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Research Article

The nil-clean 2×2 integral units

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Abstract

We prove that all trace 1, 2×2 invertible matrices over $\mathbb{Z}$ are nil-clean and, up to similarity, that there are only two trace 1, 2×2 invertible matrices over $\mathbb{Z}$.

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1. Introduction

We first recall the following.

An element a in a unital ring R is *clean* (see [5]) if $a = e + u$ with an idempotent $e \in R$ and a unit $u \in R$, and, *nil-clean* (see [4]) if $a = e + t$ with an idempotent e and a nilpotent t. It is *strongly* nil-clean if $et = te$. A nil-clean element is called *trivial* if $e \in \{0, 1\}$, the trivial idempotents. A unit u is called *unipotent* if $u = 1 + t$, for some nilpotent t.

A ring is *clean* (or *nil-clean*) if so are all its elements. Via unipotent units, it is easy to see that nil-clean rings are clean.

Though all these notions are well-known for some time, very little is known about *which clean elements of a ring are nil-clean*. Actually, besides the unipotent units (indeed, *a unit is strongly nil-clean if and only if it is unipotent*), we do not know which units of a ring are nil-clean.

We can discard the *trivial nil-clean* elements. Indeed, if $e = 0$, then there is no unit which is nilpotent (unless $R = 0$), and if $e = 1$, $a = 1 + t$, are precisely the unipotent units. Over any *commutative domain*, such 2×2 matrices M, are easily characterized by $\det(M - I_2) = \mathrm{Tr}(M - I_2) = 0$.

In this note, using an adequate (but nontrivial) Number Theory machinery, we characterize the (nontrivial) nil-clean units in the matrix ring $\mathcal{M}_2(\mathbb{Z})$.

Notice that *non-trivial* nil-clean 2×2 matrices over any commutative domain have trace 1.

As our main result, conversely, we show that *trace* 1*,* 2×2 *units over* $\mathbb{Z}$ *are nil-clean*, that is, *a* 2×2 *unit over* $\mathbb{Z}$ *is non-trivial nil-clean if and only if it has trace 1.*

Up to similarity, we also prove that all trace 1, 2×2 units are similar to $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ or to $\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$.

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2. Binary quadratic forms preliminaries

The proof of our main result requires some preparation.
First consider a particular Diophantine equation, namely

$$(x+y)^2 + xy = m \qquad (*)$$

where m is a positive integer.

Lemma 2.1. *For any divisor m of a positive integer $A(A+1)-1$, $A>1$, the equation (*) is solvable.*

Proof. From the general theory of quadratic binary forms, we know that the integer m is represented by a binary quadratic form of discriminant d only if the congruence $u^2 \equiv d(\mathrm{mod} 4k)$ is solvable, where k is the square-free part of m (see [2], Theorem 7, p. 145). In our case, i.e. for the form $G(x,y) = (x+y)^2 + xy$, $d = 5$ and the class number of $\mathbb{Q}[\sqrt{5}]$ is 1, hence the above condition becomes necessary and sufficient. The solvability of the congruence $u^2 \equiv 5(\mathrm{mod} 4k)$ is equivalent to the property that all prime factors of form $5s+2$ or $5s+3$ from the factorization of m have even exponent.

Since we have to solve this equation for a divisor m of $A(A+1)-1$, this reduces to show that if m divides $A(A+1)-1$, then m has this property. But this holds because if a prime p divides $A(A+1)-1$, then it also divides $(2A+1)^2 - 5 = 4[A(A+1)-1]$, so 5 must be a quadratic residue modulo p.

On the other hand, denoting by $\left(\frac{a}{p}\right)$ the Legendre symbol, according to the Gauss reciprocity law (see [1], Theorem 9.1.3), $\left(\frac{5}{p}\right)\left(\frac{p}{5}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{5-1}{2}} = 1$. Because $\left(\frac{5}{p}\right) = 1$, it follows $\left(\frac{p}{5}\right) = 1$ and so p is a quadratic residue modulo 5, i.e., p is congruent to $0, 1$ or 4 modulo 5, as desired. □

Next, we consider another particular Diophantine equation, namely

$$(x-y)^2 + xy = m \qquad (**)$$

where m is a positive integer.

Lemma 2.2. *For any divisor m of a positive integer $A(A+1)+1$, $A>1$, the equation (**) is solvable.*

Proof. The proof is similar to the proof of the previous lemma. Just notice that now the discriminant is -3 and the corresponding class number is also 1. Moreover, if a prime p divides $A(A+1)+1$, then it also divides $(2A+1)^2 + 3 = 4[A(A+1)+1]$, -3 must be a quadratic residue modulo p and so on. □

Secondly, we need the following

Proposition 2.3. *Suppose $A(A+1)+BC = 1$ for integers $A, B, -C > 1$. We can always chose solutions (b,d) and (a,c) of the equation (*) with $m = B$ and $m = -C$, respectively, such that $ad - bc = 1$.*

Proof. Again we use the theory of binary quadratic forms.

Consider the quadratic form $F(x,y) = Bx^2 + (2A+1)xy - Cy^2$.

Its discriminant is equal to $(2A+1)^2 + 4BC = 5$ (by our hypothesis). Using the reduction theory of quadratic forms, since the class number of $\mathbb{Q}[\sqrt{5}]$ is 1, it is well-known that (see [3]) all integer quadratic forms with discriminant 5 are $SL(2,\mathbb{Z})$-equivalent to

$G(x,y)=(x+y)^2+xy$, which has also discriminant 5. The equivalence means that there exist integers a,b,c,d with $ad-bc=1$ such that $G(ax+by,cx+dy)=F(x,y)$.

If we set $x=1,y=0$ we get $G(a,c)=B$ and if we set $x=0,y=1$ we get $G(b,d)=-C$ and we are done. □

Proposition 2.4. *Suppose $A(A+1)+BC=-1$ for integers $A,B,-C>1$. We can always chose solutions (b,d) and (a,c) of the equation (**) with $m=B$ and $m=-C$, respectively, such that $ad-bc=1$.*

Proof. We consider again the quadratic form $F(x,y)=Bx^2+(2A+1)xy-Cy^2$. Its discriminant is $(2A+1)^2+4BC=-3$ and so is the discriminant of $G(x,y)=(x-y)^2+xy$. Since the corresponding class number is 1, these are $SL(2,\mathbb{Z})$-equivalent, there exist integers a,b,c,d with $ad-bc=1$ such that $G(ax+by,cx+dy)=F(x,y)$ and we complete the proof as for the previous proposition. □

3. The main result

By E_{11} we denote the matrix with all entries zero, excepting the NW corner, which is 1. Recall that *over any principal ideal domain, every non-trivial 2×2 idempotent matrix is similar to E_{11}*. The result holds also in a more general setting (see [6]), but this hypothesis suffices for our proof below.

We first give a characterization, up to similarity, of the non-trivial nil-clean units in $\mathcal{M}_2(\mathbb{Z})$.

Proposition 3.1. *An integral 2×2 matrix U is a non-trivial nil-clean unit iff it is similar to one of the following two matrices: $V_1=\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$, $V_{-1}=\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$. More precisely, if $\det U=1$, it is similar to V_1 and if $\det U=-1$, it is similar to V_{-1}.*

Proof. Since nil-clean and unit are invariant (properties) to conjugation, up to similarity, owing to the previous paragraph, we can suppose the idempotent in the nil-clean decomposition being E_{11}. Nilpotent matrices having zero trace and zero determinant, we deal with (nil-clean) matrices $M=\begin{bmatrix} a+1 & b \\ c & -a \end{bmatrix}$ such that $a^2+bc=0$. Since $\det M=-(a+1)a-bc=-a\in\{\pm 1\}$ we distinguish two cases.

Case 1. If $a=-1$ then $bc=-1$ which give two matrices: $V_1=E_{11}+\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ and transpose (which is similar to V_1: just conjugate by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$).

Case 2. If $a=1$ then $bc=-1$ which give two matrices: $V_{-1}=E_{11}+\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ and transpose (which is similar to V_{-1}: the same conjugation). □

Example. $A=\begin{bmatrix} 8 & 5 \\ -11 & -7 \end{bmatrix}=\begin{bmatrix} 9 & 6 \\ -12 & -8 \end{bmatrix}+\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$. Here $U=\begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}$ and $U^{-1}AU=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}U=V_{-1}$, as stated.

Just taking the conjugates of these two matrices we can find the form of all the non-trivial nil-clean units in $\mathcal{M}_2(\mathbb{Z})$. This is

$$\begin{bmatrix} (a+c)(b+d)+ad & (b+d)^2+bd \\ -(a+c)^2-ac & -(a+c)(b+d)-bc \end{bmatrix}$$

for integers a,b,c,d with $ad-bc=1$.

Theorem 3.2. *Trace 1, 2×2 units over $\mathbb{Z}$ are nil-clean.*

Proof. In the sequel $M=\begin{bmatrix} A+1 & B \\ C & -A \end{bmatrix}$ denotes a trace 1, 2×2 integral matrix.

We first discuss the $\det M=-1$ case (i.e. $A(A+1)+BC=1$) and (owing to the form of the non-trivial nil-clean units deduced above) prove that there are integers a,b,c,d with $ad-bc=1$ such that

$$M=\begin{bmatrix} (a+c)(b+d)+ad & (b+d)^2+bd \\ -(a+c)^2-ac & -(a+c)(b+d)-bc \end{bmatrix}.$$

Finding the integers a,b,c,d amounts to solve the system

(i) $A=(a+c)(b+d)+bc$
(ii) $B=(b+d)^2+bd$
(iii) $C=-(a+c)^2-ac$
(iv) $1=ad-bc$, with integer unknowns a,b,c,d.

First notice that $A(A+1)-1>0$ with only two (integer) exceptions: $A=-1$ and $A=0$. The case $A=0$ reduces to $A=-1$, by conjugation with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the case $A=-1$ was already settled as Case 1, Proposition 3.1.

Hence we can assume $BC<0$ and even $B>0$, $C<0$ (otherwise we conjugate with $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$), together with $A\geq 1$ (the case $A\leq -2$ also reduces to $A\geq 1$, by conjugation with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$).

Secondly observe that (ii) and (iii) are equations of type $(x+y)^2+xy=m$, that is (*).

According to Proposition 2.3, the equations (ii), (iii) and (iv) have an integer solution.

Finally, we show that *any* solution of (ii), (iii) and (iv) (denoted again by a,b,c,d) also verifies (i) and we are done.

Indeed, $-BC=[(b+d)^2+bd][(a+c)^2+ac]=(b+d)^2(a+c)^2+ac(b+d)^2+bd(a+c)^2+abcd$ and so we have to check whether the degree 2 equation $A(A+1)=1+(b+d)^2(a+c)^2+ac(b+d)^2+bd(a+c)^2+abcd$ has $A=(a+c)(b+d)+bc$ as one root, i.e.

$$(b+d)^2(a+c)^2+bc(bc+1)+(2bc+1)(a+c)(b+d)=1+(b+d)^2(a+c)^2+ac(b+d)^2+bd(a+c)^2+abcd.$$

Equivalently $bc(bc+1-ad)+(2bc+1)(ab+ad+bc+cd)=1+ab^2c+acd^2+a^2bd+bc^2d+4abcd$ or else $(bc+1-ad)(ab+cd+3bc-1)=0$. This holds since $ad-bc=1$.

Next, we settle the $\det M=1$ case (i.e. $A(A+1)+BC=-1$) and prove that there are integers a,b,c,d with $ad-bc=1$ such that

$$M=\begin{bmatrix} (a-c)(b-d)+ad & (b-d)^2+bd \\ -(a-c)^2-ac & -(a-c)(b-d)-bc \end{bmatrix}.$$

Finding the integers a,b,c,d amounts to solve the system

(i) $A=(a-c)(b-d)+bc$
(ii) $B=(b-d)^2+bd$
(iii) $C=-(a-c)^2-ac$
(iv) $1=ad-bc$, with integer unknowns a,b,c,d.

Therefore now we deal with the equation (**). What remains for the proof is now deduced from Proposition 2.4 and a similar verification that any solution of (ii), (iii) and (iv) actually satisfies also (i). □

In closing we mention that this result fails for higher dimensions of matrices. Here is a 3×3 example:

take $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix}$ and $V = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, both with trace=determinant=1. Then $\mathrm{Tr}(U^2) = -1 \neq 1 = \mathrm{Tr}(V^2)$ and so the matrices U, V have different characteristic polynomials. Consequently, U and V are not similar.

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