



New Wilker-type and Huygens-type inequalities

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Abstract

In this paper, we first determine the relationships between the first Wilker's inequality, the second Wilker's inequality, the first Huygens inequality, and the second Huygens inequality for circular functions and for hyperbolic functions, respectively. Then, we establish new Wilker-type inequalities and Huygens-type inequalities for two function pairs, $x/\sin^{-1}x$ and $x/\tan^{-1}x$, $x/\sinh^{-1}x$ and $x/\tanh^{-1}x$. Finally, we obtain some more general conclusions than the first work of this paper, which reveal the absolute monotonicity of four functions involving the four inequalities mentioned above.

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1. Introduction

Let $0 < x < \pi/2$. Then

$$\sin x < x < \tan x, \quad (1.1)$$

which can be rewritten as

$$\frac{\sin x}{x} < 1 < \frac{\tan x}{x}, \quad (1.2)$$

or

$$\frac{x}{\tan x} < 1 < \frac{x}{\sin x}. \quad (1.3)$$

When the functions involved in (1.2) are taken into account in two forms of size relations, two famous inequalities called the first Wilker's inequality (see [7, 21, 29, 30, 37, 39]), the first Huygens inequality (see [3–5, 8, 9, 11, 28, 32, 43]), it comes to the conclusions (1.4) and (1.5). The comparison of these two inequalities (see [6]) is shown as follows in (1.6).

$$\frac{1}{2} \left(\left(\frac{\sin x}{x} \right)^2 + \frac{\tan x}{x} \right) > 1, \quad (1.4)$$

$$\frac{1}{3} \left(\frac{2 \sin x}{x} + \frac{\tan x}{x} \right) > 1, \quad (1.5)$$

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$$\frac{1}{2} \left(\left(\frac{\sin x}{x} \right)^2 + \frac{\tan x}{x} \right) > \frac{1}{3} \left(\frac{2 \sin x}{x} + \frac{\tan x}{x} \right) > 1. \quad (1.6)$$

Similar to (1.4) – (1.6), there are some conclusions (1.7) and (1.8) about the second Wilker’s inequality (see [23, 24, 32, 43]), the second Huygens inequality (see [23, 24]), and the comparison of the two inequalities as follows.

$$\frac{1}{2} \left(\left(\frac{x}{\sin x} \right)^2 + \frac{x}{\tan x} \right) > 1, \quad (1.7)$$

$$\frac{1}{3} \left(\frac{2x}{\sin x} + \frac{x}{\tan x} \right) > 1, \quad (1.8)$$

$$\frac{1}{2} \left(\left(\frac{x}{\sin x} \right)^2 + \frac{x}{\tan x} \right) > \frac{1}{3} \left(\frac{2x}{\sin x} + \frac{x}{\tan x} \right) > 1. \quad (1.9)$$

The last inequality chain is true due to

$$\begin{aligned} & \frac{1}{2} \left(\left(\frac{x}{\sin x} \right)^2 + \frac{x}{\tan x} \right) - \frac{1}{3} \left(\frac{2x}{\sin x} + \frac{x}{\tan x} \right) \\ &= \frac{1}{6} \left(2 \left(\frac{x}{\sin x} \right)^2 + \left(\frac{x}{\sin x} \right)^2 + \frac{x}{\tan x} - 4 \frac{x}{\sin x} \right) \\ &> \frac{1}{6} \left(2 \left(\frac{x}{\sin x} \right)^2 + 2 - 4 \frac{x}{\sin x} \right) = \frac{1}{3} \left(1 - \frac{x}{\sin x} \right)^2 > 0 \end{aligned}$$

and (1.8). At the same time, we find that the inequality (1.7) plays a key role in the above derivation. Furthermore, the relationships between the first and second Wilker’s inequality (see [3, 42]), the first and second Huygens inequality (see [23, 24]) are given below.

$$\frac{1}{2} \left(\left(\frac{\sin x}{x} \right)^2 + \frac{\tan x}{x} \right) > \frac{1}{2} \left(\left(\frac{x}{\sin x} \right)^2 + \frac{x}{\tan x} \right) > 1, \quad (1.10)$$

$$\frac{1}{3} \left(\frac{2 \sin x}{x} + \frac{\tan x}{x} \right) > \frac{1}{3} \left(\frac{2x}{\sin x} + \frac{x}{\tan x} \right) > 1. \quad (1.11)$$

The same case occurs in the hyperbolic functions (see [23, 24, 36, 39, 41–44]).

Now let’s turn to the discussion of similar inequalities for inverse circular functions. Let $0 < x < 1$. Then

$$\tan^{-1} x < x < \sin^{-1} x, \quad (1.12)$$

which can be rewritten as

$$\frac{\tan^{-1} x}{x} < 1 < \frac{\sin^{-1} x}{x}, \quad (1.13)$$

or

$$\frac{x}{\sin^{-1} x} < 1 < \frac{x}{\tan^{-1} x}. \quad (1.14)$$

Chen and Cheung [6] obtained an important conclusion about the inverse circular functions as follows.

$$\left(\frac{x}{\sin^{-1} x} \right)^2 + \frac{x}{\tan^{-1} x} < 2, \quad 0 < x < 1. \quad (1.15)$$

Then, they used the arithmetic–geometric–harmonic mean inequality to prove the following inequality chain for $x \in (0, 1)$:

$$\frac{1}{2} \left(\left(\frac{\sin^{-1} x}{x} \right)^2 + \frac{\tan^{-1} x}{x} \right) > \frac{1}{3} \left(\frac{2 \sin^{-1} x}{x} + \frac{\tan^{-1} x}{x} \right) \quad (1.16)$$

$$\begin{aligned}
&> \left(\left(\frac{\sin^{-1} x}{x} \right)^2 \frac{\tan^{-1} x}{x} \right)^{1/3} \\
&> \left(\frac{2}{1/((\sin^{-1} x)/x)^2 + 1/((\tan^{-1} x)/x)} \right)^{1/3} > 1.
\end{aligned}$$

They established the inverse hyperbolic version of above results for $x \in (0, 1)$:

$$\left(\frac{x}{\sinh^{-1} x} \right)^2 + \frac{x}{\tanh^{-1} x} < 2, \quad (1.17)$$

and

$$\frac{1}{2} \left(\left(\frac{\sinh^{-1} x}{x} \right)^2 + \frac{\tanh^{-1} x}{x} \right) > \frac{1}{3} \left(\frac{2 \sinh^{-1} x}{x} + \frac{\tanh^{-1} x}{x} \right) \quad (1.18)$$

$$\begin{aligned}
&> \left(\left(\frac{\sinh^{-1} x}{x} \right)^2 \frac{\tanh^{-1} x}{x} \right)^{1/3} \\
&> \left(\frac{2}{1/((\sinh^{-1} x)/x)^2 + 1/((\tanh^{-1} x)/x)} \right)^{1/3} > 1.
\end{aligned}$$

The first task of this paper is to determine the relationship between the first Wilker's inequality, the second Wilker's inequality, the first Huygens inequality and the second Huygens inequality. The second one is to consider the results according to the form of the inequality (1.6) or (1.9) for two function pairs, $x/\sin^{-1} x$ and $x/\tan^{-1} x$, $x/\sinh^{-1} x$ and $x/\tanh^{-1} x$. Finally, we obtain some more general conclusions than the first work of this paper, which reveal the absolute monotonicity of four functions involving the above four inequalities.

2. Main results

This paper obtains the following main results.

Theorem 2.1. *Let $x \in (0, \pi/2)$. Then the inequality chain*

$$\begin{aligned}
&\frac{1}{2} \left(\left(\frac{\sin x}{x} \right)^2 + \frac{\tan x}{x} \right) > \frac{1}{3} \left(\frac{2 \sin x}{x} + \frac{\tan x}{x} \right) \\
&> \frac{1}{2} \left(\left(\frac{x}{\sin x} \right)^2 + \frac{x}{\tan x} \right) > \frac{1}{3} \left(\frac{2x}{\sin x} + \frac{x}{\tan x} \right) \\
&> 1.
\end{aligned} \quad (2.1)$$

holds.

Theorem 2.2. *Let $x \in (0, \infty)$. Then the inequality chain*

$$\begin{aligned}
&\frac{1}{2} \left(\left(\frac{\sinh x}{x} \right)^2 + \frac{\tanh x}{x} \right) > \frac{1}{3} \left(\frac{2 \sinh x}{x} + \frac{\tanh x}{x} \right) \\
&> \frac{1}{2} \left(\left(\frac{x}{\sinh x} \right)^2 + \frac{x}{\tanh x} \right) > \frac{1}{3} \left(\frac{2x}{\sinh x} + \frac{x}{\tanh x} \right) \\
&> 1.
\end{aligned} \quad (2.2)$$

holds.

Theorem 2.3. *Let $x \in (0, 1)$. Then the inequality chain*

$$\frac{1}{2} \left(\left(\frac{x}{\sin^{-1} x} \right)^2 + \frac{x}{\tan^{-1} x} \right) < \frac{1}{3} \left(\frac{2x}{\sin^{-1} x} + \frac{x}{\tan^{-1} x} \right) < 1 \quad (2.3)$$

holds.

Theorem 2.4. *Let $x \in (0, 1)$. Then the inequality chain*

$$\frac{1}{2} \left(\left(\frac{x}{\sinh^{-1} x} \right)^2 + \frac{x}{\tanh^{-1} x} \right) < \frac{1}{3} \left(\frac{2x}{\sinh^{-1} x} + \frac{x}{\tanh^{-1} x} \right) < 1 \quad (2.4)$$

holds.

Then we can obtain the following corollaries.

Corollary 2.5. *Let $x \in (0, 1)$. Then*

$$\begin{aligned} \frac{1}{2} \left(\left(\frac{\sin^{-1} x}{x} \right)^2 + \frac{\tan^{-1} x}{x} \right) &> \frac{1}{3} \left(\frac{2 \sin^{-1} x}{x} + \frac{\tan^{-1} x}{x} \right) > 1 \\ &> \frac{1}{3} \left(\frac{2x}{\sin^{-1} x} + \frac{x}{\tan^{-1} x} \right) > \frac{1}{2} \left(\left(\frac{x}{\sin^{-1} x} \right)^2 + \frac{x}{\tan^{-1} x} \right). \end{aligned} \quad (2.5)$$

Corollary 2.6. *Let $x \in (0, 1)$. Then*

$$\begin{aligned} \frac{1}{2} \left(\left(\frac{\sinh^{-1} x}{x} \right)^2 + \frac{\tanh^{-1} x}{x} \right) &> \frac{1}{3} \left(\frac{2 \sinh^{-1} x}{x} + \frac{\tanh^{-1} x}{x} \right) > 1 \\ &> \frac{1}{3} \left(\frac{2x}{\sinh^{-1} x} + \frac{x}{\tanh^{-1} x} \right) > \frac{1}{2} \left(\left(\frac{x}{\sinh^{-1} x} \right)^2 + \frac{x}{\tanh^{-1} x} \right). \end{aligned} \quad (2.6)$$

3. Proofs

3.1. Proof of Theorem 2.1

We shall complete the proof of Theorem 2.1 when proving second inequality of (2.1).

Computing directly gives

$$\frac{1}{3} \left(\frac{2 \sin x}{x} + \frac{\tan x}{x} \right) - \frac{1}{2} \left(\left(\frac{x}{\sin x} \right)^2 + \frac{x}{\tan x} \right) = \frac{\sin^2 x}{6x \cos x} F(x), \quad (3.1)$$

where

$$\begin{aligned} F(x) &= \frac{4 \cos x \sin^3 x + 2 \sin^3 x - 3x^3 \cos x - 3x^2 \cos^2 x \sin x}{\sin^4 x} \\ &= 4 \cot x + 3x^2 \frac{1}{\sin x} - 3x^2 \frac{1}{\sin^3 x} + 2 \frac{1}{\sin x} + x^3 \left(-3 \frac{\cos x}{\sin^4 x} \right). \end{aligned} \quad (3.2)$$

Since

$$\begin{aligned} \left(\frac{1}{\sin x} \right)' &= -\frac{\cos x}{\sin^2 x}, \\ \left(\frac{1}{\sin x} \right)'' &= \left(-\frac{\cos x}{\sin^2 x} \right)' = \frac{2}{\sin^3 x} - \frac{1}{\sin x}, \\ \left(\frac{1}{\sin^3 x} \right)' &= -3 \frac{\cos x}{\sin^4 x}, \end{aligned}$$

from

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n-1}, \quad 0 < |x| < \pi, \quad (\text{see [12]}) \quad (3.3)$$

we have

$$\begin{aligned}
\frac{1}{\sin^3 x} &= \frac{1}{2} \left(\left(\frac{1}{\sin x} \right)'' + \frac{1}{\sin x} \right) \\
&= \frac{1}{2} \left(\frac{2}{x^3} + \sum_{n=2}^{\infty} \frac{(2^{2n} - 2)(2n-1)(2n-2)}{(2n)!} |B_{2n}| x^{2n-3} \right) \\
&\quad + \frac{1}{2} \left(\frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n-1} \right) \\
&= \frac{1}{x^3} + \sum_{n=2}^{\infty} \frac{(2^{2n} - 2)(2n-1)(2n-2)}{2(2n)!} |B_{2n}| x^{2n-3} \\
&\quad + \frac{1}{2x} + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{2(2n)!} |B_{2n}| x^{2n-1},
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
-3 \frac{\cos x}{\sin^4 x} &= \frac{1}{2} \left(-\frac{6}{x^4} + \sum_{n=2}^{\infty} \frac{(2^{2n} - 2)(2n-1)(2n-2)(2n-3)}{(2n)!} |B_{2n}| x^{2n-4} \right) \\
&\quad + \frac{1}{2} \left(-\frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{(2^{2n} - 2)(2n-1)}{(2n)!} |B_{2n}| x^{2n-2} \right) \\
&= -\frac{3}{x^4} + \sum_{n=2}^{\infty} \frac{(2^{2n} - 2)(2n-1)(2n-2)(2n-3)}{2(2n)!} |B_{2n}| x^{2n-4} \\
&\quad - \frac{1}{2x^2} + \sum_{n=1}^{\infty} \frac{(2^{2n} - 2)(2n-1)}{2(2n)!} |B_{2n}| x^{2n-2}.
\end{aligned}$$

We substitute the power series expansions of these functions into (3.2), and obtain

$$\begin{aligned}
F(x) &= 4 \cot x + 3x^2 \frac{1}{\sin x} + 2 \frac{1}{\sin x} - 3x^2 \frac{1}{\sin^3 x} + x^3 \left(-3 \frac{\cos x}{\sin^4 x} \right) \\
&= 4 \left(\frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1} \right) + 3x^2 \left(\frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n-1} \right) \\
&\quad + 2 \left(\frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n-1} \right) \\
&\quad - 3x^2 \left(\frac{1}{x^3} + \sum_{n=2}^{\infty} \frac{(2^{2n} - 2)(2n-1)(2n-2)}{2(2n)!} |B_{2n}| x^{2n-3} \right) \\
&\quad - 3x^2 \left(\frac{1}{2x} + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{2(2n)!} |B_{2n}| x^{2n-1} \right) \\
&\quad + x^3 \left(-\frac{3}{x^4} + \sum_{n=2}^{\infty} \frac{(2^{2n} - 2)(2n-1)(2n-2)(2n-3)}{2(2n)!} |B_{2n}| x^{2n-4} \right) \\
&\quad + x^3 \left(-\frac{1}{2x^2} + \sum_{n=1}^{\infty} \frac{(2^{2n} - 2)(2n-1)}{2(2n)!} |B_{2n}| x^{2n-2} \right) \\
&= \sum_{n=2}^{\infty} \frac{(2^{2n} - 2)(2n-1)(2n-2)(2n-3)}{2(2n)!} |B_{2n}| x^{2n-1} \\
&\quad - 3 \sum_{n=2}^{\infty} \frac{(2^{2n} - 2)(2n-1)(2n-2)}{2(2n)!} |B_{2n}| x^{2n-1}
\end{aligned}$$

$$\begin{aligned}
& + 2 \sum_{n=2}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n-1} - 4 \sum_{n=2}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1} \\
& + 3 \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n+1} - 3 \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{2(2n)!} |B_{2n}| x^{2n+1} \\
& + \sum_{n=1}^{\infty} \frac{(2^{2n} - 2)(2n - 1)}{2(2n)!} |B_{2n}| x^{2n+1} \\
& = \sum_{n=2}^{\infty} \frac{(8n^3 - 36n^2 + 40n - 16)2^{2n} - 16n^3 + 72n^2 + 16 - 80n}{2(2n)!} |B_{2n}| x^{2n-1} \\
& + \sum_{n=1}^{\infty} \frac{2(n+1)(2^{2n} - 2)}{2(2n)!} |B_{2n}| x^{2n+1} \\
& = \sum_{n=2}^{\infty} 4 \frac{2^{2n-1}(2n^3 + 10n - 9n^2 - 4) - (2n^3 + 10n - 9n^2 - 2)}{(2n)!} |B_{2n}| x^{2n-1} \\
& + \sum_{n=2}^{\infty} \frac{n(2^{2n-2} - 2)}{(2n-2)!} |B_{2n-2}| x^{2n-1} \\
& := \sum_{n=2}^{\infty} a_n x^{2n-1},
\end{aligned}$$

where

$$\begin{aligned}
a_n & = 4 \frac{2^{2n-1}(2n^3 - 9n^2 + 10n - 4) - (2n^3 - 9n^2 + 10n - 2)}{(2n)!} |B_{2n}| \\
& + \frac{n(2^{2n-2} - 2)}{(2n-2)!} |B_{2n-2}|
\end{aligned}$$

for $n \geq 2$.

Since

$$|B_2| = \frac{1}{6}, \quad |B_4| = \frac{1}{30}, \quad |B_6| = \frac{1}{42}, \quad |B_8| = \frac{1}{30},$$

we first compute to obtain that

$$a_2 = \frac{1}{6}, \quad a_3 = \frac{17}{315}, \quad a_4 = \frac{2509}{151200}.$$

Then using mathematical induction we can prove

$$2^{2n-1} (2n^3 - 9n^2 + 10n - 4) - (2n^3 - 9n^2 + 10n - 2) > 0$$

or

$$2^{2n-1} > \frac{2n^3 - 9n^2 + 10n - 2}{2n^3 - 9n^2 + 10n - 4} \quad (3.5)$$

for $n \geq 4$. In fact, when $n = 4$, the inequality (3.5) holds. Now, we assume that the (3.5) holds for $n = m$. Then, in order to complete the proof of (3.5) is also true for $n = m + 1$ it suffices to show that

$$4 \frac{2m^3 - 9m^2 + 10m - 2}{2m^3 - 9m^2 + 10m - 4} > \frac{2(m+1)^3 - 9(m+1)^2 + 10(m+1) - 2}{2(m+1)^3 - 9(m+1)^2 + 10(m+1) - 4},$$

which is true due to

$$\begin{aligned}
& 4 \left(2m^3 - 9m^2 + 10m - 2 \right) \left(2(m+1)^3 - 9(m+1)^2 + 10(m+1) - 4 \right) \\
& - \left(2m^3 - 9m^2 + 10m - 4 \right) \left(2(m+1)^3 - 9(m+1)^2 + 10(m+1) - 2 \right) \\
& = 12m^6 - 72m^5 + 129m^4 - 54m^3 - 3m^2 - 42m + 12
\end{aligned}$$

$$\begin{aligned}
&= 12(m-4)^6 + 216(m-4)^5 + 1569(m-4)^4 + 5850(m-4)^3 \\
&\quad + 11\,733(m-4)^2 + 11\,934(m-4) + 4788 \\
&> 0.
\end{aligned}$$

So $a_n > 0$ for $n \geq 2$. This leads to $F(x) > 0$ for all $x \in (0, \pi/2)$. The proof of (2.1) is complete via (3.1).

3.2. Proof of Theorem 2.2

Similarly, if we can prove second inequality of (2.2), we then complete the proof of Theorem 2.2.

Computing gives

$$\frac{1}{3} \left(\frac{2 \sinh x}{x} + \frac{\tanh x}{x} \right) - \frac{1}{2} \left(\left(\frac{x}{\sinh x} \right)^2 + \frac{x}{\tanh x} \right) := \frac{1}{24x \cosh x \sinh^3 x} G(x), \quad (3.6)$$

where

$$\begin{aligned}
G(x) &= \cosh 4x - 3 \cosh 3x - 4 \cosh 2x + \cosh 5x + 2 \cosh x \\
&\quad - \frac{3}{2} x^2 \cosh 4x - 6x^3 \sinh 2x + \frac{3}{2} x^2 + 3.
\end{aligned} \quad (3.7)$$

Using the power series expansions of these hyperbolic functions, we have

$$\begin{aligned}
G(x) &= \sum_{n=0}^{\infty} \frac{4^{2n}}{(2n)!} x^{2n} - 3 \sum_{n=0}^{\infty} \frac{3^{2n}}{(2n)!} x^{2n} - 4 \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{5^{2n}}{(2n)!} x^{2n} \\
&\quad + 2 \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} - \frac{3}{2} x^2 \sum_{n=0}^{\infty} \frac{4^{2n}}{(2n)!} x^{2n} - 6x^3 \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} x^{2n+1} \\
&\quad + \frac{3}{2} x^2 + 3 \\
&= \sum_{n=4}^{\infty} \frac{4^{2n} - 3 \cdot 3^{2n} - 4 \cdot 2^{2n} + 5^{2n} + 2}{(2n)!} x^{2n} \\
&\quad - \frac{3}{2} \sum_{n=3}^{\infty} \frac{4^{2n}}{(2n)!} x^{2n+2} - 6 \sum_{n=2}^{\infty} \frac{2^{2n+1}}{(2n+1)!} x^{2n+4} \\
&= \sum_{n=4}^{\infty} \frac{4^{2n} - 3 \cdot 3^{2n} - 4 \cdot 2^{2n} + 5^{2n} + 2}{(2n)!} x^{2n} \\
&\quad - \sum_{n=4}^{\infty} \frac{3 \cdot 4^{2n-2}}{2(2n-2)!} x^{2n} - \sum_{n=4}^{\infty} \frac{6 \cdot 2^{2n-3}}{(2n-3)!} x^{2n} \\
&:= \sum_{n=4}^{\infty} \frac{1}{32(2n)!} b_n x^{2n},
\end{aligned}$$

where

$$\begin{aligned}
b_n &= 32 \cdot 5^{2n} - (6n^2 - 3n - 16) 2^{4n+1} - 32 \cdot 3^{2n+1} \\
&\quad - (6n^3 - 9n^2 + 3n + 4) 2^{2n+5} + 64
\end{aligned}$$

for $n \geq 4$. We compute

$$\begin{aligned}
c_n &:= b_{n+1} - 25b_n \\
&= 1536 \cdot 3^{2n} + (108n^2 - 438n - 384) 2^{4n} \\
&\quad + (126n^3 - 261n^2 + 63n + 84) 2^{2n+5} - 1536
\end{aligned} \quad (3.8)$$

and obtain that

$$\begin{aligned} 108n^2 - 438n - 384 &> 0, \\ (126n^3 - 261n^2 + 63n + 84) 2^{2n+5} - 1536 &> 0 \end{aligned}$$

hold for all $n \geq 5$. So $c_n > 0$ for $n \geq 5$. This together with $c_4 = 17\,940\,480 > 0$ gives that $c_n > 0$ for $n \geq 4$. Then via (3.8) we have $b_{n+1} > 25b_n$ holds for $n \geq 4$. This together with $b_4 = 860\,160 > 0$ gives that $b_n > 0$ for $n \geq 4$. Then $G(x) > 0$ for all $x \in (0, \pi/2)$. The proof of (2.2) is complete via (3.6).

3.3. Proof of Theorem 2.3

In order to prove Theorem 2.3 as simple as possible, we need a tool which offers a simple but efficient criterion to determine the sign of a kind of special power series, which we call as "sign rule of a kind of special power series".

Lemma 3.1 ([34], [33]). *Let $\{a_k\}_{k=0}^\infty$ be a nonnegative real sequence with $a_m > 0$ and $\sum_{k=m+1}^\infty a_k > 0$ and let*

$$S(t) = - \sum_{k=0}^m a_k t^k + \sum_{k=m+1}^\infty a_k t^k$$

be a convergent power series on the interval $(0, r)$ ($r > 0$). (i) If $S(r^-) \leq 0$ then $S(t) < 0$ for all $t \in (0, r)$. (ii) If $S(r^-) > 0$ then there is the unique $t_0 \in (0, r)$ such that $S(t) < 0$ for $t \in (0, t_0)$ and $S(t) > 0$ for $t \in (t_0, r)$.

(1) We first prove the left hand side of (2.3).

Let $\arcsin x = t$. Then the desired inequality is equivalent to

$$\frac{1}{2} \left(\frac{\sin t}{t} \right)^2 - \frac{2 \sin t}{3t} + \frac{1}{6} \frac{\sin t}{\arctan(\sin t)} = \frac{\sin t}{6} \left(\frac{1}{\arctan(\sin t)} - \frac{4t - 3 \sin t}{t^2} \right) < 0,$$

which is in turn equivalent to

$$H(t) := \frac{t^2}{4t - 3 \sin t} - \arctan(\sin t) < 0$$

for $t \in (0, \pi/2)$. Differentiation yields

$$H'(t) = \frac{\sin^3 t}{(1 + \sin^2 t)(4t - 3 \sin t)^2} h(t),$$

where

$$h(t) = 4 \frac{t^2}{\sin t} - 6 \frac{t}{\sin^2 t} - 9 \cot t - 6t + 4 \frac{t^2}{\sin^3 t} + 24t \frac{\cos t}{\sin^2 t} + 3t^2 \cot t - 13t^2 \frac{\cos t}{\sin^3 t}.$$

From

$$\cot x = \frac{1}{x} - \sum_{n=1}^\infty \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1}, \quad 0 < |x| < \pi, \quad (\text{see [10]}) \quad (3.9)$$

and (3.3) we have

$$\begin{aligned} \frac{1}{\sin^2 t} &= -(\cot t)' = \frac{1}{t^2} + \sum_{n=1}^\infty \frac{(2n-1) 2^{2n}}{(2n)!} |B_{2n}| t^{2n-2}, \\ \frac{\cos t}{\sin^2 t} &= -\left(\frac{1}{\sin t}\right)' = \frac{1}{t^2} - \sum_{n=1}^\infty \frac{(2n-1)(2^{2n}-2)}{(2n)!} |B_{2n}| t^{2n-2}, \\ \frac{\cos t}{\sin^3 t} &= -\frac{1}{2} \left(\frac{1}{\sin^2 t}\right)' = \frac{1}{t^3} - \sum_{n=2}^\infty \frac{(2n-1)(n-1) 2^{2n}}{(2n)!} |B_{2n}| t^{2n-3}. \end{aligned} \quad (3.10)$$

The above power series expansions and (3.4) give

$$\begin{aligned}
h(t) &= 4t + 4 \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| t^{2n+1} - 6t \left(\frac{1}{t^2} + \sum_{n=1}^{\infty} \frac{(2n-1) 2^{2n}}{(2n)!} |B_{2n}| t^{2n-2} \right) \\
&\quad - 9 \left(\frac{1}{t} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| t^{2n-1} \right) + 4t^2 \left(\frac{1}{2t} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| t^{2n-1} \right) \\
&\quad + 4t^2 \left(\frac{1}{t^3} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{(2n-1)(2n-2)(2^{2n}-2)}{(2n)!} |B_{2n}| t^{2n-3} \right) \\
&\quad + 24t \left(\frac{1}{t^2} - \sum_{n=1}^{\infty} \frac{(2n-1)(2^{2n}-2)}{(2n)!} |B_{2n}| t^{2n-2} \right) \\
&\quad + 3t^2 \left(\frac{1}{t} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| t^{2n-1} \right) \\
&\quad - 13t^2 \left(\frac{1}{t^3} - \sum_{n=2}^{\infty} \frac{(2n-1)(n-1) 2^{2n}}{(2n)!} |B_{2n}| t^{2n-3} \right) - 6t \\
&= 4t + 4 \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| t^{2n+1} - \frac{6}{t} - 6 \sum_{n=1}^{\infty} \frac{(2n-1) 2^{2n}}{(2n)!} |B_{2n}| t^{2n-1} \\
&\quad - \frac{9}{t} + 9 \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| t^{2n-1} + 4t^2 \left(\frac{1}{2t} + \frac{1}{t^3} \right) + 2 \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| t^{2n+1} \\
&\quad + 2 \sum_{n=2}^{\infty} \frac{(2n-1)(2n-2)(2^{2n}-2)}{(2n)!} |B_{2n}| t^{2n-1} + \frac{24}{t} \\
&\quad - 24 \sum_{n=1}^{\infty} \frac{(2n-1)(2^{2n}-2)}{(2n)!} |B_{2n}| t^{2n-1} + 3t - 3 \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| t^{2n+1} \\
&\quad - \frac{13}{t} + 13 \sum_{n=2}^{\infty} \frac{(2n-1)(n-1) 2^{2n}}{(2n)!} |B_{2n}| t^{2n-1} - 6t \\
&= \sum_{n=1}^{\infty} \frac{3(2^{2n}-4)}{(2n)!} |B_{2n}| t^{2n+1} \\
&\quad + \sum_{n=2}^{\infty} \frac{(34n^2 - 111n + 56) 2^{2n} - 8(2n-1)(n-7)}{(2n)!} |B_{2n}| t^{2n-1} \\
&:= \sum_{n=2}^{\infty} \frac{k_n |B_{2n-2}| + l_n |B_{2n}|}{(2n)!} t^{2n-1} := \sum_{n=2}^{\infty} p_n t^{2n-1},
\end{aligned}$$

where

$$\begin{aligned}
k_n &= 24n(2n-1)(2^{2n-4} - 1), \\
l_n &= (34n^2 - 111n + 56) 2^{2n} - 8(2n-1)(n-7).
\end{aligned}$$

A simple computation shows that $p_2 = -1/2$. We claim that $p_n > 0$ for $n \geq 3$. In fact, $k_n > 0$ for $n \geq 3$. Also, since $(34n^2 - 111n + 56) > 0$, so for $n \geq 3$,

$$l_n > \left((34n^2 - 111n + 56) 8 - 8(2n-1)(n-7) \right) = 8(32n(n-3) + 49) > 0.$$

These indicate that $p_n > 0$ for $n \geq 3$.

On the other hand, we see that

$$h(\pi/2) = 2\pi(\pi - 3) > 0.$$

By Lemma 3.1, there is a $t_0 \in (0, \pi/2)$ so that $h(t) < 0$ for $t \in (0, t_0)$ and $h(t) > 0$ for $t \in (t_0, \pi/2)$, which in turn implies that $H(t)$ is decreasing on $(0, t_0)$ and increasing on $(t_0, \pi/2)$. Consequently, we obtain

$$H(t) < \lim_{t \rightarrow 0^+} H(t) = 0 \text{ for } t \in (0, t_0),$$

$$H(t) < \lim_{t \rightarrow (\pi/2)^-} H(t) = -\frac{1}{4} \frac{\pi(\pi-3)}{2\pi-3} < 0 \text{ for } t \in (t_0, \pi/2),$$

that is, $H(t) < 0$ for $t \in (0, \pi/2)$. This completes the proof of the left hand side of (2.3).

(2) We then prove the right hand side of (2.3).

The desired inequality is equivalent to

$$2 \frac{x}{\sin^{-1} x} + \frac{x}{\tan^{-1} x} < 3.$$

Since

$$\frac{x}{\sin^{-1} x} < \frac{2 + \sqrt{1-x^2}}{3}, \text{ (see [15, 16, 21, 38])}$$

$$\frac{x}{\tan^{-1} x} < 1 + \frac{1}{3}x^2, \text{ (see [6])}$$

we have

$$2 \frac{x}{\sin^{-1} x} + \frac{x}{\tan^{-1} x} < \frac{2(2 + \sqrt{1-x^2})}{3} + 1 + \frac{1}{3}x^2.$$

We can complete the proof of the right hand side of (2.3) as long as we can prove that

$$\frac{2(2 + \sqrt{1-x^2})}{3} + 1 + \frac{1}{3}x^2 < 3,$$

which is equivalent to $(1 - \sqrt{1-x^2})^2 > 0$.

3.4. Proof of Theorem 2.4

(1) We first prove the left hand side of (2.4).

Since

$$\frac{1}{3} \left(\frac{2x}{\sinh^{-1} x} + \frac{x}{\tanh^{-1} x} \right) - \frac{1}{2} \left(\left(\frac{x}{\sinh^{-1} x} \right)^2 + \frac{x}{\tanh^{-1} x} \right)$$

$$= \frac{x}{6} \left(\frac{4}{\sinh^{-1} x} - \frac{1}{\tanh^{-1} x} - 3 \frac{x}{(\sinh^{-1} x)^2} \right),$$

the desired inequality is equivalent to

$$\tanh^{-1} x > \frac{(\sinh^{-1} x)^2}{4 \sinh^{-1} x - 3x}.$$

Let $\sinh^{-1} x = t$. Then $x = \sinh t$, the above inequality is equivalent to

$$\tanh^{-1}(\sinh t) > \frac{t^2}{4t - 3 \sinh t}.$$

Let

$$Q(t) = \tanh^{-1}(\sinh t) - \frac{t^2}{4t - 3 \sinh t}.$$

Then

$$Q'(t) = \frac{q(t)}{(1 - \sinh^2 t)(4t - 3 \sinh t)^2},$$

where

$$\begin{aligned} q(t) &= (\cosh t)(4t - 3\sinh t)^2 - (1 - \sinh^2 t)(3t^2 \cosh t - 6t \sinh t + 4t^2) \\ &= \frac{9}{4} \cosh 3t - \frac{9}{4} \cosh t + 2t^2 \cosh 2t + \frac{3}{4} t^2 \cosh 3t + \frac{21}{2} t \sinh t \\ &\quad - 12t \sinh 2t - \frac{3}{2} t \sinh 3t + \frac{49}{4} t^2 \cosh t - 6t^2. \end{aligned}$$

Expanding in power series of the hyperbolic functions leads to

$$\begin{aligned} q(t) &= \frac{9}{4} \sum_{n=0}^{\infty} \frac{(3t)^{2n}}{(2n)!} - \frac{9}{4} \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} + 2t^2 \sum_{n=0}^{\infty} \frac{(2t)^{2n}}{(2n)!} + \frac{3}{4} t^2 \sum_{n=0}^{\infty} \frac{(3t)^{2n}}{(2n)!} \\ &\quad + \frac{21}{2} t \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} - 12t \sum_{n=0}^{\infty} \frac{(2t)^{2n+1}}{(2n+1)!} - \frac{3}{2} t \sum_{n=0}^{\infty} \frac{(3t)^{2n+1}}{(2n+1)!} \\ &\quad + \frac{49}{4} t^2 \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} - 6t^2 \\ &= \sum_{n=2}^{\infty} r_n t^{2n+2}, \end{aligned}$$

where

$$r_n = \frac{4n^2 - 6n + 17}{4(2n+2)!} 3^{2n+1} + \frac{2n-11}{(2n+1)!} 2^{2n+1} + \frac{196n^2 + 378n + 173}{4(2n+2)!}.$$

We find that

$$r_2 = \frac{1}{2}, r_3 = \frac{11}{30}, r_4 = \frac{411}{5600}, r_5 = \frac{403}{50400},$$

and $r_n > 0$ for $n \geq 6$ due to $4n^2 - 6n + 17 > 0$ and $2n - 11 > 0$. So $r_n > 0$ for $n \geq 2$. This leads to that $q(t) > 0$. Then $Q'(t) > 0$. So $Q(t) > Q(0^+) = 0$, which completes the proof of the left hand side of (2.4).

(2) Then we prove the right hand side of (2.4).

The desired inequality is equivalent to

$$2 \frac{x}{\sinh^{-1} x} + \frac{x}{\tanh^{-1} x} < 3.$$

Since

$$\begin{aligned} \frac{x}{\sinh^{-1} x} &< \frac{2 + \sqrt{x^2 + 1}}{3}, \quad (\text{see [40]}) \\ \frac{x}{\tanh^{-1} x} &< \frac{1 + 2\sqrt{1 - x^2}}{3}, \quad (\text{see [6]}) \end{aligned}$$

we have

$$2 \frac{x}{\sinh^{-1} x} + \frac{x}{\tanh^{-1} x} < \frac{2(2 + \sqrt{x^2 + 1})}{3} + \frac{1 + 2\sqrt{1 - x^2}}{3}.$$

In order to complete the proof of the right hand side of (2.4) it suffices to show

$$\frac{2(2 + \sqrt{x^2 + 1})}{3} + \frac{1 + 2\sqrt{1 - x^2}}{3} < 3,$$

or

$$2(2 + \sqrt{x^2 + 1}) + 1 + 2\sqrt{1 - x^2} < 9$$

$$\iff \sqrt{x^2 + 1} < 2 - \sqrt{1 - x^2}$$

$$\iff x^2 + 1 < 4 - 4\sqrt{1 - x^2} + 1 - x^2$$

$$\iff x^2 < 2 - 2\sqrt{1 - x^2}.$$

The last inequality is equivalent to $(1 - \sqrt{1 - x^2})^2 > 0$.

4. Further discussions

Let us consider a real function $f: (a, b) \rightarrow \mathbb{R}$ in case when exist finite limits $f^{(k)}(a+) = \lim_{x \rightarrow a+} f^{(k)}(x)$ (for $k = 0, 1, \dots, n$ and $n \in \mathbb{N}_0$) and $f(b-) = \lim_{x \rightarrow b-} f(x)$. We define

$$T_n^{f, a+}(x) = \sum_{k=0}^n \frac{f^{(k)}(a+)}{k!} (x - a)^k, \quad (4.1)$$

$$R_n^{f, a+}(x) = f(x) - T_n^{f, a+}(x), \quad (4.2)$$

and

$$\mathbb{T}_n^{f; a+, b-}(x) = \begin{cases} T_{n-1}^{f, a+}(x) + \frac{1}{(b-a)^n} R_{n-1}^{f, a+}(b-)(x - a)^n & , \quad n \geq 1 \\ f(b-) & , \quad n = 0 \end{cases}. \quad (4.3)$$

Then the following statement is found to be true in [20, Theorem 3] and [18, Theorem 3].

Theorem 4.1. *Let $f: (a, b) \rightarrow \mathbb{R}$ be real analytic function with the power series:*

$$f(x) = \sum_{k=0}^{\infty} c_k (x - a)^k, \quad (4.4)$$

where $c_k \in \mathbb{R}$ and $c_k \geq 0$ for every $k \in \mathbb{N}_0$. Then,

$$\begin{aligned} T_0^{f, a+}(x) &\leq \dots \leq T_n^{f, a+}(x) \leq T_{n+1}^{f, a+}(x) \leq \dots \\ \dots &\leq f(x) \leq \dots \\ \dots &\leq \mathbb{T}_{n+1}^{f; a+, b-}(x) \leq \mathbb{T}_n^{f; a+, b-}(x) \leq \dots \leq \mathbb{T}_0^{f; a+, b-}(x). \end{aligned} \quad (4.5)$$

for every $x \in (a, b)$. If $c_k \in \mathbb{R}$ and $c_k \leq 0$ for every $k \in \mathbb{N}_0$, then the reversed inequality is true.

Let us emphasize that previous theorem improves result of Theorem 2 from [31]. Inspired by [2, 13, 14, 17, 19, 22, 27], and [31], we obtain a conclusion more general than Theorem 2.1. The details are as follows.

Theorem 4.2. *Let us form the functions*

$$\varphi_1(x) = \frac{1}{2} \left(\left(\frac{\sin x}{x} \right)^2 + \frac{\tan x}{x} \right) - \frac{1}{3} \left(\frac{2 \sin x}{x} + \frac{\tan x}{x} \right) : \left(0, \frac{\pi}{2} \right) \rightarrow R,$$

$$\varphi_2(x) = \frac{1}{3} \left(\frac{2 \sin x}{x} + \frac{\tan x}{x} \right) - \frac{1}{2} \left(\left(\frac{x}{\sin x} \right)^2 + \frac{x}{\tan x} \right) : \left(0, \frac{\pi}{2} \right) \rightarrow R,$$

$$\varphi_3(x) = \frac{1}{2} \left(\left(\frac{x}{\sin x} \right)^2 + \frac{x}{\tan x} \right) - \frac{1}{3} \left(\frac{2x}{\sin x} + \frac{x}{\tan x} \right) : \left(0, \frac{\pi}{2} \right) \rightarrow R,$$

$$\varphi_4(x) = \frac{1}{3} \left(\frac{2x}{\sin x} + \frac{x}{\tan x} \right) - 1 : \left(0, \frac{\pi}{2} \right) \rightarrow R.$$

Then functions $\varphi_1(x), \varphi_2(x), \varphi_3(x), \varphi_4(x)$ are real analytic with power series

$$\varphi_1(x) = \sum_{k=2}^{\infty} s_k^{(1)} x^{2k}, \quad \varphi_2(x) = \sum_{k=2}^{\infty} s_k^{(2)} x^{2k}, \quad \varphi_3(x) = \sum_{k=2}^{\infty} s_k^{(3)} x^{2k}, \quad \varphi_4(x) = \sum_{k=2}^{\infty} s_k^{(4)} x^{2k}.$$

with positive coefficients

$$s_n^{(1)} = \frac{1}{2} \frac{(-1)^n 2^{2n+1}}{(2n+2)!} + \frac{1}{6} \frac{(2^{2n+2} - 1) 2^{2n+1}}{(2n+2)!} |B_{2n+2}| - \frac{2}{3} \frac{(-1)^n}{(2n+1)!} > 0,$$

$$\begin{aligned}
s_n^{(2)} &= \frac{2(-1)^n}{3(2n+1)!} + \frac{(2^{2n+2}-1)2^{2n+2}}{3(2n+2)!}|B_{2n+2}| - \frac{(n-1)2^{2n}}{(2n)!}|B_{2n}| > 0, \\
s_n^{(3)} &= \frac{(3n-4)2^{2n}+4}{3(2n)!}|B_{2n}| > 0, \\
s_n^{(4)} &= \frac{2^{2n}-4}{3(2n)!}|B_{2n}| > 0
\end{aligned}$$

for $n = 2, 3, \dots$. Let it be that $j \in \{1, 2, 3, 4\}$ and $c \in (0, \pi/2)$ fixed. Then the double inequality

$$\begin{aligned}
0 < T_2^{\varphi_j, 0^+}(x) &\leq T_3^{\varphi_j, 0^+}(x) \leq \dots \leq T_n^{\varphi_j, 0^+}(x) \leq T_{n+1}^{\varphi_j, 0^+}(x) \leq \dots \\
&\dots \leq \varphi_j(x) \leq \dots \\
&\dots \leq \mathbb{T}_{n+1}^{\varphi_j, 0^+, c^-}(x) \leq \mathbb{T}_n^{\varphi_j, 0^+, c^-}(x) \leq \dots \leq \mathbb{T}_3^{\varphi_j, 0^+, c^-}(x) \leq \mathbb{T}_2^{\varphi_j, 0^+, c^-}(x)
\end{aligned} \tag{4.6}$$

holds for all $x \in (0, c)$.

Proof. For example, let us consider only case $j = 2$. Since

$$\varphi(x) = \varphi_2(x) = \frac{1}{3} \left(\frac{2 \sin x}{x} + \frac{\tan x}{x} \right) - \frac{1}{2} \left(\left(\frac{x}{\sin x} \right)^2 + \frac{x}{\tan x} \right) = \sum_{k=2}^{\infty} s_k x^{2k},$$

where

$$s_n = s_n^{(2)} = \frac{2(-1)^n}{3(2n+1)!} + \frac{(2^{2n+2}-1)2^{2n+2}}{3(2n+2)!}|B_{2n+2}| - \frac{(n-1)2^{2n}}{(2n)!}|B_{2n}|, \quad n \geq 2.$$

We can prove $s_n > 0$ holds for all $n \geq 2$. In [10, 1.3.1.4] or [46, 1.3.10], we can find the following power series expansion:

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}-1}{(2n)!} 2^{2n} |B_{2n}| x^{2n-1}, \quad |x| < \frac{\pi}{2}. \tag{4.7}$$

Based on (3.9), (3.10), and (4.7) follows

$$\begin{aligned}
\varphi(x) &= \frac{2 \sin x}{3x} + \frac{1 \tan x}{3x} - \frac{1}{2} \left(\frac{x}{\sin x} \right)^2 - \frac{1}{2} \frac{x}{\tan x} \\
&= \frac{2}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} + \frac{1}{3} \sum_{n=1}^{\infty} \frac{2^{2n}-1}{(2n)!} 2^{2n} |B_{2n}| x^{2n-2} \\
&\quad - \frac{1}{2} \left[1 + \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)}{(2n)!} |B_{2n}| x^{2n} \right] - \frac{1}{2} \left[1 - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n} \right] \\
&= \frac{2}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} + \frac{1}{3} \sum_{n=0}^{\infty} \frac{2^{2n+2}-1}{(2n+2)!} 2^{2n+2} |B_{2n+2}| x^{2n} \\
&\quad - \frac{1}{2} \left[1 + \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)}{(2n)!} |B_{2n}| x^{2n} \right] - \frac{1}{2} \left[1 - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n} \right] \\
&= \sum_{n=2}^{\infty} \frac{2(-1)^n}{3(2n+1)!} x^{2n} + \sum_{n=2}^{\infty} \frac{(2^{2n+2}-1)2^{2n+2}}{3(2n+2)!} |B_{2n+2}| x^{2n} \\
&\quad - \sum_{n=2}^{\infty} \frac{2^{2n-1}(2n-1)}{(2n)!} |B_{2n}| x^{2n} + \sum_{n=2}^{\infty} \frac{2^{2n-1}}{(2n)!} |B_{2n}| x^{2n} \\
&= \sum_{n=2}^{\infty} \frac{2(-1)^n}{3(2n+1)!} x^{2n} + \sum_{n=2}^{\infty} \frac{(2^{2n+2}-1)2^{2n+2}}{3(2n+2)!} |B_{2n+2}| x^{2n} - \sum_{n=2}^{\infty} \frac{(n-1)2^{2n}}{(2n)!} |B_{2n}| x^{2n}
\end{aligned}$$

$$= \sum_{n=2}^{\infty} s_n x^{2n}.$$

Next, we shall prove that $s_n > 0$ for all $n \geq 2$.

(i) When n is even,

$$s_n = \frac{2}{3(2n+1)!} + \frac{(2^{2n+2}-1)2^{2n+2}}{3(2n+2)!}|B_{2n+2}| - \frac{(n-1)2^{2n}}{(2n)!}|B_{2n}|,$$

we complete the proof of $s_n > 0$ as long as

$$\frac{(2^{2n+2}-1)2^{2n+2}}{3(2n+2)!}|B_{2n+2}| - \frac{(n-1)2^{2n}}{(2n)!}|B_{2n}| > 0,$$

or

$$\frac{|B_{2n+2}|}{|B_{2n}|} > \frac{\frac{(n-1)2^{2n}}{(2n)!}}{\frac{(2^{2n+2}-1)2^{2n+2}}{3(2n+2)!}} = \frac{(n-1)2^{2n}}{(2n)!} \frac{3(2n+2)!}{(2^{2n+2}-1)2^{2n+2}}.$$

Since

$$\frac{|B_{2n+2}|}{|B_{2n}|} > \frac{2^{2n-1}-1}{2^{2n+1}-1} \frac{(2n+2)(2n+1)}{\pi^2}, \quad (\text{see [1, 25, 26, 35, 45]})$$

we complete the proof when proving

$$\frac{2^{2n-1}-1}{2^{2n+1}-1} \frac{(2n+2)(2n+1)}{\pi^2} > \frac{(n-1)2^{2n}}{(2n)!} \frac{3(2n+2)!}{(2^{2n+2}-1)2^{2n+2}},$$

that is,

$$2^{2n} > \frac{6[\pi^2(n-1)+3]}{8} \quad \text{for } n \geq 2.$$

It is not difficult to prove the above formula by mathematical induction.

(ii) When n is odd,

$$s_n = -\frac{2}{3(2n+1)!} + \frac{(2^{2n+2}-1)2^{2n+2}}{3(2n+2)!}|B_{2n+2}| - \frac{(n-1)2^{2n}}{(2n)!}|B_{2n}|.$$

By

$$\frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1-2^{-2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1-2^{1-2n}}, \quad n = 1, 2, \dots, \quad (\text{see [1]})$$

we have

$$\begin{aligned} s_n &> -\frac{2}{3(2n+1)!} + \frac{(2^{2n+2}-1)2^{2n+2}}{3(2n+2)!} \frac{2(2n+2)!}{(2\pi)^{2n+2}} \frac{1}{1-2^{-2n-2}} \\ &\quad - \frac{(n-1)2^{2n}}{(2n)!} \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1-2^{1-2n}} \\ &= -\frac{2}{3(2n+1)!} + \frac{(2^{2n+2}-1)2^{2n+2}}{3(2n+2)!} \frac{2(2n+2)!}{(2\pi)^{2n+2}} \frac{2^{2n+2}}{2^{2n+2}-1} \\ &\quad - \frac{(n-1)2^{2n}}{(2n)!} \frac{2(2n)!}{(2\pi)^{2n}} \frac{2^{2n-1}}{2^{2n-1}-1} \\ &= \frac{2 \cdot 2^{2n+2}}{3\pi^{2n+2}} - \frac{2^{2n}(n-1)}{(2^{2n-1}-1)\pi^{2n}} - \frac{2}{3(2n+1)!}. \end{aligned}$$

Since

$$s_n > 0 \iff \frac{2}{3} \frac{(4 \cdot 2^{2n} - 3\pi^2 n + 3\pi^2 - 8) 2^{2n}}{\pi^{2n} \pi^2 (2^{2n} - 2)} > \frac{2}{3(2n+1)!}$$

$$\begin{aligned} &\Leftrightarrow \frac{(4 \cdot 2^{2n} - 3\pi^2 n + 3\pi^2 - 8) 2^{2n}}{\pi^{2n+2} (2^{2n} - 2)} > \frac{1}{(2n+1)!} \\ &\Leftrightarrow (4 \cdot 2^{2n} - 3\pi^2 n + 3\pi^2 - 8) 2^{2n} (2n+1)! > \pi^{2n+2} (2^{2n} - 2), \end{aligned}$$

and

$$n! > \left(\frac{n}{3}\right)^n, \quad n \in \mathbb{N},$$

we have

$$(2n+1)! > \left(\frac{2n+1}{3}\right)^{2n+1} > 2^{2n+1}, \quad n \in \mathbb{N}_0,$$

and

$$(4 \cdot 2^{2n} - 3\pi^2 n + 3\pi^2 - 8) 2^{2n} (2n+1)! > (4 \cdot 2^{2n} - 3\pi^2 n + 3\pi^2 - 8) 2^{2n} 2^{2n+1}.$$

Then we complete the proof when proving

$$(4 \cdot 2^{2n} - 3\pi^2 n + 3\pi^2 - 8) 2^{2n} 2^{2n+1} > \pi^{2n+2} (2^{2n} - 2),$$

that is,

$$\begin{aligned} t_n &= (4 \cdot 2^{2n} - 3\pi^2 n + 3\pi^2 - 8) 2^{2n} 2^{2n+1} - \pi^{2n+2} (2^{2n} - 2) \\ &= 8 \cdot 8^{2n} - (2\pi)^{2n} \pi^2 - 2 \cdot 4^{2n} [3\pi^2 (n-1) + 8] + 2\pi^2 \pi^{2n} \\ &> 0 \end{aligned}$$

for all $n \geq 2$. We find

$$t_2 = 28\,672 - 14\pi^6 - 1536\pi^2 = 52.839\dots > 0,$$

and

$$\begin{aligned} t_{n+1} - 64t_n &= [4\pi^2 (4 - \pi) (\pi + 4) 2^{2n} - (128\pi^2 - 2\pi^4)] \pi^{2n} \\ &\quad + 96 \cdot 4^{2n} (3\pi^2 n - 4\pi^2 + 8) \\ &> 0. \end{aligned}$$

Then $t_n > 0$ for all $n \geq 2$. □

Remark 4.3. Obviously, Theorem 2.1 is a simple corollary of Theorem 4.2.

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