = [a^-, a^+]$, $B = [b^-, b^+] \in \mathcal{K}$, the Hausdorff-Pompeiu metric between A and B is defined by $d_H(A, B) = \max\{|a^- - b^-|, |a^+ - b^+|\}$. It is well known that $(\mathcal{K}, d_H)$ is a complete and separable metric space. Furthermore, the following relationships exist between the Hausdorff-Pompeiu metric d_H and the norm $\|\cdot\|$:

$$
\|A\| = d_H(A, \{0\}), \qquad d_H(A, B) = \|A \ominus_g B\|.
$$

In addition, for all $A, B, C, D \in \mathcal{K}$, the metric d_H has the following properties:

  (i) $d_H(A + B, A + C) = d_H(B, C)$,

  (ii) $d_H(\lambda A, \lambda, B) = |\lambda| d_H(A, B)$, $\lambda \in \mathbb{R}$,

 (iii) $d_H(A + C, B + D) \leq d_H(A, B) + d_H(C, D)$,

 (iv) $d_H(A \ominus_g B, A \ominus_g C) \leq d_H(B, C)$.

Here, we briefly recall some basic notions related to the time scale. For more details, we recommend two excellent monographs [1,2] written by Bohner and Peterson. A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, the forward jump operator σ and the back jump operator ρ are defined as $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, respectively. Especially, $\inf \emptyset = \sup \mathbb{T}$, $\sup \emptyset = \inf \mathbb{T}$.

A point $t \in \mathbb{T}$ is said to be *right-scattered*, *right-dense*, *left-scattered* and *left-dense* if $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) < t$ and $\rho(t) = t$, respectively. Given a time scale $\mathbb{T}$, the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. The set $\mathbb{T}^\kappa$ is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum γ, then $\mathbb{T}^\kappa = \mathbb{T} - \{\gamma\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. Especially, given a time scale interval $[a, b]_\mathbb{T} = \{t \in \mathbb{T} \mid a \leq t \leq b\}$, if $\rho(b) = b$, then $[a, b]^\kappa = [a, b]_\mathbb{T}$. Otherwise, $[a, b]^\kappa = [a, b)_\mathbb{T}$. In essence, $[a, b)_\mathbb{T} = [a, \rho(b)]_\mathbb{T}$.

Let $g : \mathbb{T} \to \mathbb{R}$ be a real-valued function and let $t \in \mathbb{T}^\kappa$. Given any $\varepsilon > 0$, if there exist a number α and a neighborhood U of t such that

$$
|g(\sigma(t)) - g(s) - \alpha(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|
$$

for all $s \in U$, then we say that g is delta differentiable (or in short: Δ-differentiable) at t. Correspondingly, the number α is called the Δ-derivative and it is denoted by $g^\Delta(t)$. More generally, the function g is said to be delta differentiable (Δ-differentiable) on $\mathbb{T}^\kappa$ provided the Δ-derivative $g^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Definition 2.3 (Lupulescu [7]). Let $F : \mathbb{T} \to \mathcal{K}$ be an interval-valued function. Then we say that F is *l*-nondecreasing (or *l*-nonincreasing) on $\mathbb{T}$ if the real-valued function $t \to len(F(t))$ is nondecreasing (or nonincreasing) on $\mathbb{T}$. Generally, if F is *l*-nondecreasing or *l*-nonincreasing on $\mathbb{T}$, then we say that F is *l*-monotonic on $\mathbb{T}$.

Definition 2.4 (Lupulescu [7]). Let $F : \mathbb{T} \to \mathcal{K}$ be an interval-valued function and let $A \in \mathcal{K}$. If for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|F(t) \ominus_g A\| = d_H(F(t), A) \le \varepsilon$ for all $t \in U_\mathbb{T}(t_0, \delta)$ (i.e., $U_\mathbb{T}(t_0, \delta) = (t_0 - \delta, t_0 + \delta) \cap \mathbb{T}$), then we say that A is the $\mathbb{T}$-limit of F at $t_0 \in \mathbb{T}$. If F has a $\mathbb{T}$-limit A at t_0, then it is unique and is denoted by $A = \mathbb{T} - \lim\limits_{t \to t_0} F(t)$.

An interval-valued function $F : \mathbb{T} \to \mathcal{K}$ is called *rd*-continuous if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided $\mathbb{T}$-limits exist at all left-dense points in $\mathbb{T}$.

Definition 2.5 (Lupulescu [7]). Let $F : \mathbb{T} \to \mathcal{K}$ be an interval-valued function and let $t \in \mathbb{T}^\kappa$. Then we define $F_{gH}^\Delta(t)$ to be the interval (provided it exists) with the property that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_H(F(\sigma(t)) \ominus_g F(s), (\sigma(t) - s)F_{gH}^\Delta(t)) \le \varepsilon|\sigma(t) - s|$$

for all $s \in U_\mathbb{T}(t, \delta)$. Here, $F_{gH}^\Delta(t)$ is called the delta generalized Hukuhara derivative (Δ_{gH}-derivative for short) of F at t. Meantime, if $F_{gH}^\Delta(t)$ exists for each $t \in \mathbb{T}^\kappa$, then we say that F is delta generalized Hukuhara differentiable (Δ_{gH}-differentiable for short) on $\mathbb{T}^\kappa$. In particular, the Δ_{gH}-derivative F_{gH}^Δ degenerates into the gH-derivative F_{gH}' if the time scale $\mathbb{T} = \mathbb{R}$.

Theorem 2.6 (Lupulescu [7]). *Assume that $F : \mathbb{T} \to \mathcal{K}$ is an interval-valued function and let $t \in \mathbb{T}^\kappa$. Then, the following statements are true:*

(i) *If $F : \mathbb{T} \to \mathcal{K}$ is Δ_{gH}-differentiable at $t \in \mathbb{T}^\kappa$, then it is continuous at t;*

(ii) *If F is continuous at t and t is right-scattered, then F is Δ_{gH}-differentiable at t with*

$$F_{gH}^\Delta(t) = \frac{F(\sigma(t)) \ominus_g F(t)}{\mu(t)};$$

(iii) *If t is right-dense, then F is Δ_{gH}-differentiable at t if and only if the $\mathbb{T}$-limit*

$$\mathbb{T} - \lim\limits_{s \to t} \frac{F(t) \ominus_g F(s)}{t - s}$$

exists as a closed interval. In this case

$$F_{gH}^\Delta(t) = \mathbb{T} - \lim\limits_{s \to t} \frac{F(t) \ominus_g F(s)}{t - s};$$

(iv) *If F is Δ_{gH}-differentiable at t, then*

$$F(\sigma(t)) \ominus_g F(t) = \mu(t)F_{gH}^\Delta(t).$$

Finally, the induction principle on time scales is provided, which is useful in the next section.

Theorem 2.7 (Bohner and Peterson [7]). *Let $t_0 \in \mathbb{T}$ and let $\{S(t) : t \in [t_0, +\infty)\}$ be a family of statements satisfying:*

(I) *$S(t_0)$ is true;*

(II) *If $t \in [t_0, +\infty)$ is right-scattered and $S(t)$ is true, then $S(\sigma(t))$ is also true;*

(III) *If $t \in [t_0, +\infty)$ is right-dense and $S(t)$ is true, then there is a neighborhood U of t such that $S(s)$ is true for all $s \in U \cap (t, +\infty)$;*

(IV) *If $t \in (t_0, +\infty)$ is left-dense and $S(s)$ is true for all $s \in [t_0, t)$, then $S(t)$ is true. Then $S(t)$ is true for all $t \in [t_0, +\infty)$.*

3. Mean value theorem for interval-valued functions on time scales

Based on the works of Markov [8] and Lupulescu [7], in this section, we shall establish another version of the mean value theorem for interval-valued functions on time scales.

Theorem 3.1 (Markov [8]). *Let F be a continuous interval-valued function on $[a, b]$ and gH-differentiable in (a, b). Then*

$$F(b) \ominus_g F(a) \subset (b - a)F'_{gH}([a, b]),$$

where $F'_{gH}([a, b]) = \bigcup_{\xi \in [a,b]} F'_{gH}(\xi)$.

Remark 3.2. In general, it is not true that there exists $\xi \in [a, b]$ such that $F(b) \ominus_g F(a) \subset (b - a)F'_{gH}(\xi)$.

Theorem 3.3 (Lupulescu [7]). *Let F be a continuous and l-monotonic interval-valued function on $[a, b]_{\mathbb{T}}$ and let F be Δ_{gH}-differentiable in $[a, b)_{\mathbb{T}}$. Then*

$$F(b) \ominus_g F(a) \subset (b - a)F^{\Delta}_{gH}([a, b)_{\mathbb{T}}),$$

where $F^{\Delta}_{gH}([a, b)_{\mathbb{T}}) = \bigcup_{\xi \in [a,b)_{\mathbb{T}}} F^{\Delta}_{gH}(\xi)$.

Theorem 3.4. *Let F and g be an interval-valued function and a real-valued function defined on $\mathbb{T}$, respectively. Assume that F is Δ_{gH}-differentiable and g is Δ-differentiable on $\mathbb{T}^{\kappa}$. If*

$$\|F^{\Delta}_{gH}(t)\| \le g^{\Delta}(t)$$

for all $t \in \mathbb{T}^{\kappa}$, then

$$\|F(t) \ominus_g F(r)\| \le g(t) - g(r)$$

for all $t \in [r, s]_{\mathbb{T}}$ with $r, s \in \mathbb{T}$ and $r \le s$.

Proof. Let $r, s \in \mathbb{T}$ with $r \le s$. For any $\varepsilon > 0$, we can show by the induction principle as shown in Theorem 2.7 that

$$S(t) : \|F(t) \ominus_g F(r)\| \le g(t) - g(r) + \varepsilon(t - r)$$

holds for all $t \in [r, s]_{\mathbb{T}}$. The proof is divided into four steps.

(I) If $t = r$, then the statement $S(r)$ is obviously true.

(II) Assume that t is right-scattered and $S(t)$ is satisfied. According to Definition 2.1 and Theorem 2.6 (iv), we have the following two cases:

Case (a):

$$\begin{aligned}
\|F(\sigma(t)) \ominus_g F(r)\| &= d_H(F(\sigma(t)), F(r)) \\
&= d_H(F(t) + \mu(t)F^{\Delta}_{gH}(t), F(r)) \\
&\le d_H(F(t), F(r)) + d_H(\mu(t)F^{\Delta}_{gH}(t), \{0\}) \\
&= d_H(F(t), F(r)) + \mu(t)d_H(F^{\Delta}_{gH}(t), \{0\}) \\
&= d_H(F(t), F(r)) + \mu(t)\|F^{\Delta}_{gH}(t)\| \\
&\le d_H(F(t), F(r)) + \mu(t)g^{\Delta}(t) \\
&\le g(t) - g(r) + \varepsilon(t - r) + g(\sigma(t)) - g(t) \\
&= g(\sigma(t)) - g(r) + \varepsilon(t - r) \\
&\le g(\sigma(t)) - g(r) + \varepsilon(\sigma(t) - r).
\end{aligned}$$

Case (b):

$$\|F(\sigma(t)) \ominus_g F(r)\| = d_H(F(\sigma(t)), F(r))$$
$$= d_H(F(\sigma(t)) + (-1)\mu(t)F_{gH}^\Delta(t), F(r) + (-1)\mu(t)F_{gH}^\Delta(t))$$
$$= d_H(F(t), F(r) + (-1)\mu(t)F_{gH}^\Delta(t))$$
$$\le d_H(F(t), F(r)) + d_H(\{0\}, (-1)\mu(t)F_{gH}^\Delta(t))$$
$$= d_H(F(t), F(r)) + \mu(t)d_H(\{0\}, F_{gH}^\Delta(t))$$
$$= d_H(F(t), F(r)) + \mu(t)\|F_{gH}^\Delta(t)\|$$
$$\le g(\sigma(t)) - g(r) + \varepsilon(\sigma(t) - r).$$

Thus, the statement $S(\sigma(t))$ is satisfied.

(III) Suppose that $S(t)$ holds and $t \ne s$ is right-dense. Clearly, $\sigma(t) = t$. Since F is Δ_{gH}-differentiable and g is Δ-differentiable at t, there exists a neighborhood $U_{\mathbb{T}}$ of t such that

$$d_H(F(t) \ominus_g F(s), F_{gH}^\Delta(t)(t - s)) \le \frac{\varepsilon}{2}|t - s|$$

for all $s \in U_{\mathbb{T}}$ and

$$|g(t) - g(s) - g^\Delta(t)(t - s)| \le \frac{\varepsilon}{2}|t - s|$$

for all $s \in U_{\mathbb{T}}$. Therefore, we can obtain that

$$d_H(F(t), F(s)) = d_H(F(t) \ominus_g F(s), \{0\})$$
$$\le d_H(F(t) \ominus_g F(s), F_{gH}^\Delta(t)(t - s)) + d_H(\{0\}, F_{gH}^\Delta(t)(t - s))$$
$$\le \left(\|F_{gH}^\Delta(t)\| + \frac{\varepsilon}{2}\right)|t - s|$$

and

$$g(s) - g(t) - g^\Delta(t)(s - t) \ge -\frac{\varepsilon}{2}|t - s|$$

for all $s \in U_{\mathbb{T}}$. Hence, for all $s \in U_{\mathbb{T}} \cap (t, \infty)$, we have

$$\|F(s) \ominus_g F(r)\| = d_H(F(s), F(r))$$
$$\le d_H(F(s), F(t)) + d_H(F(t), F(r))$$
$$\le \left(\|F_{gH}^\Delta(t)\| + \frac{\varepsilon}{2}\right)|t - s| + d_H(F(t), F(r))$$
$$\le \left(g^\Delta(t) + \frac{\varepsilon}{2}\right)|t - s| + d_H(F(t), F(r))$$
$$\le \left(g^\Delta(t) + \frac{\varepsilon}{2}\right)|t - s| + g(t) - g(r) + \varepsilon(t - r)$$
$$= g^\Delta(t)(s - t) + \frac{\varepsilon}{2}(s - t) + g(t) - g(r) + \varepsilon(t - r)$$
$$\le g(s) - g(t) + \frac{\varepsilon}{2}|t - s| + \frac{\varepsilon}{2}(s - t) + g(t) - g(r) + \varepsilon(t - r)$$
$$= g(s) - g(r) + \varepsilon(s - r),$$

which implies that $S(s)$ holds for all $s \in U_{\mathbb{T}} \cap (t, \infty)$.

(IV) Let t be left-dense and assume that $S(\tau)$ holds for all $\tau < t$. By the continuity of F and g, we then obtain that

$$\|F(t) \ominus_g F(r)\| = \lim_{\tau \to t_-} \|F(\tau) \ominus_g F(r)\|$$
$$\le \lim_{\tau \to t_-} g(\tau) - g(r) + \varepsilon(\tau - r)$$
$$= g(t) - g(r) + \varepsilon(t - r),$$

which means that the statement $S(t)$ is true.

Due to the arbitrariness of ε, we have obtained the desired result and completed the proof of this theorem. $\qquad\square$

As an application of Theorem 3.4, we can obtain the following results.

Corollary 3.5. *Let $F, G : \mathbb{T} \to \mathcal{K}$ be two Δ_{gH}-differentiable interval-valued functions on $\mathbb{T}^{\kappa}$. Then*

(i) *If D is a compact interval with endpoints $r, s \in \mathbb{T}$, then*

$$\|F(s) \ominus_g F(r)\| \leq \Big(\sup_{t \in D^{\kappa}} \|F_{gH}^{\Delta}(t)\| \Big)|s - r|.$$

(ii) *If $F_{gH}^{\Delta}(t) = \{0\}$ for all $t \in \mathbb{T}^{\kappa}$, then F is a constant interval.*

(ii) *If both F and G are l-nondecreasing or l-nonincreasing, and $F_{gH}^{\Delta}(t) = G_{gH}^{\Delta}(t)$ for all $t \in \mathbb{T}^{\kappa}$, then*

$$F(t) \ominus_g G(t) = C$$

for all $t \in \mathbb{T}$, where C is a constant interval.

(iv) *If F and G are such that one is l-nondecreasing and the other is l-nonincreasing, and $F_{gH}^{\Delta}(t) = -G_{gH}^{\Delta}(t)$ for all $t \in \mathbb{T}^{\kappa}$, then*

$$F(t) + G(t) = C$$

for all $t \in \mathbb{T}$, where C is a constant interval.

Proof. (i) Let $r, s \in \mathbb{T}$ with $r \leq s$. Define

$$g(t) := \Big(\sup_{\tau \in [r,s]^{\kappa}} \|F_{gH}^{\Delta}(\tau)\| \Big)(t - r)$$

for $t \in \mathbb{T}$. Then, it is easy to know that

$$g^{\Delta}(t) = \sup_{\tau \in [r,s]^{\kappa}} \|F_{gH}^{\Delta}(\tau)\| \geq \|F_{gH}^{\Delta}(t)\|$$

for all $\tau \in [r, s]^{\kappa}$. By Theorem 3.4, the desired result can be obtained.
(ii) It is a direct consequence of part (i).
(iii) By Theorem 4 in [7], we have

$$(F(t) \ominus_g G(t))_{gH}^{\Delta} = F_{gH}^{\Delta}(t) \ominus_g G_{gH}^{\Delta}(t) = F_{gH}^{\Delta}(t) \ominus_g F_{gH}^{\Delta}(t) = \{0\}$$

for $t \in \mathbb{T}^{\kappa}$. The desired result follows immediately from (ii).
(iv) Similar to part (iii), since

$$(F(t) + G(t))_{gH}^{\Delta} = F^{\Delta}(t) \ominus_g (-G_{gH}^{\Delta}(t)) = F_{gH}^{\Delta}(t) \ominus_g F_{gH}^{\Delta}(t) = \{0\}$$

for $t \in \mathbb{T}^{\kappa}$. $\qquad\square$

Remark 3.6. If F and G are differently l-monotonic in (iii) of Corollary 3.5, in general, there is no constant interval C such that $F(t) \ominus_g G(t) = C$. Analogously, F and G are equally l-monotonic in (iv), then the result is not necessarily true.

Remark 3.7. The results (iii) and (iv) of Corollary 3.4 coincide with Corollary 2 in [7].

Example 3.8. (i) Let $\mathbb{T} = [0, 1]$ and let $F(t) = [t, 2t]$ and $G(t) = [2t - 1, t]$. Note that $len(F(t)) = t$ is nondecreasing on $\mathbb{T}$ and $len(G(t)) = 1 - t$ is nonincreasing on $\mathbb{T}$. It is easy to check that $F(t)$ and $G(t)$ are Δ_{gH}-differentiable on $\mathbb{T}^{\kappa} = [0, 1]$ and $F_{gH}^{\Delta}(t) = F_{gH}'(t) = [1, 2] = G_{gH}'(t) = G_{gH}^{\Delta}(t)$ for each $t \in [0, 1]$(Only consider the unilateral derivative at the endpoints 0 and 1). However, there is no constant interval C such that $F(t) \ominus_g G(t) = C$.

(ii) Let $\mathbb{T} = [0, 1]$ and let $F(t) = [-t, 2t]$ and $G(t) = [t - 1, 2(1 - t)]$. Clearly, $len(F(t)) = 3t$ is nondecreasing and $len(G(t)) = 3(1 - t)$ in nonincreasing on $\mathbb{T}$. It can easily be verified that $F(t)$ and $G(t)$ are Δ_{gH}-differentiable on $\mathbb{T}^{\kappa} = [0, 1]$. Moreover, $F_{gH}^{\Delta}(t) = F_{gH}'(t) =$

$[-1, 2]$, $G_{gH}^{\Delta}(t) = G'_{gH}(t) = [-2, 1]$ for each $t \in [0, 1]$. Then, we have $F_{gH}^{\Delta}(t) = -G_{gH}^{\Delta}(t)$ for each $t \in [0, 1]$. By Corollary 3.5, there exists an interval $C = [-1, 2]$ such that $F(t) + G(t) = [-1, 2] = C$.

Example 3.9. Let $\mathbb{T} = h\mathbb{Z}_0^+ = \{hk : k \in \mathbb{Z}_0^+\}$, $h > 0$. Suppose $F(t) = [t, t^2]$ and $G(t) = [t + a, t^2 + b]$, where a and b are two fixed constants with $a \leq b$. Obviously, both $len(F(t)) = t(t-1)$ and $len(G(t)) = t(t-1) + b - a$ are l-nondecreasing on $\mathbb{T}$. By Theorem 2.6, we can obtain $F_{gH}^{\Delta}(t) = [1, 2t] = G_{gH}^{\Delta}(t)$ for each $t \in \mathbb{T}$. Therefore, we can find an interval $C = [-b, -a]$ such that $F(t) \ominus_g G(t) = C$ on $\mathbb{T}$.

Example 3.10. Let $\mathbb{T} = \mathbb{R}$ and let $F(t) = [-2e^{-t} - 1, e^{-t} + 2]$ and $G(t) = [-2e^{-t}, e^{-t} + 1]$. Obviously, $len(F(t)) = 3 + 3e^{-t}$ and $len(G(t)) = 1 + 3e^{-t}$ are nonincreasing on $\mathbb{R}$. It is easy to know that $F(t)$ and $G(t)$ are Δ_{gH}-differentiable on $\mathbb{R}$ and $F_{gH}^{\Delta}(t) = F'_{gH}(t) = [-1, 2]e^{-t} = G'_{gH}(t) = G_{gH}^{\Delta}(t)$ for each $t \in \mathbb{R}$. By Corollary 3.5, we can find an interval $C = [-1, 1]$ such that $F(t) \ominus_g G(t) = C$.

Example 3.11. Let $\mathbb{T} = q^{\mathbb{Z}} = \{q^k | k \in \mathbb{Z}\}$, where $q > 1$. Assume $F(t) = [-t, 2t^2]$ and $G(t) = [-2t^2 + 1, t + 2]$. According to Theorem 2.6, for each $t \in \mathbb{T}$, it follows that

$$
\begin{aligned}
F_{gH}^{\Delta}(t) &= \frac{F(\sigma(t)) \ominus_g F(t)}{\mu(t)} \\
&= \frac{F(qt) \ominus_g F(t)}{(q-1)t} \\
&= \frac{[-qt, 2q^2 t^2] \ominus_g [-t, 2t^2]}{(q-1)t} \\
&= \frac{[-(q-1)t, 2(q^2 - 1)t^2]}{(q-1)t} \\
&= [-1, 2(q+1)t].
\end{aligned}
$$

Using the similar method, we can obtain $G_{gH}^{\Delta}(t) = [-2(q+1)t, 1] = -F_{gH}^{\Delta}(t)$. However, $len(F(t)) = 2t^2 + t$ and $len(G(t)) = 2t^2 + t + 1$ are nondecreasing on $\mathbb{T}$. Therefore, the conditions of Corollary 3.5 are not satisfied. Indeed, there does not exist an interval C such that $F(t) + G(t) = C$.

Example 3.12. Let $\mathbb{T} = \mathbb{N}_0^2 = \{n^2 | n \in \mathbb{N}_0\}$ and let $F(t) = [-\sqrt{t}, \sqrt{t}]$ and $G(t) = [\min\{1 - \sqrt{t}, \sqrt{t}\}, \max\{1 - \sqrt{t}, \sqrt{t}\}]$. For every $t \in \mathbb{T}$, it is easy to know that t is right-scattered. By Theorem 2.6, we can obtain

$$
\begin{aligned}
F_{gH}^{\Delta}(t) &= \frac{F(\sigma(t)) \ominus_g F(t)}{\mu(t)} \\
&= \frac{F((\sqrt{t} + 1)^2) \ominus_g F(t)}{2\sqrt{t} + 1} \\
&= \frac{[-\sqrt{t} - 1, \sqrt{t} + 1] \ominus_g [-\sqrt{t}, \sqrt{t}]}{2\sqrt{t} + 1} \\
&= \frac{1}{2\sqrt{t} + 1}[-1, 1].
\end{aligned}
$$

Similarly, we can infer that $G_{gH}^{\Delta}(t) = \frac{1}{2\sqrt{t}+1}[-1, 1] = F_{gH}^{\Delta}(t)$. Although $len(F(t)) = 2\sqrt{t}$ is nondecreasing on $\mathbb{T}$, $len(G(t)) = |2\sqrt{t} - 1|$ is not monotonic on $\mathbb{T}$. Therefore, the conditions of Corollary 3.5 are not satisfied. In fact, there does not exist an interval C such that $F(t) \ominus_g G(t) = C$.

4. C_0-Semigroups for interval-valued functions on time scales

In this section, we shall introduce some basic notions and results associated with semigroups of operators for interval-valued functions on time scales.

Definition 4.1. Let $\widetilde{A} : \mathcal{K} \to \mathcal{K}$. $\widetilde{A}$ is said to be a linear operator on $\mathcal{K}$ if

$$\widetilde{A}(\alpha \cdot x + \beta \cdot y) = \alpha \cdot \widetilde{A}(x) + \beta \cdot \widetilde{A}(y)$$

for all $x, y \in \mathcal{K}$ and $\alpha, \beta \in \mathbb{R}$.

Remark 4.2. Unlike the property of linear operators on a linear space, it should be noticed that the continuity of a linear operator $\widetilde{A}$ at $\{0\} \in \mathcal{K}$ does not imply the continuity of $\widetilde{A}$ at each $x \in \mathcal{K}$, because $(\mathcal{K}, +, \cdot)$ is not a linear space, in general, the equality $x_0 = (x_0 \ominus_g x) + x$ does not hold, unless $len(x_0) \geq len(x)$.

Lemma 4.3. *Let $\widetilde{A}$ be a linear operator on $\mathcal{K}$. Then, for all $x, y \in \mathcal{K}$, we have*

$$\widetilde{A}(x \ominus_g y) = \widetilde{A}(x) \ominus_g \widetilde{A}(y).$$

Proof. Let $z = x \ominus_g y$. Then, we get $x = y + z$ or $y = x + (-z)$. According to Definition 4.1, it follows that

$$\begin{cases} \widetilde{A}(x) = \widetilde{A}(y + z) = \widetilde{A}(y) + \widetilde{A}(z), \\ or \ \widetilde{A}(y) = \widetilde{A}(x + (-z)) = \widetilde{A}(x) + \widetilde{A}(-z), \end{cases}$$

which is equivalent to

$$\begin{cases} \widetilde{A}(x) = \widetilde{A}(y) + \widetilde{A}(z), \\ or \ \widetilde{A}(y) = \widetilde{A}(x) + (-1)\widetilde{A}(z). \end{cases}$$

Therefore, $\widetilde{A}(x \ominus_g y) = \widetilde{A}(z) = \widetilde{A}(x) \ominus_g \widetilde{A}(y)$. $\qquad\qquad\square$

$$L(\mathcal{K}) = \{\widetilde{A} : \mathcal{K} \to \mathcal{K}| \ \widetilde{A} \ is \ linear \ and \ continuous \ at \ each \ x \in \mathcal{K}\}.$$

Let us introduce the addition and scalar multiplication in $L(\mathcal{K})$ as follows

$$(\widetilde{A} + \widetilde{B})(x) = \widetilde{A}(x) + \widetilde{B}(x), \quad (\lambda \cdot \widetilde{A})(x) = \lambda \cdot \widetilde{A}(x),$$

for $\widetilde{A}, \widetilde{B} \in L(\mathcal{K})$ and $\lambda \in \mathbb{R}$. Consider the metric $D_H : L(\mathcal{K}) \times L(\mathcal{K}) \to [0, +\infty)$ defined by

$$D_H(\widetilde{A}, \widetilde{B}) = \sup\{d_H(\widetilde{A}(x), \widetilde{B}(x)) : \|x\| \leq 1\},$$

where $\|x\| = d_H(x, 0)$. From the properties of d_H, it can easily be verified that

(i) $D_H(\widetilde{A} + \widetilde{B}, \widetilde{C} + \widetilde{D}) \leq D_H(\widetilde{A}, \widetilde{C}) + D_H(\widetilde{B}, \widetilde{D})$;
(ii) $D_H(\lambda \cdot \widetilde{A}, \lambda \cdot \widetilde{B}) = |\lambda| D_H(\widetilde{A}, \widetilde{B})$;
(iii) $D_H(\widetilde{A}, \widetilde{B}) \leq D_H(\widetilde{A}, 0) + D_H(0, \widetilde{B}) = \|\widetilde{A}\| + \|\widetilde{B}\|$;
(iv) $D_H(\widetilde{A} + \widetilde{B}, \widetilde{C}) \leq D_H(\widetilde{A}, \widetilde{C}) + D_H(\widetilde{B}, \widetilde{C})$,

where $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D} \in L(\mathcal{K})$ and $\lambda \in \mathbb{R}$.

As a special case of Corollary 3.6 in [4], it is easy to know that $(L(\mathcal{K}), D_H)$ is a complete metric space.

Definition 4.4. Let $\mathbb{T} \subseteq \mathbb{R}_0^+$ be a semigroup time scale. A C_0-semigroup T on $\mathcal{K}$ is a family of continuous linear operators $\{T(t) : t \in \mathbb{T}\} \subset L(\mathcal{K})$, which satisfies

(i) $T(0) = I$, I is the identity operator on $\mathcal{K}$;
(ii) $T(t + s) = T(t)T(s)$ for every $t, s \in \mathbb{T}$;
(iii) $\lim\limits_{t \to 0+} T(t)x = x$ for each $x \in \mathcal{K}$, i.e., $T(\cdot)x : \mathbb{T} \to \mathcal{K}$ is continuous at 0.

Definition 4.5. Let T be a C_0-semigroup on $\mathcal{K}$. A linear operator $\widetilde{A}$ is called the generator of the C_0-semigroup T if for all $x \in \mathcal{K}$, the limit

$$\lim_{s \to 0+} \frac{T(\mu(t))x \ominus_g T(s)x}{\mu(t) - s} = \widetilde{A}x$$

exists uniformly in t. Here the limit are considered in the metric d_H.

Example 4.6. Let $\mathbb{T} = h\mathbb{Z}_0^+ = \{hk : k \in \mathbb{Z}_0^+\}$, $h > 0$ and $\widetilde{A}$ be a continuous linear operator on $\mathcal{K}$. Then $\widetilde{A}$ is the generator of $T(t) = (I + t\widetilde{A})^{t/h}$ for $t \in h\mathbb{Z}_0^+$. In fact, for $x \in \mathcal{K}$, we have

$$\lim_{s \to 0+} \frac{T(\mu(t))x \ominus_g T(s)x}{\mu(t) - s}$$
$$= \lim_{s \to 0+} \frac{T(h)x \ominus_g T(s)x}{h - s}$$
$$= \frac{T(h)x \ominus_g Ix}{h}$$
$$= \frac{(I + h\widetilde{A})x \ominus_g x}{h} = \widetilde{A}x.$$

Lemma 4.7. Let $\widetilde{A} \in L(\mathcal{K})$ and $\widetilde{A}^0 = I$, $\widetilde{A}^{k+1} = \widetilde{A}^k \widetilde{A}$, $k = 0, 1, 2, \ldots$. Then the sequence of operators $\{S_n(t)\}$, $t \in \mathbb{R}_0^+$, is a Cauchy sequence in $L(\mathcal{K})$, where $S_n(t) = \sum_{k=0}^{n} \frac{t^k}{k!} \cdot \widetilde{A}^k$.

Proof. It is a direct consequence of Theorem 3.9 in [4]. $\qquad\square$

In view of the completeness of $L(\mathcal{K})$ and Lemma 4.3, there exists $T(t) \in L(\mathcal{K})$ such that the sequence of operators $\{S_n(t)\}$ converges to $T(t)$ for each $t \in \mathbb{R}_0^+$. Formally, we denote $T(t)$ by

$$e^{t \cdot \widetilde{A}} \triangleq \sum_{k=0}^{\infty} \frac{t^k}{k!} \cdot \widetilde{A}^k.$$

Lemma 4.8. Let $\mathbb{T} = \mathbb{R}_0^+$ and $\widetilde{A} \in L(\mathcal{K})$. Define $T(t) = e^{t \cdot \widetilde{A}}$, $t \in \mathbb{T}$, then

    (i) $T(t + s) = T(t)T(s)$ for all $t, s \in \mathbb{T}$;

    (ii) $\lim_{s \to 0+} \dfrac{T(s)x \ominus_g x}{s} = \widetilde{A}x$ for each $x \in \mathcal{K}$.

Proof. (i) By Theorem 3.9 (ii) in [4], it is obvious.
(ii) According to Proposition 5 in [7], this result can be proved in a similar way as in Theorem 3.9 in [4]. $\qquad\square$

Example 4.9. Let $\mathbb{T} = \mathbb{R}_0^+$ and $\widetilde{A} \in L(\mathcal{K})$. Then $\widetilde{A}$ is the generator of $T(t) = e^{t \cdot \widetilde{A}}$ for $t \in \mathbb{R}_0^+$. In fact, by Lemma 4.8, for $x \in \mathcal{K}$, we have

$$\lim_{s \to 0+} \frac{T(\mu(t))x \ominus_g T(s)x}{\mu(t) - s}$$
$$= \lim_{s \to 0+} \frac{T(s)x \ominus_g T(0)x}{s}$$
$$= \widetilde{A}x.$$

Lemma 4.10. Let $\mathbb{T} \subseteq \mathbb{R}_0^+$ be a semigroup time scale. Then for each $x \in \mathcal{K}$, the function $T(\cdot)x : t \mapsto T(t)x$ is continuous from $\mathbb{T}$ into $\mathcal{K}$.

Proof. Let $t \in \mathbb{T}$. For all $0 < s \in \mathbb{T}$, we get

$$
\begin{aligned}
d_H(T(t+s)x, T(t)x) &= d_H(T(t+s)x \ominus_g T(t)x, 0) \\
&= d_H(T(t)T(s)x \ominus_g T(t)x, 0) \\
&= \|T(t)(T(s)x \ominus_g x)\| \\
&\leq \|T(t)\| \|T(s)x \ominus_g x\|.
\end{aligned}
$$

Letting $s \to 0+$, $\|T(s)x \ominus_g x\| \to 0$, which implies the continuity of $T(t)x$ at $t \in \mathbb{T}$. $\qquad \square$

Theorem 4.11. *Let $\mathbb{T} \subseteq \mathbb{R}_0^+$ be a semigroup time scale with the constant graininess function $\mu(t) = h$. Suppose that T is a C_0-semigroup on $\mathcal{K}$. Then $T(t)$ is Δ_{gH}-differentiable in $t \in \mathbb{T}$, and*

$$
T_{gH}^\Delta(t) = \widetilde{A}[T(t)].
$$

Proof. (i) If $\mu(t) = h > 0$, then t is right-scattered. By Lemma 2.3 in [5], we know $T = h\mathbb{Z}_0^+$. Furthermore, according to Lemma 4.4, $T(t)$ is continuous at t, so $T(t)$ is Δ_{gH}-differentiable. From Example 4.6, we can obtain

$$
\begin{aligned}
T_{gH}^\Delta(t) &= \frac{T(\sigma(t)) \ominus_g T(t)}{\mu(t)} \\
&= \frac{T(t+h) \ominus_g T(t)}{h} \\
&= \frac{T(h)T(t) \ominus_g T(t)}{h} \\
&= \widetilde{A}[T(t)].
\end{aligned}
$$

(ii) If $\mu(t) = h = 0$, then t is right-dense. In view of Lemma 2.3 in [5], $T = \mathbb{R}_0^+$. Based on Lemma 4.8, we can obtain the above result by using a similar argument as in Theorem 3.9 (iv) in [4]. $\qquad \square$

Definition 4.12. Let $\mathbb{T} \subseteq \mathbb{R}_0^+$ be a semigroup time scale and let T be a C_0-semigroup on $\mathcal{K}$. We say that T is a l-monotonic C_0-semigroup on $\mathcal{K}$ if the interval-valued function $T(\cdot)x : \mathbb{T} \to \mathcal{K}$ is l-monotonic for every $x \in \mathcal{K}$.

Lemma 4.13. *Let $\mathbb{T} \subseteq \mathbb{R}_0^+$ be a semigroup time scale and let T be a C_0-semigroup on $\mathcal{K}$. Assume that $g : \mathbb{T} \to \mathcal{K}$ is rd-continuous on $\mathbb{T}$. Define $F(t) = \int_0^t T(t-s)g(s)\Delta s$. If T is l-nondecreasing on $\mathcal{K}$, then F is also l-nondecreasing on $\mathcal{K}$.*

Proof. Let $t_1, t_2 \in \mathbb{T}$ with $t_1 < t_2$. Then, we have

$$
\begin{aligned}
T(t_2 - t_1)F(t_1) &= \int_0^{t_1} T(t_2 - s)g(s)\Delta s \\
&\subseteq \int_0^{t_1} T(t_2 - s)g(s)\Delta s + \int_{t_1}^{t_2} T(t_2 - s)g(s)\Delta s \\
&= \int_0^{t_2} T(t_2 - s)g(s)\Delta s = F(t_2).
\end{aligned}
$$

Since T is l-nondecreasing on $\mathcal{K}$, it follows that

$$
F(t_1) \subseteq T(t_2 - t_1)F(t_1) \subseteq F(t_2),
$$

which implies $len(F(t_1)) \leq len(F(t_2))$. Namely, F is l-nondecreasing on $\mathcal{K}$. $\qquad \square$

Theorem 4.14. *Let $\mathbb{T} \subseteq \mathbb{R}_0^+$ be a semigroup time scale with the constant graininess function $\mu(t) = h$. Assume that $x_0 \in \mathcal{K}$ and $g : \mathbb{T} \to \mathcal{K}$ is rd-continuous on $\mathbb{T}$. If T is a l-nondecreasing C_0-semigroup on $\mathcal{K}$, then*

$$
x(t) = T(t)(x_0) + \int_0^t T(t-s)g(s)\Delta s \tag{4.1}
$$

is Δ_{gH}-differentiable on $\mathbb{T}^\kappa$. And then, $x(t)$ satisfies

$$\begin{cases} x_{gH}^\Delta(t) = \widetilde{A}[x(t)] + T(\mu(t))(g(t)), \\ x(0) = x_0, \end{cases} \qquad t \in \mathbb{T}^\kappa, \qquad (4.2)$$

where the integral (including the integral in Lemma 4.13) for interval-valued functions defined on $[0, t)_\mathbb{T}$ is considered in the Riemann sense (the detailed definition can be seen in [7]).

Proof. For every $\mathbb{T}^\kappa$, we set

$$F(t) = \int_0^t T(t - s)g(s)\Delta s.$$

Since T is a l-nondecreasing C_0-semigroup on $\mathcal{K}$, by Lemma 4.13, it is easy to know that $F(t)$ is l-nondecreasing on $\mathbb{T}^\kappa$. Now, we distinguish two cases.
(i) If $t \in \mathbb{T}^\kappa$ is right-scattered, then we get

$$\begin{aligned} F(\sigma(t)) = F(t + h) &= \int_0^{t+h} T(t - s + h)g(s)\Delta s \\ &= T(h)\Big(\int_0^{t+h} T(t - s)g(s)\Delta s \Big) \\ &= T(h)\Big(F(t) + \int_t^{t+h} T(t - s)g(s)\Delta s \Big) \qquad (4.3) \\ &= T(h)(F(t)) + T(h)\Big(\int_t^{t+h} T(t - s)g(s)\Delta s \Big) \\ &= T(h)(F(t)) + T(h)(hT(0)(g(t))) \\ &= T(h)(F(t)) + hT(h)((g(t))) \end{aligned}$$

By Lemma 2.2, it follows from (4.3), Theorems 2.6 and 4.11 that

$$F_{gH}^\Delta(t) = \frac{F(t + h) \ominus_g F(t)}{h} = \widetilde{A}[F(t)] + T(h)g(t), \qquad (4.4)$$

since F is l-nondecreasing. By Theorem 4.11, we know that $x(t)$ is Δ_{gH}-differentiable. Furthermore, we can infer from (4.1), (4.4) and Theorem 4 in [7] that

$$\begin{aligned} x^\Delta(t) &= \Big(T(t)(x_0) + \int_0^t T(t - s)g(s)\Delta s \Big)_{gH}^\Delta \\ &= \Big(T(t)(x_0) \Big)_{gH}^\Delta + \Big(\int_0^t T(t - s)g(s)\Delta s \Big)_{gH}^\Delta \\ &= \widetilde{A}[T(t)(x_0)] + \widetilde{A}[F(t)] + T(h)g(t) \\ &= \widetilde{A}[T(t)(x_0) + F(t)] + T(h)g(t) \\ &= \widetilde{A}[x(t)] + T(\mu(t))g(t), \end{aligned}$$

which means that $x(t)$ satisfies (4.2).
(ii) If $t \in \mathbb{T}^\kappa$ is right-dense, the proof is similar to Theorem 3.9 in [4] and so is omitted. $\square$

Remark 4.15. From Lemma 4.13, we know that F is l-nondecreasing on $\mathbb{T}$ if T is l-nondecreasing C_0-semigroup on $\mathcal{K}$. Apparently, a question that deserves further consideration is whether F is l-monotonic on $\mathbb{T}$ if T is l-nonincreasing C_0-semigroup on $\mathcal{K}$. Furthermore, if F is l-monotonic on $\mathbb{T}$, then we can consider another question from Theorem 4.14. In detail, what is the solution to the initial value problem (4.2) when T is l-nonincreasing C_0-semigroup on $\mathcal{K}$?

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