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SPECIAL HELICES ON EQUIFORM DIFFERENTIAL GEOMETRY OF SPACELIKE CURVES IN MINKOWSKI SPACE-TIME

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Abstract. In this paper, we establish k-type helices for equiform differential geometry of spacelike curves in 4-dimensional Minkowski space E_1^4. Also we obtain (k, m)-type slant helices for equiform differential geometry of spacelike curves in Minkowski space-time.

1. Introduction

Helices, which are an important subject of the theory of curves in differential geometry, are studied by physicists, engineers and biologists. Helix (or general helix) is described as an in 3-dimensional Euclidean space (or Minkowski) tangent vector field forming a constant angle with a fixed direction of the curve. So, many authors were interested in helices to study it in Euclidean (or Minkowski) 3- and 4-space and they gave new characterizations for an helix. In the 4-dimensional Minkowski space k-type slant helices were defined in a study by Ali et al. [1]. In addition, M.Y. Yılmaz and M.Bektaş in [6] defined (k, m)-type slant helices in 4-dimensional Euclidean space.

In our study, we establish k-type helices and (k, m)-type slant helices for equiform differential geometry of spacelike curves in 4-dimensional Minkowski space E_1^4 and give some new characterizations for these helices.

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2. Geometric Preliminaries

Let $E^4 = \{(x_1, x_2, x_3, x_4) | x_1, x_2, x_3, x_4 \in R\}$ be a 4-dimensional vector space. For any two vectors $x = (x_1, x_2, x_3, x_4)$, $y = (y_1, y_2, y_3, y_4)$ in E^4, the pseudo scalar product of x and y is defined by $\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$. We call $(E^4, \langle ., .\rangle)$ a Minkowski 4-space and denote it by E_1^4. We say that a vector x in $E_1^4 \setminus \{0\}$ is a spacelike vector, a lightlike vector or a timelike vector if $\langle x, x\rangle$ is positive, zero, negative respectively.

The norm of a vector $x \in E_1^4$ is defined by $\|x\| = \sqrt{|\langle x, x\rangle|}$. For any two vectors a, b in E_1^4, we say that a is pseudo-perpendicular to b if $\langle a, b\rangle = 0$. Let $\alpha : I \subset R \to E_1^4$ be an arbitrary curve in E_1^4, we say that a curve α is a spacelike curve if $\langle \dot{\alpha}(t), \dot{\alpha}(t)\rangle > 0$ for any $t \in I$. The arclength of a spacelike curve γ measured from $\alpha(t_0)$ $(t_0 \in I)$ is

$$s(t) = \int_{t_0}^{t} \|\dot{\alpha}(t)\| \mathrm{d}t. \tag{1}$$

Hence a parameter s is determined such that $\|\alpha'(s)\| = 1$, where $\alpha'(s) = d\alpha/ds$. Consequently, we say that a spacelike curve α is parameterized by arclength if $\|\alpha'(s)\| = 1$. Throughout the rest of this paper s is assumed arclength parameter. For any $x, y, z \in E_1^4$, we define a vector $x \times y \times z$ by

$$x \times y \times z = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ x_1^1 & x_1^2 & x_1^3 & x_1^4 \\ x_2^1 & x_2^2 & x_2^3 & x_2^4 \\ x_3^1 & x_3^2 & x_3^3 & x_3^4 \end{vmatrix}, \tag{2}$$

where $x_i = (x_i^1, x_i^2, x_i^3, x_i^4)$. Let $\alpha : I \longrightarrow E_1^4$ be a spacelike curve in E_1^4. Therefore we can construct a pseudo-orthogonal frame $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b_1}(s), \mathbf{b_2}(s)\}$, which satisfies the following Frenet-Serret type formula of E_1^4 along α.

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b_1} \\ \mathbf{b_2} \end{bmatrix}' = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ \mu_1\kappa_1 & 0 & \mu_2\kappa_2 & 0 \\ 0 & \mu_3\kappa_2 & 0 & \mu_4\kappa_3 \\ 0 & 0 & \mu_5\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b_1} \\ \mathbf{b_2} \end{bmatrix}, \tag{3}$$

where κ_1, κ_2 and κ_3 are respectively, first, second and third curvature of the spacelike curve α and we have

$$\kappa_1(s) = \|\alpha''(s)\|,$$
$$\mathbf{n}(s) = \frac{\alpha''(s)}{\kappa_1(s)},$$
$$\mathbf{b_1}(s) = \frac{\mathbf{n}'(s) + \mu_1\kappa_1(s)\mathbf{t}(s)}{\|\mathbf{n}'(s) + \mu_1\kappa_1(s)\mathbf{t}(s)\|},$$

$$\mathbf{b_2}(s) = \mathbf{t}(s) \times \mathbf{n}(s) \times \mathbf{b_1}(s).$$

Denote by $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b_1}(s), \mathbf{b_2}(s)\}$ the moving Frenet frame along the spacelike curve α, where s is a pseudo arclength parameter [1,2,3,5,7].

3. Equiform Differential Geometry of Curves

3.1. **Spacelike Curves:**

Definition 3.1. *Unless otherwise stated, we use the same terminology such as [2,4]. Let $\alpha : I \longrightarrow E_1^4$ be* **a spacelike curve***. We define the equiform parameter of $\alpha(s)$ by*

$$\sigma = \int \frac{ds}{\rho} = \int \kappa_1 ds, \tag{4}$$

where $\rho = \dfrac{1}{\kappa_1}$ is the radius of curvature of the curve α.

It follows

$$\frac{ds}{d\sigma} = \rho. \tag{5}$$

Let h be a homothety with the center in the origin and the coefficient λ. If we put $\alpha^* = h(\alpha)$ then it follows

$$s^* = \lambda s, \ \textit{and} \ \rho^* = \lambda\rho, \tag{6}$$

where s^* is the arclength parameter of α^* and ρ^* the radius of curvature of α^*. Hence α is an equiform invariant parameter of α.

Notation 3.1. Let us note that κ_1, κ_2 and κ_3 are not invariants of the homothety group, it follows $\kappa_1^* = \dfrac{1}{\lambda}\kappa_1$, $\kappa_2^* = \dfrac{1}{\lambda}\kappa_2$ and $\kappa_3^* = \dfrac{1}{\lambda}\kappa_3$. The vector

$$\mathbf{V}_1 = \frac{d\alpha(s)}{d\sigma}, \tag{7}$$

is called a tangent vector of the curve α in the equiform geometry. From (5) and (7), we get

$$\mathbf{V}_1 = \frac{d\alpha(s)}{d\sigma} = \rho\frac{d\alpha(s)}{ds} = \rho\mathbf{t}. \tag{8}$$

Furthermore, we define the tri-normals by

$$\mathbf{V}_2 = \rho\mathbf{n}, \quad \mathbf{V}_3 = \rho\mathbf{b_1}, \quad \mathbf{V}_4 = \rho\mathbf{b_2}. \tag{9}$$

It is easy to check that the tetrahedron $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4\}$ is an equiform invariant tetrahedron of the curve α. Now, we will find the derivatives of these vectors with respect to σ using by (5), (7) and (9) as follows:

$$\mathbf{V}_1' = \frac{d}{d\sigma}(\mathbf{V}_1) = \rho\frac{d}{ds}(\rho\mathbf{t}) = \dot{\boldsymbol{\rho}}\mathbf{V_1} + \mathbf{V_2}$$

where the derivative with respect to the arclength s is denoted by a dot and respect to σ by a dash. Similarly, we obtain

$$\begin{aligned}
\mathbf{V}_2' &= \frac{d}{d\sigma}(\mathbf{V}_2) = \rho\frac{d}{ds}(\rho\mathbf{n}) = \boldsymbol{\mu_1}\mathbf{V_1} + \dot{\boldsymbol{\rho}}\mathbf{V_2} + \boldsymbol{\mu_2}\left(\frac{\boldsymbol{\kappa_2}}{\boldsymbol{\kappa_1}}\right)\mathbf{V_3},\\
\mathbf{V}_3' &= \frac{d}{d\sigma}(\mathbf{V}_3) = \rho\frac{d}{ds}(\rho\mathbf{b_1}) = \boldsymbol{\mu_3}\left(\frac{\boldsymbol{\kappa_2}}{\boldsymbol{\kappa_1}}\right)\mathbf{V_2} + \dot{\boldsymbol{\rho}}\mathbf{V_3} + \boldsymbol{\mu_4}\left(\frac{\boldsymbol{\kappa_3}}{\boldsymbol{\kappa_1}}\right)\mathbf{V_4},\\
\mathbf{V}_4' &= \frac{d}{d\sigma}(\mathbf{V}_4) = \rho\frac{d}{ds}(\rho\mathbf{b_2}) = \boldsymbol{\mu_5}\left(\frac{\boldsymbol{\kappa_3}}{\boldsymbol{\kappa_1}}\right)\mathbf{V_3} + \dot{\boldsymbol{\rho}}\mathbf{V_4}.
\end{aligned} \tag{10}$$

Definition 3.2. *The functions* $\mathbf{K}_i : I \longrightarrow R$ $(i = 1, 2, 3)$ *defined by*

$$\mathbf{K}_1 = \dot{\rho},\ \mathbf{K}_2 = \frac{\kappa_2}{\kappa_1},\ \mathbf{K}_3 = \frac{\kappa_3}{\kappa_1} \tag{11}$$

are called i^{th} *equiform curvatures of the curve* α*.*

These functions $\mathbf{K}_i$ are differential invariant of the group of equiform transformations, too. Therefore, the formulas analogous to famous the Frenet formulas in the equiform geometry of the Minkowski space E_1^4 have the following form:

$$\begin{aligned}
\mathbf{V}_1' &= \mathbf{K}_1\mathbf{V}_1 + \mathbf{V}_2,\\
\mathbf{V}_2' &= \mu_1\mathbf{V}_1 + \mathbf{K}_1\mathbf{V}_2 + \mu_2\mathbf{K}_2\mathbf{V}_3,\\
\mathbf{V}_3' &= \mu_3\mathbf{K}_2\mathbf{V}_2 + \mathbf{K}_1\mathbf{V}_3 + \mu_4\mathbf{K}_3\mathbf{V}_4,\\
\mathbf{V}_4' &= \mu_5\mathbf{K}_3\mathbf{V}_3 + \mathbf{K}_1\mathbf{V}_4.
\end{aligned} \tag{12}$$

Notation 3.2. The equiform parameter $\sigma = \int \kappa_1(s)ds$ for closed curves is called the total curvature, and it plays an important role in global differential geometry of the Euclidean space. Also, the functions $\frac{\kappa_2}{\kappa_1}$ and $\frac{\kappa_3}{\kappa_1}$ have been already known as conical curvatures and they also have interesting geometric interpretation.

Because of the equiform Frenet formulas (12), the following equalities regarding equiform curvatures can be given

$$\begin{aligned}
\mathbf{K}_1 &= \frac{1}{\rho^2}\langle \mathbf{V}_j', \mathbf{V}_j\rangle;\ (j = 1, 2, 3, 4),\\
\mathbf{K}_2 &= \frac{1}{\mu_2\rho^2}\langle \mathbf{V}_2', \mathbf{V}_3\rangle = \frac{1}{\mu_3\rho^2}\langle \mathbf{V}_3', \mathbf{V}_2\rangle,\\
\mathbf{K}_3 &= \frac{1}{\mu_4\rho^2}\langle \mathbf{V}_3', \mathbf{V}_4\rangle = \frac{1}{\mu_5\rho^2}\langle \mathbf{V}_4', \mathbf{V}_3\rangle.
\end{aligned} \tag{13}$$

Definition 3.3. *Let* α *be a spacelike curve in* E_1^4 *with equiform Frenet frame* $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4\}$*. If there exists a non-zero constant vector field* U *in* E_1^4 *such that* $< \mathbf{V}_i, U >=$ *constant for* $1 \leq i \leq 4$*, then* α *is said to be a* $k-$*type slant helix and* U *is called the slope axis of* α*.*

Theorem 3.1. *Let α be a spacelike curve with Frenet formulas in equiform geometry of the Minkowski space E_1^4. Then, if the curve α is a 1-type helix (or general helix), then we have*

$$\langle \mathbf{V}_2, U\rangle = -\mathbf{K}_1 c, \tag{14}$$

where c is a constant.

Proof. Assume that α is a 1-type helix. Then for a constant field U such that $\langle \mathbf{V}_1, U\rangle = c$ is a constant. Differentiating this equation with respect to σ, we get

$$\langle \mathbf{V}_1', U\rangle = 0,$$

and using equiform Frenet equations, we find

$$\mathbf{K}_1\langle \mathbf{V}_1, U\rangle + \langle \mathbf{V}_2, U\rangle = 0$$

and using $\langle \mathbf{V}_1, U\rangle = c$,

$$\mathbf{K}_1 c + \langle \mathbf{V}_2, U\rangle = 0. \tag{15}$$

From (15), it is written as follows:

$$\langle \mathbf{V}_2, U\rangle = -\mathbf{K}_1 c,$$

thus, the proof is completed. □

Theorem 3.2. *Let α be a spacelike curve with Frenet formulas in equiform geometry of the Minkowski space E_1^4. Then, if the curve α is a 2-type helix, then we have*

$$\mu_1\langle \mathbf{V}_1, U\rangle + \mu_2\mathbf{K}_2\langle \mathbf{V}_3, U\rangle = -\mathbf{K}_1 c_1 \tag{16}$$

where c_1 is a constant.

Proof. If the curve α is a 2-type helix. Therefore for a constant field U such that $\langle \mathbf{V}_2, U\rangle = c_1$ is a constant. Differentiating this equation with respect to σ, we get

$$\langle \mathbf{V}_2', U\rangle = 0,$$

and using equiform Frenet equations, we have

$$\mu_1\langle \mathbf{V}_1, U\rangle + \mathbf{K}_1\langle \mathbf{V}_2, U\rangle + \mu_2\mathbf{K}_2\langle \mathbf{V}_3, U\rangle = 0$$

and using $\langle \mathbf{V}_2, U\rangle = c_1$, we find

$$\mu_1\langle \mathbf{V}_1, U\rangle + \mathbf{K}_1 c_1 + \mu_2\mathbf{K}_2\langle \mathbf{V}_3, U\rangle = 0. \tag{17}$$

From (17), we obtain

$$\mu_1\langle \mathbf{V}_1, U\rangle + \mu_2\mathbf{K}_2\langle \mathbf{V}_3, U\rangle = -\mathbf{K}_1 c_1.$$

The proof is completed. □

Theorem 3.3. *Let α be a spacelike curve with Frenet formulas in equiform geometry of the Minkowski space E_1^4. In that case, if the curve α is a 3-type helix, then we have*

$$\mu_3\mathbf{K}_2\langle \mathbf{V}_2, U\rangle + \mu_4\mathbf{K}_3\langle \mathbf{V}_4, U\rangle = -\mathbf{K}_1 c_2, \tag{18}$$

where c_2 is a constant.

Proof. If the curve α is a 3-type helix. Thus, for a constant field U such that

$$\langle \mathbf{V}_3, U\rangle = c_2 \tag{19}$$

is a constant. Differentiating this equation with respect to σ, we get

$$\langle \mathbf{V}_3', U\rangle = 0$$

and using equiform Frenet equations, we have

$$\mu_3 \mathbf{K}_2 \langle \mathbf{V}_2, U\rangle + \mathbf{K}_1 \langle \mathbf{V}_3, U\rangle + \mu_4 \mathbf{K}_3 \langle \mathbf{V}_4, U\rangle = 0, \tag{20}$$

and by setting (19) in (20), we can write

$$\mu_3 \mathbf{K}_2 \langle \mathbf{V}_2, U\rangle + \mathbf{K}_1 c_2 + \mu_4 \mathbf{K}_3 \langle \mathbf{V}_4, U\rangle = 0, \tag{21}$$

and from the last equation, we find

$$\mu_3 \mathbf{K}_2 \langle \mathbf{V}_2, U\rangle + \mu_4 \mathbf{K}_3 \langle \mathbf{V}_4, U\rangle = -\mathbf{K}_1 c_2,$$

the proof is completed. □

Theorem 3.4. *Let α be a spacelike curve with Frenet formulas in equiform geometry of the Minkowski space E_1^4. Then, if the curve α is a 4-type helix, in that case, we have*

$$\langle \mathbf{V}_3, U\rangle = -\frac{\mathbf{K}_1}{\mathbf{K}_3 \mu_5} c_3, \tag{22}$$

where c_3 is a constant.

Proof. If the curve α is a 4-type helix. Then for a constant field U such that

$$\langle \mathbf{V}_4, U\rangle = c_3, \tag{23}$$

is a constant. By differentiating of this last equation with respect to σ, we get

$$\langle \mathbf{V}_4', U\rangle = 0,$$

and using equiform Frenet equations, we obtain

$$\langle \mu_5 \mathbf{K}_3 \mathbf{V}_3 + \mathbf{K}_1 \mathbf{V}_4, U\rangle = 0. \tag{24}$$

From (24), we get

$$\mu_5 \mathbf{K}_3 \langle \mathbf{V}_3, U\rangle + \mathbf{K}_1 \langle \mathbf{V}_4, U\rangle = 0. \tag{25}$$

Substituting (23) in (25), we obtain

$$\langle \mathbf{V}_3, U\rangle = -\frac{\mathbf{K}_1}{\mathbf{K}_3 \mu_5} c_3.$$

The proof is completed. □

4. (k, m)-type slant helices in $\mathbf{E}_1^4$

In this section, we will define (k, m) type slant helices for spacelike curve with equiform Frenet frame in E_1^4 such as [6].

Definition 4.1. *Let α be a spacelike curve in E_1^4 with equiform Frenet frame $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4\}$. We call α is a (k, m)- type slant helix if there exists a non-zero constant vector field $U \in E_1^4$ satisfies $\langle \mathbf{V}_k, U\rangle = c_1$ (c_1 is a constant) and $\langle \mathbf{V}_m, U\rangle = c_2$ (c_2 is a constant) for $1 \leq k, m \leq 4$, $k \neq m$. The constant vector U is on axis of α.*

Theorem 4.1. *If the curve α is a $(1, 2)$-type slant helix in E_1^4, then we have*

$$\langle \mathbf{V}_3, U\rangle = -\frac{\mu_1 c_1 + \mathbf{K}_1 c_2}{\mu_2 \mathbf{K}_2},$$

and

$$\mathbf{K}_1 = -\frac{c_2}{c_1} \text{ is a constant.}$$

Proof. If the curve α is a $(1, 2)$-type slant helix in E_1^4, then for a constant field U. We can write

$$\langle \mathbf{V}_1, U\rangle = c_1, \tag{26}$$

and

$$\langle \mathbf{V}_2, U\rangle = c_2 \tag{27}$$

is a constant. Differentiating (26) and (27) with respect to σ, we have that

$$\langle \mathbf{V}_1', U\rangle = 0$$

and

$$\langle \mathbf{V}_2', U\rangle = 0.$$

Using equiform Frenet equations, the following equations can be obtained:

$$\mathbf{K}_1\langle \mathbf{V}_1, U\rangle + \langle \mathbf{V}_2, U\rangle = 0, \tag{28}$$

$$\mu_1\langle \mathbf{V}_1, U\rangle + \mathbf{K}_1\langle \mathbf{V}_2, U\rangle + \mu_2\mathbf{K}_2\langle \mathbf{V}_3, U\rangle = 0. \tag{29}$$

By setting (26) and (27) in (28), we find

$$\mathbf{K}_1 c_1 + c_2 = 0, \tag{30}$$

and substituting (26) and (27) in (28), we obtain

$$\mu_1 c_1 + \mathbf{K}_1 c_2 + \mu_2\mathbf{K}_2\langle \mathbf{V}_3, U\rangle = 0. \tag{31}$$

Finally, we have the following equations:

$$\begin{aligned} \mathbf{K}_1 &= -\frac{c_2}{c_1}, \\ \langle \mathbf{V}_3, U\rangle &= -\frac{\mu_1 c_1 + \mathbf{K}_1 c_2}{\mu_2\mathbf{K}_2}. \end{aligned}$$

The proof is completed. □

Theorem 4.2. *If the curve α is a $(1,3)$-type slant helix in E_1^4, then there exists a constant such that*

$$\langle \mathbf{V}_4, U\rangle = \frac{\mu_3\mathbf{K}_2\mathbf{K}_1c_1 - \mathbf{K}_1c_3}{\mu_4\mathbf{K}_3}. \tag{32}$$

where c_1 and c_3 are constant.

Proof. If the curve α is a $(1,3)$-type slant helix in E_1^4, then for a constant field U. We can write

$$\langle \mathbf{V}_1, U\rangle = c_1 \tag{33}$$

and

$$\langle \mathbf{V}_3, U\rangle = c_3 \tag{34}$$

is a constant. Differentiating (33) and (34) with respect to σ, we get

$$\langle \mathbf{V}_1', U\rangle = 0,$$

and

$$\langle \mathbf{V}_3', U\rangle = 0.$$

Using equiform Frenet equations, we have

$$\mathbf{K}_1\langle \mathbf{V}_1, U\rangle + \langle \mathbf{V}_2, U\rangle = 0, \tag{35}$$

$$\mu_3\mathbf{K}_2\langle \mathbf{V}_2, U\rangle + \mathbf{K}_1\langle \mathbf{V}_3, U\rangle + \mu_4\mathbf{K}_3\langle \mathbf{V}_4, U\rangle = 0. \tag{36}$$

By setting (33) in (35), we obtain

$$\mathbf{K}_1c_1 + \langle \mathbf{V}_2, U\rangle = 0. \tag{37}$$

From (37), we find as follows:

$$\langle \mathbf{V}_2, U\rangle = -\mathbf{K}_1c_1. \tag{38}$$

Substituting (34) and (38) in (36), we find

$$\langle \mathbf{V}_4, U\rangle = \frac{\mu_3\mathbf{K}_2\mathbf{K}_1c_1 - \mathbf{K}_1c_3}{\mu_4\mathbf{K}_3}.$$

The proof is completed. □

Theorem 4.3. *If the curve α is a $(1,4)$-type slant helix in E_1^4, then there exists a constant such that*

$$\langle \mathbf{V}_2, U\rangle = -\mathbf{K}_1c_1$$

and

$$\langle \mathbf{V}_3, U\rangle = -\frac{\mathbf{K}_1c_4}{\mu_5\mathbf{K}_3}.$$

Proof. If the curve α is a $(1,4)$-type slant helix in E_1^4, then for a constant field U. We can write

$$\langle \mathbf{V}_1, U\rangle = c_1 \tag{39}$$

and

$$\langle \mathbf{V}_4, U\rangle = c_4 \tag{40}$$

is a constant. Differentiating (39) and (40) with respect to σ, we get

$$\langle \mathbf{V}_1', U\rangle = 0,$$

and it follows

$$\langle \mathbf{V}_4', U\rangle = 0.$$

Using equiform Frenet equations, we have

$$\mathbf{K}_1\langle \mathbf{V}_1, U\rangle + \langle \mathbf{V}_2, U\rangle = 0, \tag{41}$$

and

$$\mu_5\mathbf{K}_3\langle \mathbf{V}_3, U\rangle + \mathbf{K}_1\langle \mathbf{V}_4, U\rangle = 0. \tag{42}$$

By setting (39) in (41), we obtain as below:

$$\mathbf{K}_1 c_1 + \langle \mathbf{V}_2, U\rangle = 0. \tag{43}$$

Substituting (40) in (42), we can write

$$\mu_5\mathbf{K}_3\langle \mathbf{V}_3, U\rangle + \mathbf{K}_1 c_4 = 0. \tag{44}$$

From (43) and (44), we get

$$\langle \mathbf{V}_2, U\rangle = -\mathbf{K}_1 c_1,$$

and

$$\langle \mathbf{V}_3, U\rangle = -\frac{\mathbf{K}_1 c_4}{\mu_5\mathbf{K}_3}.$$

The proof is completed. □

Theorem 4.4. *If the curve α is a $(2,3)$-type slant helix in E_1^4, then there exist constants such that*

$$\langle \mathbf{V}_1, U\rangle = -\frac{\mathbf{K}_1 c_2 + \mu_2\mathbf{K}_2 c_3}{\mu_1}, \tag{45}$$

and

$$\langle \mathbf{V}_4, U\rangle = -\frac{\mu_3\mathbf{K}_2 c_2 + \mathbf{K}_1 c_3}{\mu_4\mathbf{K}_3}. \tag{46}$$

Proof. If the curve α is a $(2,3)$-type slant helix in E_1^4, thus for a constant field U. We can write as below:

$$\langle \mathbf{V}_2, U\rangle = c_2 \tag{47}$$

is a constant and

$$\langle \mathbf{V}_3, U\rangle = c_3 \tag{48}$$

is a constant. Differentiating (47) and (48) with respect to σ, we get

$$\langle \mathbf{V}_2', U\rangle = 0,$$

and

$$\langle \mathbf{V}_3', U\rangle = 0.$$

Using equiform Frenet formulas, we have the following equations:

$$\mu_1\langle \mathbf{V}_1, U\rangle + \mathbf{K}_1\langle \mathbf{V}_2, U\rangle + \mu_2\mathbf{K}_2\langle \mathbf{V}_3, U\rangle = 0, \tag{49}$$

$$\mu_3\mathbf{K}_2\langle \mathbf{V}_2, U\rangle + \mathbf{K}_1\langle \mathbf{V}_3, U\rangle + \mu_4\mathbf{K}_3\langle \mathbf{V}_4, U\rangle = 0. \tag{50}$$

Substituting (47) and (48) in (49), we can write

$$\langle \mathbf{V}_1, U\rangle = -\frac{\mathbf{K}_1 c_2 + \mu_2\mathbf{K}_2 c_3}{\mu_1},$$

and by setting (47) and (48) in (50), we obtain

$$\langle \mathbf{V}_4, U\rangle = -\frac{\mu_3\mathbf{K}_2 c_2 + \mathbf{K}_1 c_3}{\mu_4\mathbf{K}_3}.$$

The proof is completed. □

Theorem 4.5. *If the curve α is a $(2,4)$-type slant helix in E_1^4, then there exists constant such that*

$$\langle \mathbf{V}_1, U\rangle = \frac{\mu_2\mathbf{K}_2\mathbf{K}_1 c_4 \ - \mu_5\mathbf{K}_3\mathbf{K}_1 c_2}{\mu_1\mu_5\mathbf{K}_3},$$

where c_2 and c_4 are constants.

Proof. If the curve α is a $(2,4)$-type slant helix in E_1^4, then for a constant field U. We can write the following equations:

$$\langle \mathbf{V}_2, U\rangle = c_2, \tag{51}$$

and

$$\langle \mathbf{V}_4, U\rangle = c_4 \tag{52}$$

is a constant. By differentiating (51) and (52) with respect to σ, we get the following equations:

$$\langle \mathbf{V}_2', U\rangle = 0$$

and

$$\langle \mathbf{V}_4', U\rangle = 0.$$

Using equiform Frenet equations, we have as below:

$$\mu_1\langle \mathbf{V}_1, U\rangle + \mathbf{K}_1\langle \mathbf{V}_2, U\rangle + \mu_2\mathbf{K}_2\langle \mathbf{V}_3, U\rangle = 0, \tag{53}$$

$$\mu_5\mathbf{K}_3\langle \mathbf{V}_3, U\rangle + \mathbf{K}_1\langle \mathbf{V}_4, U\rangle = 0. \tag{54}$$

Using (52), in the last equation, we get

$$\langle \mathbf{V}_3, U\rangle = -\frac{\mathbf{K}_1 c_4}{\mu_5\mathbf{K}_3}. \tag{55}$$

Substituting (51) and (55) in (53), we obtain

$$\langle \mathbf{V}_1, U\rangle = \frac{\mu_2\mathbf{K}_2\mathbf{K}_1 c_4 \ - \mu_5\mathbf{K}_3\mathbf{K}_1 c_2}{\mu_1\mu_5\mathbf{K}_3},$$

the proof is completed. □

Theorem 4.6. *If the curve α is a $(3,4)$-type slant helix in E_1^4, then we have*

$$\langle\langle \mathbf{V}_2, U\rangle = \frac{\mathbf{K}_3}{\mathbf{K}_2}\frac{\left(\mu_5 c_3^2 - \mu_4 c_4^2\right)}{\mu_3 c_4} \tag{56}$$

and

$$\mathbf{K}_1 = -\mu_5 \mathbf{K}_3 \frac{c_3}{c_4}.$$

Proof. If the curve α is a $(3,4)$-type slant helix in E_1^4, then for a constant field U. We can write

$$\langle \mathbf{V}_3, U\rangle = c_3 \tag{57}$$

is a constant and

$$\langle \mathbf{V}_4, U\rangle = c_4 \tag{58}$$

is a constant. By differentiating (57) and (58) with respect to σ, we get

$$\langle \mathbf{V}_3', U\rangle = 0$$

and it follows

$$\langle \mathbf{V}_4', U\rangle = 0.$$

Using equiform Frenet formulas, we get as follows:

$$\mu_3 \mathbf{K}_2 \langle \mathbf{V}_2, U\rangle + \mathbf{K}_1 \langle \mathbf{V}_3, U\rangle + \mu_4 \mathbf{K}_3 \langle \mathbf{V}_4, U\rangle = 0, \tag{59}$$

$$\mu_5 \mathbf{K}_3 \langle \mathbf{V}_3, U\rangle + \mathbf{K}_1 \langle \mathbf{V}_4, U\rangle = 0. \tag{60}$$

By setting (57) and (58) in (60), we have the following equation:

$$\mathbf{K}_1 = -\mu_5 \mathbf{K}_3 \frac{c_3}{c_4}, \tag{61}$$

and substituting (57) and (58) in (59), we obtain

$$\langle \mathbf{V}_2, U\rangle = -\frac{\mathbf{K}_1 c_3}{\mu_3 \mathbf{K}_2} - \frac{\mu_4 \mathbf{K}_3 c_4}{\mu_3 \mathbf{K}_2}. \tag{62}$$

Using (61), in the last equation, we find

$$\langle \mathbf{V}_2, U\rangle = \frac{\mathbf{K}_3}{\mathbf{K}_2}\frac{\left(\mu_5 c_3^2 - \mu_4 c_4^2\right)}{\mu_3 c_4}. \tag{63}$$

The proof is completed. □

References

[1] Ali, A. T., López, R., Turgut, M., k-type partially null and pseudo null slant helices in Minkowski 4-space, *Math.Commun.,* 17 (2012), 93-103.

[2] Abdel-Aziz, H.S., Saad, M.K. Abdel-Salam, A.A., Equiform Differential Geometry of Curves in Minkowski Space-Time, arXiv.org/math/ arXiv:1501.02283.

[3] İlarslan, K., Nešović, E., Spacelike and Timelike Normal Curves in Minkowski Space-Time, *Publications de L'Institut Mathematique, Nouvelle série,* tome 85(99) (2009), 111-118.

[4] Aydın, M.E., Ergüt, M., The equiform differential geometry of curves in 4-dimensional Galilean space G4, *Stud. Univ. Babeş-Bolyai Math.,* 58(3) (2013), 399–406.

[5] Turgut, M., Yilmaz, S., Characterizations of Some Special Space-like Curves in Minkowski Space-time, *International J.Math. Combin.,* 2 (2008), 17-22.

[6] Yılmaz, M.Y., Bektaş, M., Slant helices of (k, m)-type in E^4 , *Acta Univ. Sapientiae, Mathematica,* 10(2) (2018), 395–401.

[7] Yilmaz, S., Turgut, M., On the characterizations of inclined curves in Minkowski space-time E_1^4, *International Mathematical Forum,* 3(16) (2008), 783-792.