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Research Article

# The multiplicative norm convergence in normed Riesz algebras

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## Abstract

A net $(x_\alpha)_{\alpha\in A}$ in an $f$-algebra $E$ is called multiplicative order convergent to $x \in E$ if $|x_\alpha - x| \cdot u \xrightarrow{o} 0$ for all $u \in E_+$. This convergence was introduced and studied on $f$-algebras with the order convergence. In this paper, we study a variation of this convergence for normed Riesz algebras with respect to the norm convergence. A net $(x_\alpha)_{\alpha\in A}$ in a normed Riesz algebra $E$ is said to be multiplicative norm convergent to $x \in E$ if $\big\||x_\alpha - x| \cdot u\big\| \to 0$ for each $u \in E_+$. We study this concept and investigate its relationship with the other convergences, and also we introduce the $mn$-topology on normed Riesz algebras.

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## 1. Introduction and preliminaries

Let us recall some notations and terminologies used in this paper. An ordered vector space $E$ is said to be *vector lattice* (or, *Riesz space*) if, for each pair of vectors $x, y \in E$, the supremum $x \vee y = \sup\{x, y\}$ and the infimum $x \wedge y = \inf\{x, y\}$ both exist in E. For $x \in E$, $x^+ := x \vee 0$, $x^- := (-x) \vee 0$, and $|x| := x \vee (-x)$ are called the *positive* part, the *negative* part, and the *absolute value* of $x$, respectively. A vector lattice $E$ is called *order complete* if every nonempty bounded above subset has a supremum (or, equivalently, whenever every nonempty bounded below subset has an infimum). A vector lattice is order complete if and only if $0 \le x_\alpha \uparrow\le x$ implies the existence of the $\sup x_\alpha$. A partially ordered set $A$ is called *directed* if, for each $a_1, a_2 \in A$, there is another $a \in A$ such that $a \ge a_1$ and $a \ge a_2$ (or, equivalently, $a \le a_1$ and $a \le a_2$). A function from a directed set $A$ into a set $E$ is called a *net* in $E$. A net $(x_\alpha)_{\alpha\in A}$ in a vector lattice $E$ is *order convergent* (or *o-convergent*, for short) to $x \in E$, if there exists another net $(y_\beta)_{\beta\in B}$ satisfying $y_\beta \downarrow 0$, and for any $\beta \in B$ there exists $\alpha_\beta \in A$ such that $|x_\alpha - x| \le y_\beta$ for all $\alpha \ge \alpha_\beta$. In this case, we write $x_\alpha \xrightarrow{o} x$. An operator $T : E \to F$ between two vector lattices is called *order continuous* whenever $x_\alpha \xrightarrow{o} 0$ in $E$ implies $Tx_\alpha \xrightarrow{o} 0$ in $F$. A vector $e \ge 0$ in a vector lattice $E$ is said to be a *weak order unit* whenever the band generated by $e$ satisfies $B_e = E$, or equivalently, whenever for each $x \in E_+$ we have $x \wedge ne \uparrow x$; see much more information of vector lattices for example [1, 2, 16, 17]. Recall that a net $(x_\alpha)_{\alpha\in A}$ in a vector lattice $E$ is

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*unbounded order convergent* (or shortly, *uo-convergent*) to $x \in E$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for every $u \in E_+$. In this case, we write $x_\alpha \xrightarrow{uo} x$, we refer the reader for an exposition on *uo*-convergence to [3, 5–11].

A vector lattice $E$ under an associative multiplication is said to be a *Riesz algebra* (or, shortly, $l$-algebra) whenever the multiplication makes $E$ an algebra (with the usual properties), and besides, it satisfies the following property: $x \cdot y \in E_+$ for every $x, y \in E_+$. A Riesz algebra $E$ is called *commutative* whenever $x \cdot y = y \cdot x$ for all $x, y \in E$. Also, a subset $A$ of an $l$-algebra $E$ is called *$l$-subalgebre* of $E$ whenever it is also an $l$-algebra under the multiplication operation in $E$

An $l$-algebra $X$ is called: *$d$-algebra* whenever $u \cdot (x \wedge y) = (u \cdot x) \wedge (u \cdot y)$ and $(x \wedge y) \cdot u = (x \cdot u) \wedge (y \cdot u)$ holds for all $u, x, y \in X_+$; *almost $f$-algebra* if $x \wedge y = 0$ implies $x \cdot y = 0$ for all $x, y \in X_+$; *$f$-algebra* if, for all $u, x, y \in X_+$, $x \wedge y = 0$ implies $(u \cdot x) \wedge y = (x \cdot u) \wedge y = 0$; *semiprime* whenever the only nilpotent element in $X$ is zero; *unital* if $X$ has a multiplicative unit. Moreover, any $f$-algebra is both $d$- and almost $f$-algebra (cf. [2, 12, 13, 17]). A vector lattice $E$ is called *Archimedean* whenever $\frac{1}{n} x \downarrow 0$ holds in $E$ for each $x \in E_+$. Every Archimedean $f$-algebra is commutative; see for example [13, p.7]. Assume $E$ is an Archimedean $f$-algebra with a multiplicative unit vector $e$. Then, by applying [17, Thm.142.1(v)], in view of $e = e \cdot e = e^2 \geq 0$, it can be seen that $e$ is a positive vector. On the other hand, since $e \wedge x = 0$ implies $x = x \wedge x = (x \cdot e) \wedge x = 0$, it follows that $e$ is a weak order unit (cf.[12, Cor.1.10]). In this article, unless otherwise, all vector lattices are assumed to be real and Archimedean and all $l$-algebras are assumed to be commutative.

A net $(x_\alpha)_{\alpha \in A}$ in an $f$-algebra $E$ is called *multiplicative order convergent* (or shortly, *mo-convergent*) to $x \in E$ whenever $|x_\alpha - x| \cdot u \xrightarrow{o} 0$ for all $u \in E_+$. Also, it is called *mo-Cauchy* if the net $(x_\alpha - x_{\alpha'})_{(\alpha, \alpha') \in A \times A}$ *mo*-converges to zero. $E$ is called *mo-complete* if every *mo*-Cauchy net in $E$ is *mo*-convergent, and it is also called *mo-continuous* if $x_\alpha \xrightarrow{o} 0$ implies $x_\alpha \xrightarrow{\text{mo}} 0$; see much more detail information [4]. Recall that a norm $\|\cdot\|$ on a vector lattice is said to be a *lattice norm* whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. A vector lattice equipped with a lattice norm is known as a *normed Riesz space* or *normed vector lattice*. Moreover, a normed complete vector lattice is called *Banach lattice*. A net $(x_\alpha)_{\alpha \in A}$ in a Banach lattice $E$ is *unbounded norm convergent* (or *un-convergent*) to $x \in E$ if $\| |x_\alpha - x| \wedge u \| \to 0$ for all $u \in E_+$ (cf. [8–10, 15]). We routinely use the following fact: $y \leq x$ implies $u \cdot y \leq u \cdot x$ for all positive elements $u$ in $l$-algebras. So, we can give the following notion.

**Definition 1.1.** An $l$-algebra $E$ which is at the same time a normed Riesz space is called a *normed $l$-algebra* whenever $\|x \cdot y\| \leq \|x\|.\|y\|$ holds for all $x, y \in E$.

Motivated by the above definitions, we give the following notion.

**Definition 1.2.** A net $(x_\alpha)_{\alpha \in A}$ in a normed $l$-algebra $E$ is said to be *multiplicative norm convergent* (or shortly, *mn-convergent*) to $x \in E$ if $\| |x_\alpha - x| \cdot u \| \to 0$ for all $u \in E_+$. Abbreviated as $x_\alpha \xrightarrow{\text{mn}} x$. If the condition holds only for sequences then it is called sequentially *mn*-convergence.

In this paper, we study only the *mn*- cases because the sequential cases are analogous in general.

**Remark 1.3.** (i) For a net $(x_\alpha)_{\alpha \in A}$ in a normed $l$-algebra $E$, $x_\alpha \xrightarrow{\text{mn}} x$ implies $x_\alpha \cdot y \xrightarrow{\text{mn}} x \cdot y$ for all $y \in E$ because of $\| |x_\alpha \cdot y - x \cdot y| \cdot u \| \leq \| |x_\alpha - x| \cdot |y| \cdot u \|$ for all $u \in E_+$; see for example [12, p.1]. The converse holds true in normed $l$-algebras with the multiplication unit. Indeed, assume $x_\alpha \cdot y \xrightarrow{\text{mn}} x \cdot y$ for each $y \in E$. Fix $u \in E_+$. So, $\| |x_\alpha - x| \cdot u \| = \| |x_\alpha \cdot e - x \cdot e| \cdot u \| \xrightarrow{\text{mn}} 0$.

(ii) In normed $l$-algebras, the norm convergence implies the $mn$-convergence. Indeed, by considering the inequality $\||x_\alpha - x| \cdot u\| \leq \|x_\alpha - x\|.\|u\|$ for any net $x_\alpha \xrightarrow{\text{mn}} x$, we can get the desired result.

(iii) If a net $(x_\alpha)_{\alpha\in A}$ is order Cauchy and $x_\alpha \xrightarrow{\text{mn}} x$ in a normed $l$-algebra then we have $x_\alpha \xrightarrow{\text{mo}} x$. Indeed, since the order Cauchy norm convergent net is order convergent to its norm limit, we can get the desired result.

(iv) In order continuous normed $l$-algebras, it is clear that the $mo$-convergence implies the $mn$-convergence.

(v) In order continuous normed $l$-algebras, following from the inequality $\||x_\alpha - x| \cdot u\| \leq \|x_\alpha - x\|.\|u\|$, the order convergence implies the $mn$-convergence.

(vi) In atomic and order continuous Banach lattice $l$-algebras, an order bounded and $mn$-convergent to zero sequence is sequentially $mo$-convergent to zero; see [9, Lem.5.1.].

(vii) For an $mn$-convergent to zero sequence $(x_n)$ in a Banach lattice $l$-algebra, there is a subsequence $(x_{n_k})$ which sequentially $mo$-converges to zero; see [11, Lem.3.11.].

**Example 1.4.** Let $E$ be a Banach lattice. Fix an element $x \in E$. Then the principal ideal $I_x = \{y \in E : \exists\lambda > 0 \text{ with } |y| \leq \lambda x\}$, generated by $x$ in $E$ under the norm $\|\cdot\|_\infty$ which is defined by $\|y\|_\infty = \inf\{\lambda > 0 : |y| \leq \lambda x\}$, is an $AM$-space; see [2, Thm.4.21.].

Recall that a vector $e > 0$ is called order unit whenever for each $x$ there exists some $\lambda > 0$ with $|x| \leq \lambda e$ (cf. [1, p.20]). Thus, we have $(I_x, \|\cdot\|_\infty)$ is $AM$-space with the unit $|x|$. Since every $AM$-space with the unit, besides being a Banach lattice, has also an $l$-algebra structure (cf. [2, p.259]). So, we can say that $(I_x, \|\cdot\|_\infty)$ is a Banach lattice $l$-algebra. Therefore, for a net $(x_\alpha)_{\alpha\in A}$ in $I_x$ and $y \in I_x$, by applying [2, Cor.4.4.], we get $x_\alpha \xrightarrow{\text{mn}} y$ in the original norm of $E$ on $I_x$ if and only if $x_\alpha \xrightarrow{\text{mn}} y$ in the norm $\|\cdot\|_\infty$. In particular, take $x$ as the unit element $e$ of $E$. Then we have $E_e = E$. Thus, for a net $(x_\alpha)_{\alpha\in A}$ in $E$, we have $x_\alpha \xrightarrow{\text{mn}} y$ in the $(E, \|\cdot\|_\infty)$ if and only if $x_\alpha \xrightarrow{\text{mn}} y$ in the $(E, \|\cdot\|)$.

## 2. The $mn$-convergence on normed $l$-algebras

We begin the section with the next list of properties of $mn$-convergence which follows directly from the inequalities $|x - y| \leq |x - x_\alpha| + |x_\alpha - y|$ and $||x_\alpha| - |x|| \leq |x_\alpha - x|$ for arbitrary net in $(x_\alpha)_{\alpha\in A}$ in vector lattice.

**Lemma 2.1.** *Let $(x_\alpha)_{\alpha\in A}$ and $(y_\beta)_{\beta\in B}$ be two nets in a normed $l$-algebra $E$. Then the followings hold:*

(i) $x_\alpha \xrightarrow{\text{mn}} x \iff (x_\alpha - x) \xrightarrow{\text{mn}} 0 \iff |x_\alpha - x| \xrightarrow{\text{mn}} 0$;

(ii) *if $x_\alpha \xrightarrow{\text{mn}} x$ then $y_\beta \xrightarrow{\text{mn}} x$ for each subnet $(y_\beta)$ of $(x_\alpha)$;*

(iii) *suppose $x_\alpha \xrightarrow{\text{mn}} x$ and $y_\beta \xrightarrow{\text{mn}} y$, then $ax_\alpha + by_\beta \xrightarrow{\text{mn}} ax + by$ for any $a, b \in \mathbb{R}$;*

(iv) *if $x_\alpha \xrightarrow{\text{mn}} x$ then $|x_\alpha| \xrightarrow{\text{mn}} |x|$.*

The lattice operations in normed $l$-algebras are $mn$-continuous in the following sense.

**Proposition 2.2.** *Let $(x_\alpha)_{\alpha\in A}$ and $(y_\beta)_{\beta\in B}$ be two nets in a normed $l$-algebra $E$. If $x_\alpha \xrightarrow{\text{mn}} x$ and $y_\beta \xrightarrow{\text{mn}} y$ then $(x_\alpha \vee y_\beta)_{(\alpha,\beta)\in A\times B} \xrightarrow{\text{mn}} x \vee y$.*

***Proof.*** Assume $x_\alpha \xrightarrow{\text{mn}} x$ and $y_\beta \xrightarrow{\text{mn}} y$. Then, for a given $\varepsilon > 0$, there exist indexes $\alpha_0 \in A$ and $\beta_0 \in B$ such that $\||x_\alpha - x| \cdot u\| \leq \frac{1}{2}\varepsilon$ and $\||y_\beta - y| \cdot u\| \leq \frac{1}{2}\varepsilon$ for every $u \in E_+$ and for all $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$. It follows from the inequality $|a \vee b - a \vee c| \leq |b - c|$ in vector lattices (cf. [2, Thm.1.9(2)]) that

$$\||x_\alpha \vee y_\beta - x \vee y| \cdot u\| \leq \||x_\alpha \vee y_\beta - x_\alpha \vee y| \cdot u + |x_\alpha \vee y - x \vee y| \cdot u\|$$
$$\leq \||y_\beta - y| \cdot u\| + \||x_\alpha - x| \cdot u\| \leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

for all $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$ and for every $u \in E_+$. That is, $(x_\alpha \vee y_\beta)_{(\alpha,\beta)\in A\times B} \xrightarrow{\text{mn}} x \vee y$. □

The following proposition is similar to [4, Prop.2.7.], and so we omit its proof.

**Proposition 2.3.** *Let $B$ be a projection band in a normed $l$-algebra $E$ and $P_B$ be the corresponding band projection. Then $x_\alpha \xrightarrow{\text{mn}} x$ in $E$ implies $P_B(x_\alpha) \xrightarrow{\text{mn}} P_B(x)$ in both $E$ and $B$.*

A positive vector $e$ in a normed vector lattice $E$ is called *quasi-interior point* if and only if $\|x - x \wedge ne\| \to 0$ for each $x \in E_+$. If $(x_\alpha)$ is a net in a vector lattice with a weak unit $e$ then $x_\alpha \xrightarrow{\text{uo}} 0$ if and only if $|x_\alpha| \wedge e \xrightarrow{\text{o}} 0$; see [10, Lem. 3.5]. Also, there exist some results for the quasi-interior point case in [9, Lem. 2.11] and for $p$-unit case in [5, Thm. 3.2]. We give an expansion to normed $l$-algebras with the $mn$-convergence for quasi-interior points in the next result.

**Proposition 2.4.** *Let $(x_\alpha)_{\alpha\in A}$ be a positive and decreasing net in a normed $l$-algebra $E$ with a quasi-interior point $e$. Then $x_\alpha \xrightarrow{\text{mn}} 0$ if and only if $(x_\alpha \cdot e)_{\alpha\in A}$ norm converges to zero.*

***Proof.*** The forward implication is immediate because of $e \in E_+$. For the converse implication, fix a positive vector $u \in E_+$ and $\varepsilon > 0$. Thus, for a fixed index $\alpha_1$, we have $x_\alpha \leq x_{\alpha_1}$ for all $\alpha \geq \alpha_0$ because of $(x_\alpha)_{\alpha\in A}\downarrow$. Then we have

$$x_\alpha \cdot u \leq x_\alpha \cdot (u - u \wedge ne) + x_\alpha \cdot (u \wedge ne) \leq x_{\alpha_1} \cdot (u - u \wedge ne) + n(x_\alpha \cdot e)$$

for all $\alpha \geq \alpha_1$ and each $n \in \mathbb{N}$. Hence, we get

$$\|x_\alpha \cdot u\| \leq \|x_{\alpha_1}\|.\|u - u \wedge ne\| + n\|x_\alpha \cdot e\|$$

for every $\alpha \geq \alpha_1$ and each $n \in \mathbb{N}$. So, we can find $n$ such that $\|u - u \wedge ne\| < \frac{\varepsilon}{2\|x_{\alpha_1}\|}$ because $e$ is a quasi-interior point. On the other hand, it follows from $x_\alpha \cdot e \xrightarrow{\|\cdot\|} 0$ that there exists an index $\alpha_2$ such that $\|x_\alpha \cdot e\| < \frac{\varepsilon}{2n}$ whenever $\alpha \geq \alpha_2$. Since index set $A$ is directed, there exists another index $\alpha_0 \in A$ such that $\alpha_0 \geq \alpha_1$ and $\alpha_0 \geq \alpha_2$. Therefore, we get

$$\|x_\alpha \cdot u\| < \|x_{\alpha_0}\| \frac{\varepsilon}{2\|x_{\alpha_0}\|} + n\frac{\varepsilon}{2n} = \varepsilon,$$

and so $\|x_\alpha \cdot u\| \to 0$. □

**Remark 2.5.** A positive and decreasing net $(x_\alpha)_{\alpha\in A}$ in an order continuous Banach $l$-algebra $E$ with weak unit $e$ is $mn$-convergent to zero if and only if $x_\alpha \cdot e \xrightarrow{\|\cdot\|} 0$. Indeed, it is known that $e$ is a weak unit if and only if $e$ is a quasi-interior point in an order continuous Banach lattice; see for example [1, p.135]. Thus, following from Proposition 2.4, one can get the desired result.

The $mn$-convergence passes obviously to any normed $l$-subalgebra $Y$ of a normed $l$-algebra $E$, i.e., for any net $(y_\alpha)_{\alpha\in A}$ in $Y$ with $y_\alpha \xrightarrow{\text{mn}} 0$ in $E$ implies $y_\alpha \xrightarrow{\text{mn}} 0$ in $Y$. For the converse, we give the following theorem whose proof is similar to [4, Thm. 2.10], and so we omit it.

**Theorem 2.6.** *Let $Y$ be a normed $l$-subalgebra of a normed $l$-algebra $E$ and $(y_\alpha)_{\alpha\in A}$ be a net in $Y$. If $y_\alpha \xrightarrow{\text{mn}} 0$ in $Y$ then it mn-converges to zero in $E$ for both of the following cases hold;*

(i) *$Y$ is majorizing in $E$;*
(ii) *$Y$ is a projection band in $E$.*

It is known that every Archimedean vector lattice has a unique order completion; see [2, Thm. 2.24]. Moreover, Archimedean commutative $l$-algebra admits the unique extension multiplication to the order completion of it.

**Theorem 2.7.** *Let $E$ and $E^\delta$ be order continuous normed $l$-algebras with $E^\delta$ being order completion of $E$. Then, for a sequence $(x_n)$ in $E$, the followings hold true:*

(i) *If $x_n \xrightarrow{\text{mn}} 0$ in $E$ then there is a subsequence $(x_{n_k})$ of $(x_n)$ such that $x_{n_k} \xrightarrow{\text{mn}} 0$ in $E^\delta$;*

(ii) *If $x_n \xrightarrow{\text{mn}} 0$ in $E^\delta$ then there is a subsequence $(x_{n_k})$ of $(x_n)$ such that $x_{n_k} \xrightarrow{\text{mn}} 0$ in $E$.*

***Proof.*** Let $x_n \xrightarrow{\text{mn}} 0$ in $E$, i.e., $|x_n| \cdot u \xrightarrow{\|\cdot\|} 0$ in $E$ for all $u \in E_+$. Now, let's fix $v \in E^\delta_+$. Then there exists $u_v \in E_+$ such that $v \le u_v$ because $E$ majorizes $E^\delta$. Since $|x_n| \cdot u_v \xrightarrow{\|\cdot\|} 0$, by the standard fact in [1, Exer.13., p.25], there exists a subsequence $(x_{n_k})$ of $(x_n)$ such that $(|x_{n_k}| \cdot u_v)$ order converges to zero in $E$. Thus, we get $|x_{n_k}| \cdot u_v \xrightarrow{\text{o}} 0$ in $E^\delta$; see [10, Cor.2.9.]. Then it follows from the inequality $|x_{n_k}| \cdot v \le |x_{n_k}| \cdot u_v$ that we have $|x_{n_k}| \cdot v \xrightarrow{\text{o}} 0$ in $E^\delta$. That is, $x_{n_k} \xrightarrow{\text{mo}} 0$ in the order completion $E^\delta$ because $v \in E^\delta_+$ is arbitrary. It follows from the order continuous norm that $x_{n_k} \xrightarrow{\text{mn}} 0$ in the order completion $E^\delta$.

For the converse, put $x_n \xrightarrow{\text{mn}} 0$ in $E^\delta$. Then, for all $u \in E^\delta_+$, we have $|x_n| \cdot u \xrightarrow{\|\cdot\|} 0$ in $E^\delta$. In particular, for all $w \in E_+$, $\||x_n| \cdot w\| \to 0$ in $E^\delta$. Fix $w \in E_+$. Then, again by the standard fact in [1, Exer.13., p.25], we have a subsequence $(x_{n_k})$ of $(x_n)$ such that $(x_{n_k})$ is order convergent to zero in $E^\delta$. Thus, we get $|x_{n_k}| \cdot w \xrightarrow{\text{o}} 0$ in $E$. As a result, since $w$ is arbitrary, $x_{n_k} \xrightarrow{\text{mo}} 0$ in $E$. Therefore, one can get the result by using order continuous norm. □

Recall that a subset $A$ in a normed lattice $(E, |\cdot\|)$ is said to *almost order bounded* if, for any $\epsilon > 0$, there is $u_\epsilon \in E_+$ such that $\|(|x| - u_\epsilon)^+\| = \||x| - u_\epsilon \wedge |x|\| \le \epsilon$ for any $x \in A$. For a given normed $l$-algebra $E$, one can give the following definition: a subset $A$ of $E$ is called an *$l$-almost order bounded* if, for any $\epsilon > 0$, there is $u_\varepsilon \in E_+$ such that $\||x| - u_\epsilon \cdot |x|\| \le \epsilon$ for any $x \in A$. Similar to [11, Prop.3.7.], we give the following work.

**Proposition 2.8.** *Let $E$ be a normed $l$-algebra. If $(x_\alpha)_{\alpha\in A}$ is $l$-almost order bounded and $mn$-converges to $x$, then $(x_\alpha)_{\alpha\in A}$ converges to $x$ in norm.*

***Proof.*** Assume $(x_\alpha)_{\alpha\in A}$ is an $l$-almost order bounded net. Then the net $(|x_\alpha - x|)_{\alpha\in A}$ is also $l$-almost order bounded. For any fixed $\varepsilon > 0$, there exists $u_\varepsilon > 0$ such that

$$\||x_\alpha - x| - u_\epsilon \cdot |x_\alpha - x|\| \le \epsilon.$$

Since $x_\alpha \xrightarrow{\text{mn}} x$, we have $\||x_\alpha - x| \cdot u_\varepsilon\| \to 0$. Therefore, following from Proposition 2.2, we get $\|x_\alpha - x\| \le \varepsilon$, i.e., $x_\alpha \to x$ in the norm. □

**Proposition 2.9.** *In an order continuous Banach $l$-algebra, every $l$-almost order bounded mo-Cauchy net converges mn and in norm to the same limit.*

***Proof.*** Assume a net $(x_\alpha)_{\alpha\in A}$ is $l$-almost order bounded and $mo$-Cauchy in an order continuous Banach $l$-algebra $E$. Then the net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha')\in A\times A}$ is $l$-almost order bounded and is $mo$-convergent to zero. Thus, it $mn$-converges to zero by the order continuity of the norm. Hence, by applying Proposition 2.8, we get that the net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha')\in A\times A}$ converges to zero in the norm. It follows that the net $(x_\alpha)$ is norm Cauchy, and so it is norm convergent because $E$ is Banach lattice. As a result, we have that $(x_\alpha)$ $mn$-converges to its norm limit by Remark 1.3($ii$). □

The multiplication in normed $l$-algebra is $mn$-continuous in the following sense.

**Theorem 2.10.** *Let $E$ be a normed $l$-algebra, and $(x_\alpha)_{\alpha\in A}$ and $(y_\beta)_{\beta\in B}$ be two nets in $E$. If $x_\alpha \xrightarrow{\text{mn}} x$ and $y_\beta \xrightarrow{\text{mn}} y$ for some $x, y \in E$ and each positive element of $E$ can be written as a multiplication of two positive elements then we have $x_\alpha \cdot y_\beta \xrightarrow{\text{mn}} x \cdot y$.*

***Proof.*** Assume $x_\alpha \xrightarrow{\text{mn}} x$ and $y_\beta \xrightarrow{\text{mn}} y$. Then $|x_\alpha - x| \cdot u \xrightarrow{\|\cdot\|} 0$ and $|y_\beta - y| \cdot u \xrightarrow{\|\cdot\|} 0$ for every $u \in E_+$. Let's fix $u \in E_+$ and $\varepsilon > 0$. So, there exist indexes $\alpha_0$ and $\beta_0$ such that $\||x_\alpha - x| \cdot u\| \leq \varepsilon$ and $\||y_\beta - y| \cdot u\| \leq \varepsilon$ for all $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$.

Next, we show the $mn$-convergence of $(x_\alpha \cdot y_\beta)$ to $x \cdot y$. By considering the equality $|x \cdot y| \leq |x| \cdot |y|$ (cf. [12, p.1]), we have

$$\begin{aligned} \||x_\alpha \cdot y_\beta - x \cdot y|u\| &= \||x_\alpha \cdot y_\beta - x_\alpha \cdot y + x_\alpha \cdot y - x \cdot y| \cdot u\| \\ &\leq \||x_\alpha| \cdot |y_\beta - y| \cdot u\| + \||x_\alpha - x| \cdot |y| \cdot u\| \\ &\leq \||x_\alpha - x| \cdot |y_\beta - y| \cdot u\| + \||y_\beta - y| \cdot |x| \cdot u\| + \||x_\alpha - x| \cdot |y| \cdot u\|. \end{aligned}$$

The second and the third terms in the last inequality both order converge to zero as $\beta \to \infty$ and $\alpha \to \infty$ respectively because of $|x| \cdot u, |y| \cdot u \in E_+$ and $x_\alpha \xrightarrow{\text{mn}} x$ and $y_\beta \xrightarrow{\text{mn}} y$. Now, let's show the $mn$-convergence of the first term of last inequality. For fixed $u$, we can find two positive elements $u_1, u_2 \in E_+$ such that $u = u_1 \cdot u_2$ because the positive element of $E$ can be written as a multiplication of two positive elements. So, we can get

$$\||x_\alpha - x| \cdot |y_\beta - y| \cdot u\| = \|(|x_\alpha - x| \cdot u_1) \cdot (|y_\beta - y| \cdot u_2)\| \leq \||x_\alpha - x| \cdot u_1\|.\||y_\beta - y| \cdot u_2\|.$$

Therefore, we see $|x_\alpha - x| \cdot |y_\beta - y| \cdot u \xrightarrow{\|\cdot\|} 0$. Hence, we get $x_\alpha \cdot y_\beta \xrightarrow{\text{mn}} x \cdot y$. □

In Theorem 2.10, the case of each positive element of $E$ can be written as a multiplication of two positive elements is called *the factorization property* for $f$-algebras in [13, Def.12.10]. But, instead of that property, we can give another easy condition in the following result.

**Corollary 2.11.** *Let $E$ be a normed $l$-algebra, and $(x_\alpha)_{\alpha\in A}$ and $(y_\beta)_{\beta\in B}$ be two nets in $E$. If $x_\alpha \xrightarrow{\text{mn}} x$ and $y_\beta \xrightarrow{\text{mn}} y$ for some $x, y \in E$ and at least one of two nets is eventually norm bounded then we have $x_\alpha \cdot y_\beta \xrightarrow{\text{mn}} x \cdot y$.*

***Proof.*** Modify Theorem 2.10. □

We give some basic notions motivated by their analogies from vector lattice theory.

**Definition 2.12.** Let $(x_\alpha)_{\alpha\in A}$ be a net in a normed $l$-algebra $E$. Then

(1) $(x_\alpha)$ is said to be *mn-Cauchy* if the net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha')\in A\times A}$ $mn$-converges to 0,
(2) $E$ is called *mn-complete* if every $mn$-Cauchy net in $E$ is $mn$-convergent,
(3) $E$ is called *mn-continuous* if $x_\alpha \xrightarrow{o} 0$ implies that $x_\alpha \xrightarrow{\text{mn}} 0$,

**Proposition 2.13.** *A normed $l$-algebra is mn-continuous if and only if $x_\alpha \downarrow 0$ implies $x_\alpha \xrightarrow{\text{mn}} 0$.*

***Proof.*** Suppose any decreasing to zero net is $mn$-convergent to zero. We show $mn$-continuity. Let $(x_\alpha)_{\alpha\in A}$ be an order convergent to zero net in a normed $l$-algebra $E$. Then there exists another net $z_\beta \downarrow 0$ in $E$ such that, for any $\beta$ there exists $\alpha_\beta$ so that $|x_\alpha| \leq z_\beta$, and so $\|x_\alpha\| \leq \|z_\beta\|$ for all $\alpha \geq \alpha_\beta$. Since $z_\beta \downarrow 0$, by assumption, we have $z_\beta \xrightarrow{\text{mn}} 0$, i.e., for fixed $\varepsilon > 0$ and $u \in E_+$, there is $\beta_0$ such that $\|z_\beta \cdot u\| < \varepsilon$ for all $\beta \geq \beta_0$. Thus, there exists an index $\alpha_{\beta_0}$ so that $\||x_\alpha| \cdot u\| \leq \varepsilon$ for all $\alpha \geq \alpha_{\beta_0}$. Hence, $x_\alpha \xrightarrow{\text{mn}} 0$. The other case is obvious. □

**Proposition 2.14.** *Let $E$ be an mn-continuous and mn-complete normed $l$-algebra. Then every $l$-almost order bounded and order Cauchy net is mn-convergent.*

***Proof.*** Let $(x_\alpha)_{\alpha\in A}$ be an $l$-almost order bounded order Cauchy net. Then the net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha')\in A\times A}$ is $l$-almost order bounded and is order convergent to zero. Since $E$ is $mn$-continuous, $x_\alpha - x_{\alpha'} \xrightarrow{\text{mn}} 0$. By using Proposition 2.8, we have $x_\alpha - x_{\alpha'} \xrightarrow{\|\cdot\|} 0$. Hence, we get that $(x_\alpha)_{\alpha\in A}$ is $mn$-Cauchy, and so it is $mn$-convergent because of $mn$-completeness. □

## 3. The $mn$-topology on normed $l$-algebra

In this section, we now turn our attention to topology on normed $l$-algebras. We show that the $mn$-convergence in a normed $l$-algebra is topological. While $mo$- and $uo$-convergence need not be given by a topology. But, it was observed in [9] that the $un$-convergence is topological. Motivated from that definition of the $mn$-convergence, we give the following construction of the $mn$-topology.

Let $\varepsilon > 0$ be given. For a non-zero positive vector $u \in E_+$, we put

$$V_{u,\varepsilon} = \{x \in E : \||x| \cdot u\| < \varepsilon\}.$$

Let $\mathcal{N}$ be the collection of all the sets of this form. We claim that $\mathcal{N}$ is a base of neighborhoods of zero for some Hausdorff linear topology. It is obvious that $x_\alpha \xrightarrow{\text{mn}} 0$ if and only if every set of $\mathcal{N}$ contains a tail of this net, hence the $mn$-convergence is the convergence induced by the mentioned topology.

We have to show that $\mathcal{N}$ is a base of neighborhoods of zero. To show this we apply [14, Thm.3.1.10.]. First, note that every element in $\mathcal{N}$ contains zero. Now, we show that for every two elements of $\mathcal{N}$, their intersection is again in $\mathcal{N}$. Take any two set $V_{u_1,\varepsilon_1}$ and $V_{u_2,\varepsilon_2}$ in $\mathcal{N}$. Put $\varepsilon = \varepsilon_1 \wedge \varepsilon_2$ and $u = u_1 \vee u_2$. We show that $V_{u,\varepsilon} \subseteq V_{u_1,\varepsilon_1} \cap V_{u_2,\varepsilon_2}$. For any $x \in V_{u,\varepsilon}$, we have $\||x| \cdot u\| < \varepsilon$. Thus, it follows from $|x| \cdot u_1 \le |x| \cdot u$ that

$$\||x| \cdot u_1\| \le \||x| \cdot u\| < \varepsilon \le \varepsilon_1.$$

Thus, we get $x \in V_{u_1,\varepsilon_1}$. By a similar way, we also have $x \in V_{u_2,\varepsilon_2}$.

Next, it is not a hard job to see that $V_{u,\varepsilon} + V_{u,\varepsilon} \subseteq V_{u,2\varepsilon}$, so that for each $U \in \mathcal{N}$, there is another $V \in \mathcal{N}$ such that $V + V \subseteq U$. In addition, one can easily verify that, for every $U \in \mathcal{N}$ and every scalar $\lambda$ with $|\lambda| \le 1$, we have $\lambda U \subseteq U$.

Now, we show that, for each $U \in \mathcal{N}$ and each $y \in U$, there exists $V \in \mathcal{N}$ with $y + V \subseteq U$. Suppose $y \in V_{u,\varepsilon}$. We should find $\delta > 0$ and a non-zero $v \in E_+$ such that $y + V_{v,\delta} \subseteq V_{u,\varepsilon}$. Take $v := u$. Hence, since $y \in V_{u,\varepsilon}$, we have $\||y| \cdot u\| < \varepsilon$. Put $\delta := \varepsilon - \||y| \cdot u\|$. We claim that $y + V_{v,\delta} \subseteq V_{u,\varepsilon}$. Let's take $x \in V_{v,\delta}$. We show that $y + x \in V_{u,\varepsilon}$. Consider the inequality $|y + x| \cdot u \le |y| \cdot u + |x| \cdot u$. Then we have

$$\||y + x| \cdot u\| \le \||y| \cdot u\| + \||x| \cdot u\| < \||y| \cdot u\| + \delta = \varepsilon.$$

Finally, we show that this topology is Hausdorff. It is enough to show that $\bigcap \mathcal{N} = \{0\}$. Suppose that it is not hold true, i.e., assume that $0 \ne x \in V_{u,\varepsilon}$ for all non-zero $u \in E_+$ and for all $\varepsilon > 0$. In particular, take $x \in V_{|x|,\varepsilon}$. Thus, we have $\||x|^2\| < \varepsilon$. Since $\varepsilon$ is arbitrary, we get $|x|^2 = 0$, i.e., $x = 0$ by using [17, Thm.142.3.]; a contradiction.

Recall that the statement $V_{u,\varepsilon}$ is either contained in $[-u, u]$ or contains a non-trivial ideal holds true for the $un$-topology. However, it is not true for the $mn$-topology. To see this, we give the following counterexample.

**Example 3.1.** Consider the $l$-algebra $E = C[0, 1]$ with the sup-norm topology $\tau$. Take $a = \mathbb{1}$ and $A = B(0, 10)$. The set $U_{a,A} = \{x \in E : |x| \cdot a \in A\} = B(0, 10)$ is neither contained in $[-a, a] = [-\mathbb{1}, \mathbb{1}] = B(0, 1)$ nor contains a non-trivial ideal.

**Lemma 3.2.** *If $V_{u,\varepsilon}$ is contained in $[-u, u]$, then $u$ is a strong unit.*

***Proof.*** Take a positive element $x \in E_+$. Then we have a positive scalar $\lambda$ such that $(\lambda x) \cdot a \in A$. Thus we get $\lambda x \in U_{a,A}$ and so, $\lambda x \in [-a, a]$. Then one can see that $a$ is a strong unit. □

## 4. The $mn$-convergence on semiprime normed $f$-algebras

Recall that an element $x$ in an $f$-algebra $E$ is called *nilpotent* whenever $x^n = 0$ for some natural number $n \in \mathbb{N}$. The algebra $E$ is called *semiprime* if the only nilpotent element in $E$ is the null element ([17, p.670]). We begin the section with the next useful result.

**Proposition 4.1.** *Let $(x_\alpha)_{\alpha\in A}$ be a net in nilpotent elements of a normed $f$-algebra $E$. If $x_\alpha \xrightarrow{\text{mn}} x$ then $x$ is also a nilpotent element.*

***Proof.*** Take a fixed positive element $u \in E_+$. Then, by using [13, Prop.10.2(iii)] and [17, Thm.142.1(ii)], we get

$$\| |x_\alpha - x| \cdot u \| = \| |x_\alpha \cdot u - x \cdot u| \| = \| x_\alpha \cdot u - x \cdot u \| = \| x \cdot u \| \to 0.$$

Thus $\|x \cdot u\| = 0$ and hence $x \cdot u = 0$ for every $u \in X_+$. Then $y \cdot x = 0$ for all $y \in E$. It follows now from [12, p.157] that $x$ is nilpotent in $E$. □

**Remark 4.2.** By considering Proposition 4.1, it is easy to see that $mn$-convergence in normed $f$-algebra $E$ has an unique limit if and only if $E$ is semiprime normed $f$-algebra.

Unless stated otherwise, we will assume that $E$ is a semiprime normed $f$-algebra and all nets and vectors lie in $E$.

**Proposition 4.3.** *Let $(x_\alpha)_{\alpha\in A}$ be a net in $E$. Then we have that*

(i) $0 \le x_\alpha \xrightarrow{\text{mn}} x$ *implies* $x \in E_+$,

(ii) *if* $(x_\alpha)$ *is monotone and* $x_\alpha \xrightarrow{\text{mn}} x$ *then* $x_\alpha \xrightarrow{\text{o}} x$.

***Proof.*** (i) Assume $(x_\alpha)_{\alpha\in A}$ consists of non-zero elements and $mn$-converges to $x \in E$. Then, by using Proposition 2.2, we have $x_\alpha = x_\alpha^+ \xrightarrow{\text{mn}} x^+$. Also, following from Remark 4.2, we get $x^+ = x$. Therefore, we get $x \in E_+$.

(ii) For the order convergence of $(x_\alpha)_{\alpha\in A}$, it is enough to show that $x_\alpha \uparrow$ and $x_\alpha \xrightarrow{\text{mn}} x$ implies $x_\alpha \xrightarrow{\text{o}} x$. For a fixed index $\alpha$, we have $x_\beta - x_\alpha \in X_+$ for all $\beta \ge \alpha$. By applying (i), we can see $x_\beta - x_\alpha \xrightarrow{\text{mn}} x - x_\alpha \in X_+$ as $\beta \to \infty$. Therefore, $x \ge x_\alpha$ for the index $\alpha$. Since $\alpha$ is arbitrary, $x$ is an upper bound of $(x_\alpha)$. Assume $y$ is another upper bound of $(x_\alpha)$, i.e., $y \ge x_\alpha$ for all $\alpha$. So, $y - x_\alpha \xrightarrow{\text{mn}} y - x \in X_+$, or $y \ge x$, and so $x_\alpha \uparrow x$. □

**Theorem 4.4.** *The following statements are equivalent:*

(i) *$E$ is mn-continuous;*

(ii) *if* $0 \le x_\alpha \uparrow \le x$ *holds in $E$ then* $(x_\alpha)$ *is an mn-Cauchy net;*

(iii) $x_\alpha \downarrow 0$ *implies* $x_\alpha \xrightarrow{\text{mn}} 0$ *in $E$.*

***Proof.*** (i)⇒(ii) Take a net $0 \le x_\alpha \uparrow \le x$ in $E$. Then there exists another net $(y_\beta)$ in $E$ such that $(y_\beta - x_\alpha)_{\alpha,\beta} \downarrow 0$; see [2, Lem.4.8]. Thus, by applying Proposition 2.13, we have $(y_\beta - x_\alpha)_{\alpha,\beta} \xrightarrow{\text{mn}} 0$ because $E$ is $mn$-continuous. Therefore, the net $(x_\alpha)$ is $mn$-Cauchy because of $\|x_\alpha - x_{\alpha'}\|_{\alpha,\alpha'\in A} \le \|x_\alpha - y_\beta\| + \|y_\beta - x_{\alpha'}\|$.

(ii)⇒(iii) Put $x_\alpha \downarrow 0$ in $E$ and fix arbitrary $\alpha_0$. Thus, we have $x_\alpha \le x_{\alpha_0}$ for all $\alpha \ge \alpha_0$, and so we can get $0 \le (x_{\alpha_0} - x_\alpha)_{\alpha\ge\alpha_0} \uparrow \le x_{\alpha_0}$. Then it follows from $(ii)$ that the net $(x_{\alpha_0} - x_\alpha)_{\alpha\ge\alpha_0}$ is $mn$-Cauchy, i.e., $(x_{\alpha'} - x_\alpha) \xrightarrow{\text{mn}} 0$ as $\alpha_0 \le \alpha, \alpha' \to \infty$. Since $E$ is $mn$-complete, there exists an element $x \in E$ satisfying $x_\alpha \xrightarrow{\text{mo}} x$ as $\alpha_0 \le \alpha \to \infty$. It follows from Proposition 4.3 that $x_\alpha \downarrow 0$ because of $x_\alpha \downarrow$ and $x_\alpha \xrightarrow{\text{mn}} 0$, and so, following from Remark 4.2 tha we have $x = 0$. Therefore, we get $x_\alpha \xrightarrow{\text{mn}} 0$.

(iii)⇒(i) It is just the implication of Proposition 2.13. □

**Corollary 4.5.** *Every mn-continuous and mn-complete normed f-algebra E is order complete.*

***Proof.*** Suppose $E$ is $mn$-continuous and $mn$-complete. For $y \in E_+$, put a net $0 \leq x_\alpha \uparrow\leq y$ in $E$. By applying Theorem 4.4 $(ii)$, the net $(x_\alpha)$ is $mn$-Cauchy. Thus, there exists an element $x \in E$ such that $x_\alpha \xrightarrow{\text{mn}} x$ because of $mn$-completeness. Since $x_\alpha \uparrow$ and $x_\alpha \xrightarrow{\text{mo}} x$, it follows from Lemma 4.3 that $x_\alpha \uparrow x$. Therefore, $E$ is order complete. □

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