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Research Article

On new classes of chains of evolution algebras

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Abstract

The paper is devoted to studying new classes of chains of evolution algebras and their time-depending dynamics and property transition.

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1. Introduction

In the 1920s and 1930s, a new object, the general genetic algebra, was introduced into mathematics as a consequence of the synergy between Mendelian genetics and mathematics. Recognizing algebraic structures and properties in Mendelian genetics was one of the essential steps to start to study genetic algebras. Firstly, Mendel made use of some symbols [17], which expressed his genetic laws in an entirely algebraic manner. They were later named "Mendelian algebras" by several authors. Mendel's laws were mathematically formulated by Serebrowsky [25], who was the first to provide an algebraic interpretation of the sign "×", which suggested sexual reproduction. Later, Glivenkov [10] introduced the so-called Mendelian algebras. Independently, Kostitzin [15] also set forth a "symbolic multiplication" to express Mendel's laws. Etherington [6–8] made a systematic study of the algebras occurring in genetics and introduced the formal language of abstract algebra in the field of genetics. These algebras, in general, are non-associative.

The research on several classes of non-associative algebras (baric, evolution, Bernstein, train, stochastic, etc.) has rendered a notable enrichment to theoretical population genetics. Such classes have been defined at different times by various authors, and all algebras included in these classes are generally referred to as "genetic".

Essential contributions have also been made by Gonshor [11], Schafer [24], Holgate [13, 14], Heuch [12], Reiersöl [21], Abraham [1]. Until the 1980s, the most extensive reference in this area was Wörz-Busekros' book [28]. More recent results, such as evolution theory in genetic algebras, can be seen in Lyubich's book [16]. An excellent survey article is Reed's paper [20]. All algebras studied by these authors are generally called "genetic".

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In the present days, non-Mendelian genetics has become an essential language for molecular geneticists. Some questions arise naturally in this context, such as what new subjects non-Mendelian genetics brings to mathematics, or what type of mathematics leads to a better understanding of non-Mendelian genetics. The systematic formulation of reproduction in non-Mendelian genetics as multiplication in algebras was introduced in [27], leading to the so-called "evolution algebras". These are algebras in which the multiplication tables are motivated by evolution laws of genetics.

Tian in [26] develops the framework of evolution algebra theory and applications in non-Mendelian genetics and Markov chains. The concept of evolution algebra is situated between algebras and dynamical systems. Evolution algebras associated with function spaces defined by graphs, state spaces, and Gibbs measures are studied in [23].

A notion of a chain of evolution algebras was introduced in [4], where the sequence of matrices of structural constants of the chain of evolution algebras satisfies an analogue of the Chapman-Kolmogorov equation. In [22], twenty-five distinct examples of chains of two-dimensional evolution algebras are constructed.

In this paper, we present examples of chains of two-dimensional evolution algebras other than those of [22], by studying the behavior of the baric property, of the set of absolute nilpotent elements and the time-depending dynamics of the set of idempotent elements.

The paper is organized as follows. In Section 2, we give the main concepts related to a chain of evolution algebras. In Section 3, we construct new chains of evolution algebras (CEAs) and study their time-depending dynamics. Finally, in Section 4, we analyze the property transitions of the new CEAs.

2. Chain of evolution algebras

Evolution algebras are defined as follows.

Definition 2.1. Let $(E, \cdot)$ be an algebra over a field K. If it admits a basis $\{e_1, e_2, \dots\}$, such that

$$e_i \cdot e_j = \begin{cases} 0, & \text{if } \; i \neq j; \\ \sum_k a_{ik} e_k, & \text{if } \; i = j, \end{cases}$$

then this algebra is called an *evolution algebra*. The basis is called a natural basis.

The matrix $M = (a_{ij})$ is called the matrix of structural constants.

Evolution algebras have the following primary properties (see [26]). Evolution algebras are not associative, in general; they are commutative, flexible, but not power-associative, in general; direct sums of evolution algebras are also evolution algebras; Kronecker products of evolutions algebras are also evolution algebras.

Let $\{e_1, e_2\}$ be a basis of the two-dimensional evolution algebra E. It is evident that if $\dim E^2 = 0$, then E is an abelian algebra, i.e. an algebra with all products equal to zero. The next theorem gives the classification of the real two-dimensional evolution algebras.

Theorem 2.2 ([19])**.** *Any two-dimensional real evolution algebra E is isomorphic to one of the following pairwise non-isomorphic algebras:*

(i) $\dim E^2 = 1$.

$$E_1 : e_1e_1 = e_1, \quad e_2e_2 = 0, \quad \textit{with matrix } M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};$$

$$E_2 : e_1e_1 = e_1, \quad e_2e_2 = e_1, \quad \textit{with matrix } M_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix};$$

$$E_3 : e_1e_1 = e_1 + e_2, \quad e_2e_2 = -e_1 - e_2, \quad \textit{with matrix } M_3 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix};$$

$E_4: e_1e_1 = e_2, \quad e_2e_2 = 0, \quad$ *with matrix* $M_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$;

$E_5: e_1e_1 = e_2, \quad e_2e_2 = -e_2, \quad$ *with matrix* $M_5 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$;

(ii) $\dim E^2 = 2$.

$E_6(a_2, a_3): e_1e_1 = e_1 + a_2e_2, \quad e_2e_2 = a_3e_1 + e_2, \; 1 - a_2a_3 \neq 0, \; a_2, a_3 \in \mathbb{R},$ *with matrix* $M_6 = \begin{pmatrix} 1 & a_3 \\ a_2 & 1 \end{pmatrix}$. *Moreover,* $E_6(a_2, a_3)$ *is isomorphic to* $E_6(a_3, a_2)$.

$E_7(a_4): e_1e_1 = e_2, \quad e_2e_2 = e_1 + a_4e_2,$ *where* $a_4 \in \mathbb{R}$, *with matrix* $M_7 = \begin{pmatrix} 0 & 1 \\ 1 & a_4 \end{pmatrix}$.

Different authors performed the classification of two-dimensional evolution algebras over several fields. In [5] for the field of complex numbers, in [2] over a field that is closed under all square and cubic roots, and in [3, 9] without restrictions on the underlying field.

Remark 2.3. We notice that the classification of the two-dimensional real evolution algebras consists of an alternative of the complex case [5] or the case [3]. E_5 only appears in the real case. Observe that E_5 is isomorphic to the algebra with matrix $\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$. In the proof of [3, Theorem 3.3], case 1.2.2, the algebra E_5 does not appear since the author considers $c_1 \neq 0$, but if c_1 is negative there is no $\sqrt{c_1}$, and therefore there is one more case. Moreover, the cases (f), (g) and (h) of [3, Theorem 3.3] correspond to $E_6(0, a_3)$ with $a_3 \neq 0$, $E_6(0, 0)$, and $E_7(0)$, respectively.

Following [4] we consider a family $\left\{E^{[s,t]} : \; s, t \in \mathbb{R}, \; 0 \leq s \leq t\right\}$ of n-dimensional evolution algebras over the field $\mathbb{R}$, with basis $e_1, \dots, e_n$, and the multiplication table

$$e_ie_i = \sum_{j=1}^{n} a_{ij}^{[s,t]} e_j, \quad i = 1, \dots, n; \qquad e_ie_j = 0, \quad i \neq j.$$

Here parameters s, t are considered as time, and we define $\mathcal{T} = \{(s, t) : 0 \leq s \leq t, \text{ where } s, t \in \mathbb{R}\}$.

Denote by $M^{[s,t]} = \left(a_{ij}^{[s,t]}\right)_{i,j=1,\dots,n}$ the matrix of structural constants.

Definition 2.4. A family $\left\{E^{[s,t]} : \; s, t \in \mathbb{R}, \; 0 \leq s \leq t\right\}$ of n-dimensional evolution algebras over the field $\mathbb{R}$ is called a *chain of evolution algebras* (CEA) if the matrix $M^{[s,t]}$ of structural constants satisfies the Chapman-Kolmogorov equation

$$M^{[s,t]} = M^{[s,\tau]} M^{[\tau,t]}, \quad \text{for any} \quad s < \tau < t. \tag{2.1}$$

3. Construction of chains of evolution algebras

To construct a chain of two-dimensional evolution algebras, we need to solve equation (2.1) for the 2×2 matrix $\mathcal{M}^{[s,t]}$. This equation provides the following system of functional equations (with four unknown functions):

$$\begin{aligned} a_{11}^{[s,t]} &= a_{11}^{[s,\tau]} a_{11}^{[\tau,t]} + a_{12}^{[s,\tau]} a_{21}^{[\tau,t]}, \\ a_{12}^{[s,t]} &= a_{11}^{[s,\tau]} a_{12}^{[\tau,t]} + a_{12}^{[s,\tau]} a_{22}^{[\tau,t]}, \\ a_{21}^{[s,t]} &= a_{21}^{[s,\tau]} a_{11}^{[\tau,t]} + a_{22}^{[s,\tau]} a_{21}^{[\tau,t]}, \\ a_{22}^{[s,t]} &= a_{21}^{[s,\tau]} a_{12}^{[\tau,t]} + a_{22}^{[s,\tau]} a_{22}^{[\tau,t]}. \end{aligned} \tag{3.1}$$

But the general analysis of system (3.1) is complicated.

In [18] we studied the classification dynamics of known two-dimensional chains of evolution algebras constructed in [22] and showed that known chains of evolution algebras never contain an evolution algebra isomorphic to E_4 in any time s,t (see Theorem 2.2). In this section, we will construct CEAs, including E_4 for some period of time.

To construct a CEA that will be isomorphic to E_4 at some time interval, we need the following theorem.

Theorem 3.1 ([18]). *An evolution algebra $E_{\mathcal{M}}$ is isomorphic to E_4 if and only if $E_{\mathcal{M}}$ has the matrix of structural constants in the following form:*

$$\mathcal{M}_1 = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \mathcal{M}_2 = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}, \quad \text{where} \quad \beta, \gamma \in \mathbb{R}. \tag{3.2}$$

Thus, we should construct CEAs with the matrix of structural constants that are listed in (3.2).

Consider (3.1) with $a_{11}^{[s,t]} = \alpha(s,t)$, $a_{12}^{[s,t]} = \beta(s,t)$, $a_{21}^{[s,t]} = \gamma(s,t)$, $a_{22}^{[s,t]} = \delta(s,t)$. Therefore, to find a CEA, we should solve the next equation:

$$\begin{pmatrix} \alpha(s,\tau) & \beta(s,\tau) \\ \gamma(s,\tau) & \delta(s,\tau) \end{pmatrix} \cdot \begin{pmatrix} \alpha(\tau,t) & \beta(\tau,t) \\ \gamma(\tau,t) & \delta(\tau,t) \end{pmatrix} = \begin{pmatrix} \alpha(s,t) & \beta(s,t) \\ \gamma(s,t) & \delta(s,t) \end{pmatrix}. \tag{3.3}$$

Case 1.1. If we consider in (3.3), $\alpha(s,t) = \gamma(s,t) \equiv 0, \beta(s,t) \neq 0, \delta(s,t) \neq 0$, then we have the following:

$$\begin{pmatrix} 0 & \beta(s,\tau) \\ 0 & \delta(s,\tau) \end{pmatrix} \cdot \begin{pmatrix} 0 & \beta(\tau,t) \\ 0 & \delta(\tau,t) \end{pmatrix} = \begin{pmatrix} 0 & \beta(s,t) \\ 0 & \delta(s,t) \end{pmatrix}. \tag{3.4}$$

From (3.4), we get the following system of functional equations:

$$\begin{cases} \beta(s,\tau)\delta(\tau,t) = \beta(s,t), \\ \delta(s,\tau)\delta(\tau,t) = \delta(s,t). \end{cases} \tag{3.5}$$

The second equation of system (3.5) is known as Cantor's second equation, which has the following solutions:

(1) $\delta(s,t) \equiv 0$;
(2) $\delta(s,t) = \frac{\phi(t)}{\phi(s)}$, where ϕ is an arbitrary function with $\phi(s) \neq 0$;
(3) $\delta(s,t) = \begin{cases} 1, & \text{if } 0 < s \leq t < a; \\ 0, & \text{if } t \geq a. \end{cases}$

Substituting these solutions into the first equation of (3.5), we find $\beta(s,t)$:

(1) $\beta(s,t) \equiv 0$;
(2) $\beta(s,t) = \rho(s)\phi(t)$, where ρ is an arbitrary function;
(3) $\beta(s,t) = \begin{cases} \sigma(s), & \text{if } 0 < s \leq t < a; \\ 0, & \text{if } t \geq a, \end{cases}$

where σ is an arbitrary function;

From these solutions, we have the following matrices of structural constants of CEAs:

$$\mathcal{M}_0^{[s,t]} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$\mathcal{M}_1^{[s,t]} = \begin{pmatrix} 0 & \rho(s)\phi(t) \\ 0 & \frac{\phi(t)}{\phi(s)} \end{pmatrix},$$

where ρ, ϕ are arbitrary functions, with $\phi(s) \neq 0$;

$$\mathcal{M}_2^{[s,t]} = \begin{cases} \begin{pmatrix} 0 & \sigma(s) \\ 0 & 1 \end{pmatrix}, & \text{if } 0 < s \leq t < a; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } t \geq a, \end{cases}$$

where $a > 0$ and σ is an arbitrary function.

Case 1.2. Consider the case $\alpha(s,t) = \beta(s,t) \equiv 0, \gamma(s,t) \neq 0, \delta(s,t) \neq 0$. Then from (3.3), we have the following:

$$\begin{pmatrix} 0 & 0 \\ \gamma(s,\tau) & \delta(s,\tau) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ \gamma(\tau,t) & \delta(\tau,t) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \gamma(s,t) & \delta(s,t) \end{pmatrix}.$$

From the last equality, we have the following system of equations:

$$\begin{cases} \delta(s,\tau)\gamma(\tau,t) = \gamma(s,t), \\ \delta(s,\tau)\delta(\tau,t) = \delta(s,t). \end{cases} \tag{3.6}$$

The second equation (Cantor's second equation) of system (3.6) has the following solutions:

(1) $\delta(s,t) \equiv 0$;

(2) $\delta(s,t) = \frac{\varphi(t)}{\varphi(s)}$, where φ is an arbitrary function with $\varphi(s) \neq 0$;

(3) $\delta(s,t) = \begin{cases} 1, & \text{if } 0 < s \leq t < a; \\ 0, & \text{if } t \geq a. \end{cases}$

Substituting these solutions into the first equation of (3.6), we find $b(s,t)$:

(1) $\gamma(s,t) \equiv 0$;

(2) $\gamma(s,t) = \frac{f(t)}{\varphi(s)}$, where f is an arbitrary function;

(3) $\gamma(s,t) = \begin{cases} g(t), & \text{if } 0 < s \leq t < a; \\ 0, & \text{if } t \geq a. \end{cases}$ where g is an arbitrary function.

From these solutions, we have the next matrices of structural constants of CEAs:

$$\mathcal{M}_0^{[s,t]} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathcal{M}_3^{[s,t]} = \begin{pmatrix} 0 & 0 \\ \frac{f(t)}{\varphi(s)} & \frac{\varphi(t)}{\varphi(s)} \end{pmatrix},$$

where f, φ are arbitrary functions, $\varphi(s) \neq 0$;

$$\mathcal{M}_4^{[s,t]} = \begin{cases} \begin{pmatrix} 0 & 0 \\ g(t) & 1 \end{pmatrix}, & \text{if } 0 < s \leq t < a; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } t \geq a, \end{cases}$$

where $a > 0$ and g is an arbitrary function.

Case 1.3. Let us try to find the solution satisfying the following:

$$\begin{pmatrix} \alpha(s,\tau) & \beta(s,\tau) \\ \gamma(s,\tau) & \delta(s,\tau) \end{pmatrix} \cdot \begin{pmatrix} \alpha(\tau,t) & \beta(\tau,t) \\ \gamma(\tau,t) & \delta(\tau,t) \end{pmatrix} = \begin{pmatrix} 0 & \beta(s,t) \\ 0 & 0 \end{pmatrix}. \tag{3.7}$$

From (3.7) we have the next system of functional equations:

$$\begin{cases} \alpha(s,\tau)\alpha(\tau,t)+\beta(s,\tau)\gamma(\tau,t)=0, \\ \alpha(s,\tau)\beta(\tau,t)+\beta(s,\tau)\delta(\tau,t)=\beta(s,t), \\ \gamma(s,\tau)\alpha(\tau,t)+\delta(s,\tau)\gamma(\tau,t)=0, \\ \gamma(s,\tau)\beta(\tau,t)+\delta(s,\tau)\delta(\tau,t)=0. \end{cases} \tag{3.8}$$

Let $\alpha(s,t)=\gamma(s,t)=0$. Then we get:

$$\begin{cases} \beta(s,\tau)\delta(\tau,t)=\beta(s,t), \\ \delta(s,\tau)\delta(\tau,t)=0. \end{cases} \tag{3.9}$$

To find a non-zero solution of the system of equations (3.9), we should prove that the equation

$$\delta(s,\tau)\delta(\tau,t)=0, \quad \text{for all } \ s<\tau<t, \tag{3.10}$$

has a non-zero solution. Indeed, take $C>0$ and

$$\delta(s,t)=\begin{cases} 0, & \text{if } \ 0<C\le s<t \ \text{ or } \ 0<s<t\le C; \\ f(s,t), & \text{if } \ 0<s<C<t, \end{cases} \tag{3.11}$$

where $f(s,t)$ is an arbitrary non-zero function.

Now, we show that independently on $f(s,t)$ the function (3.11) satisfies (3.10): for a given $C>0$, we only have two possibilities by taking an arbitrary τ such that $s<\tau<t$:

Case 1.3.1. Let $\tau\le C$. By the defined function (3.11), we have that $\delta(s,\tau)=0$ and for $\delta(\tau,t)$:

$$\delta(\tau,t)=\begin{cases} 0, & \text{if } \ t\le C; \\ f(\tau,t), & \text{if } \ t>C, \end{cases} \tag{3.12}$$

where $f(\tau,t)$ is the function fixed in (3.11).

Therefore, $\delta(s,\tau)\delta(\tau,t)=0$.

Case 1.3.2. $\tau>C$. Also from (3.11), we have that $\delta(\tau,t)=0$ and for $\delta(s,\tau)$:

$$\delta(s,\tau)=\begin{cases} f(s,\tau), & \text{if } \ s<C; \\ 0, & \text{if } \ s\ge C, \end{cases}$$

where $f(s,\tau)$ is the function fixed in (3.11).

Therefore, $\delta(s,\tau)\delta(\tau,t)=0$.

Thus, we have proved that the function (3.11) satisfies equation (3.10).

Now we should find solutions to the first equation of system (3.9):

$$\beta(s,\tau)\delta(\tau,t)=\beta(s,t), \qquad s<\tau<t, \tag{3.13}$$

where $\delta(\tau,t)$ is given by (3.11).

To find a solution, we have the next possibilities:

Case 1.3.3. Let $\tau\le C$. Then by the defined function (3.11) we have that $\delta(s,\tau)=0$ and from (3.12) in a period of time $t\le C$, $\delta(\tau,t)=0$, and so from (3.13) we have $\beta(s,t)=0$. When $t>C$, $\delta(\tau,t)=f(\tau,t)$ and by (3.13) we have to solve the next equation:

$$\beta(s,\tau)f(\tau,t)=\beta(s,t), \qquad s<\tau<t. \tag{3.14}$$

We solve (3.14) for some particular cases:

Case 1.3.3.1 Consider $\beta(s,t)=f(s,t)$. Then from (3.14), we have $f(s,\tau)f(\tau,t)=f(s,t)$, which is Cantor's second equation. As $f(s,t)$ is a non-zero function, then we have the next solution:

$$f(s,t) = \frac{\Phi(t)}{\Phi(s)},$$

where Φ is an arbitrary function, with $\Phi(s) \neq 0$.

Thus we have the next solution of system (3.8):

$$\alpha(s,t) \equiv 0,$$
$$\beta(s,t) = \begin{cases} 0, & \text{if } s < t \leq C; \\ \frac{\Phi(t)}{\Phi(s)}, & \text{if } t > C, \end{cases}$$
$$\gamma(s,t) \equiv 0,$$
$$\delta(s,t) = \begin{cases} 0, & \text{if } 0 < C \leq s < t \text{ or } 0 < s < t \leq C; \\ \frac{\Phi(t)}{\Phi(s)}, & \text{if } s < C < t, \end{cases}$$

where $C > 0$ and Φ is an arbitrary function, with $\Phi(s) \neq 0$.

Then we have the next matrix of structural constants:

$$\mathcal{M}_5^{[s,t]} = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } s < t \leq C; \\ \begin{pmatrix} 0 & \frac{\Phi(t)}{\Phi(s)} \\ 0 & 0 \end{pmatrix}, & \text{if } t > C, \end{cases}$$

where $C > 0$ and Φ is an arbitrary function, with $\Phi(t) \neq 0$.

Case 1.3.3.2. Let $\beta(s,t) \neq f(s,t)$. As $f(\tau,t)$ is an arbitrary non-zero function, consider $f(\tau,t) = \frac{\phi(\tau)}{\phi(t)}$, with $\phi(t) \neq 0$. Then from (3.14) we have the following:

$$\beta(s,\tau) \cdot \frac{\phi(\tau)}{\phi(t)} = \beta(s,t),$$
$$\beta(s,t)\phi(t) = \beta(s,\tau)\phi(\tau).$$

From the last equality, we can see $\beta(s,t)\phi(t)$ does not depend on t, i.e. there exists a function $\rho(s)$ such that $\beta(s,t)\phi(t) = \rho(s)$. Therefore, $\beta(s,t) = \frac{\rho(s)}{\phi(t)}$.

Then we get the next solution of system (3.8):

$$\alpha(s,t) \equiv 0,$$
$$\beta(s,t) = \begin{cases} 0, & \text{if } s < t \leq C; \\ \frac{\rho(s)}{\phi(t)}, & \text{if } t > C, \end{cases}$$
$$\gamma(s,t) \equiv 0,$$
$$\delta(s,t) = \begin{cases} 0, & \text{if } 0 < C \leq s < t \text{ or } 0 < s < t \leq C; \\ \frac{\phi(s)}{\phi(t)}, & \text{if } s < C < t, \end{cases}$$

where $C > 0$ and ϕ, ρ are arbitrary functions with $\phi(t) \neq 0$.

Then we have, respectively, the next matrix of structural constants to the solution:

$$\mathcal{M}_6^{[s,t]} = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } s < t \leq C; \\ \begin{pmatrix} 0 & \frac{\rho(s)}{\phi(t)} \\ 0 & 0 \end{pmatrix}, & \text{if } t > C, \end{cases}$$

where $C > 0$ and ϕ, ρ are arbitrary functions with $\phi(t) \neq 0$.

Case 1.3.4. When $\tau > C$, then by the defined function (3.11) we have that $\delta(\tau,t) = 0$. So from (3.13), we have $\beta(s,t) = 0$. Thus we get the trivial CEA.

Case 1.4. Let us try to find the solution satisfying:

$$\begin{pmatrix}\alpha(s,\tau) & \beta(s,\tau)\\ \gamma(s,\tau) & \delta(s,\tau)\end{pmatrix}\cdot\begin{pmatrix}\alpha(\tau,t) & \beta(\tau,t)\\ \gamma(\tau,t) & \delta(\tau,t)\end{pmatrix}=\begin{pmatrix}0 & 0\\ \gamma(s,t) & 0\end{pmatrix}. \tag{3.15}$$

From equality (3.15) we have the next system of functional equations:

$$\begin{cases}\alpha(s,\tau)\alpha(\tau,t)+\beta(s,\tau)\gamma(\tau,t)=0,\\ \alpha(s,\tau)\beta(\tau,t)+\beta(s,\tau)\delta(\tau,t)=0,\\ \gamma(s,\tau)\alpha(\tau,t)+\delta(s,\tau)\gamma(\tau,t)=\gamma(s,t),\\ \gamma(s,\tau)\beta(\tau,t)+\delta(s,\tau)\delta(\tau,t)=0.\end{cases}$$

Let $\alpha(s,t)=\beta(s,t)=0$. Then we have the next system:

$$\begin{cases}\delta(s,\tau)\gamma(\tau,t)=\gamma(s,t),\\ \delta(s,\tau)\delta(\tau,t)=0.\end{cases}$$

The analysis of this system is similar to (3.9), and we get the following CEAs:

$$\mathcal{M}_7^{[s,t]}=\begin{cases}\begin{pmatrix}0 & 0\\ \frac{\Psi(t)}{\Psi(s)} & 0\end{pmatrix}, & \text{if} \quad s<C;\\ \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, & \text{if} \quad s\geq C,\end{cases}$$

where $C>0$ and Ψ is an arbitrary function, with $\Psi(t)\neq 0$;

$$\mathcal{M}_8^{[s,t]}=\begin{cases}\begin{pmatrix}0 & 0\\ \frac{\sigma(t)}{\varphi(s)} & 0\end{pmatrix}, & \text{if} \quad s<C;\\ \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, & \text{if} \quad s\geq C,\end{cases}$$

where $C>0$ and φ,σ are arbitrary functions with $\varphi(s)\neq 0$.

Denote by $E_i^{[s,t]}$ the CEA with matrix $\mathcal{M}_i^{[s,t]}$.

Remark 3.2. We should note that from the CEAs $E_i^{[s,t]}, i=1,\dots,8$, only $E_3^{[s,t]}$ coincides with the CEA $E_{16}^{[s,t]}$ constructed in [22] and it has the same dynamic. All other CEAs are different from CEAs constructed in [22] and have different dynamics.

Now, we provide the time-depending dynamics of these CEAs:

Theorem 3.3. *For the next CEAs hold:*

$$E_1^{[s,t]} \simeq \begin{cases} E_1 & \textit{for all} \ \ (s,t) \in \{(s,t) : s < t, \ \ \rho(s) = 0\}\,, \\ E_2 & \textit{for all} \ \ (s,t) \in \{(s,t) : s < t, \ \ \rho(s) \neq 0\}\,; \end{cases}$$

$$E_2^{[s,t]} \simeq \begin{cases} E_1 & \textit{for all} \ \ (s,t) \in \{(s,t) : s < t < a, \ \ \sigma(s) = 0\}\,, \\ E_2 & \textit{for all} \ \ (s,t) \in \{(s,t) : s < t < a, \ \ \sigma(s) \neq 0\}\,, \\ E_0 & \textit{for all} \ \ (s,t) \in \{(s,t) : t \geq a\}\,; \end{cases}$$

$E_3^{[s,t]} \simeq E_1$ *for any* $(s,t) \in \mathfrak{T}$;

$$E_4^{[s,t]} \simeq \begin{cases} E_1 & \textit{for all} \ \ (s,t) \in \{(s,t) : s < t < a\}\,, \\ E_0 & \textit{for all} \ \ (s,t) \in \{(s,t) : t \geq a\}\,; \end{cases}$$

$$E_5^{[s,t]} \simeq \begin{cases} E_0 & \textit{for all} \ \ (s,t) \in \{(s,t) : s < t \leq C\}\,, \\ E_4 & \textit{for all} \ \ (s,t) \in \{(s,t) : t > C\}\,; \end{cases}$$

$$E_6^{[s,t]} \simeq \begin{cases} E_0 & \textit{for all} \ \ (s,t) \in \{(s,t) : s < t \leq C\}\,, \\ E_0 & \textit{for all} \ \ (s,t) \in \{(s,t) : t > C, \ \ \rho(s) = 0\}\,, \\ E_4 & \textit{for all} \ \ (s,t) \in \{(s,t) : t > C, \ \ \rho(s) \neq 0\}\,; \end{cases}$$

$$E_7^{[s,t]} \simeq \begin{cases} E_4 & \textit{for all} \ \ (s,t) \in \{(s,t) : s < C\}\,, \\ E_0 & \textit{for all} \ \ (s,t) \in \{(s,t) : s \geq C\}\,; \end{cases}$$

$$E_8^{[s,t]} \simeq \begin{cases} E_0 & \textit{for all} \ \ (s,t) \in \{(s,t) : s < C, \ \ \sigma(t) = 0\}\,, \\ E_4 & \textit{for all} \ \ (s,t) \in \{(s,t) : s < C, \ \ \sigma(t) \neq 0\}\,, \\ E_0 & \textit{for all} \ \ (s,t) \in \{(s,t) : s \geq C\}\,. \end{cases}$$

Proof. When $\rho(s) = 0$, then $E_1^{[s,t]} \simeq E_1$, for all $s,t \in \mathfrak{T}$ by the change of basis $e_1' = e_1,\ e_2' = \frac{\phi(s)}{\phi(t)} e_2$, and when $\rho(s) \neq 0$, it is isomorphic to E_2, for all $s,t \in \mathfrak{T}$ by the change of basis $e_1' = \frac{1}{\rho(s)\phi(t)} e_1,\ e_2' = \frac{\phi(s)}{\phi(t)} e_2$.

When $\sigma(s) = 0$, then $E_2^{[s,t]} \simeq E_1$, for all $s,t \in \mathfrak{T},\ s < t < a$, by the change of basis $e_1' = e_1,\ e_2' = e_2$, and when $\sigma(s) \neq 0$, it is isomorphic to E_2, for all $s,t \in \mathfrak{T},\ s < t < a$, by the change of basis $e_1' = \frac{1}{\sigma(s)} e_1,\ e_2' = e_2$. In the period of time $t \geq a$, it will be isomorphic to the trivial evolution algebra E_0.

$E_3^{[s,t]} \simeq E_1$, for all $s,t \in \mathfrak{T}$ by the change of basis $e_2' = \frac{f(t)\varphi(s)}{\varphi^2(t)} e_1 + \frac{\varphi(s)}{\varphi(t)} e_2,\ e_2' = e_1$.

$E_4^{[s,t]} \simeq E_1$, for all $s,t \in \mathfrak{T},\ s < t < a$, by the change of basis $e_1' = \sigma(t)e_1 + e_2,\ \ e_2' = e_1$, in the period of time $t \geq a$, it will be isomorphic to the trivial evolution algebra E_0.

$E_5^{[s,t]} \simeq E_4$, for all $s,t \in \mathfrak{T},\ t > C$, by the change of basis $e_1' = \frac{\Phi(s)}{\Phi(t)} e_1,\ \ e_2' = e_2$, in the period of time $s < t \leq C$, it will be isomorphic to the trivial evolution algebra E_0.

When $\rho(s) \neq 0$, then $E_6^{[s,t]} \simeq E_4$, for all $s,t \in \mathfrak{T},\ t > C$, by the change of basis $e_1' = \frac{\phi(t)}{\rho(s)} e_1,\ \ e_2' = e_2$, in the period of time $s < t \leq C$, and when $\rho(s) = 0$, then it will be isomorphic to the trivial evolution algebra E_0.

$E_7^{[s,t]} \simeq E_4$, for all $s,t \in \mathfrak{T},\ s < C$, by the change of basis $e_1' = \frac{\Psi(s)}{\Psi(t)} e_1,\ \ e_2' = e_2$, in the period of time $s \geq C$, it will be isomorphic to the trivial evolution algebra E_0.

When $\sigma(t) \neq 0$, then $E_6^{[s,t]} \simeq E_4$, for all $s,t \in \mathfrak{T},\ s < C$, by the change of basis $e_1' = \frac{\varphi(s)}{\sigma(t)} e_1,\ \ e_2' = e_2$, in the period of time $s \geq C$, and when $\sigma(t) = 0$, then it will be isomorphic to the trivial evolution algebra E_0. □

Thus, we proved that there exist CEAs that for some values of time will be isomorphic to E_4.

4. Property transition

In this section, we will study property transitions of the CEAs $E_i^{s,t}, i = 0.\dots, 8$.

In [4], we provided the ideas of property transition for CEAs. We recall these definitions.

Definition 4.1. Assume a CEA, $E^{[s,t]}$, has a property, say P, at pair of times (s_0, t_0); one says that the CEA has P *property transition* if there is a pair $(s,t) \neq (s_0, t_0)$ at which the CEA has no property P.

Denote

$$\mathcal{T} = \{(s,t) : 0 \leq s \leq t\};$$
$$\mathcal{T}_P = \{(s,t) \in \mathcal{T} : E^{[s,t]} \text{ has property } P\};$$
$$\mathcal{T}_P^0 = \mathcal{T} \setminus \mathcal{T}_P = \{(s,t) \in \mathcal{T} : E^{[s,t]} \text{ has no property } P\}.$$

The sets have the following meaning:

- $\mathcal{T}_P$-the duration of the property P;
- $\mathcal{T}_P^0$-the lost duration of the property P.

The partition $\{\mathcal{T}_P, \mathcal{T}_P^0\}$ of the set $\mathcal{T}$ is called the P property diagram.

For example, if P =commutativity, then we determine that any CEA has not commutativity property transition because any evolution algebra is commutative.

4.1. Baric property transition

A *character* for an algebra A is a nonzero multiplicative linear form on A, i.e. a nonzero algebra homomorphism $\sigma\colon A \to \mathbb{R}$ (see [16]). Not every algebra carries a character. For example, an algebra with the zero multiplication has no character.

Definition 4.2. A pair (A, σ) consisting of an algebra A and a character σ on A is called a *baric algebra*. The homomorphism σ is called the weight (or baric) function of A and $\sigma(x)$ the weight (baric value) of x.

There is a character $\sigma(x) = \sum_i x_i$ for the evolution algebra of a free population (see [16]); therefore, that algebra is baric. But the evolution algebra E introduced in [26] is not baric, in general. The following theorem provides a criterion for an evolution algebra E to be baric.

Theorem 4.3 ([4])**.** *An n-dimensional evolution algebra E, over the field $\mathbb{R}$, is baric if and only if there is a column $(a_{1i_0}, \dots, a_{ni_0})^T$ of its structural constants matrix $\mathcal{M} = (a_{ij})_{i,j=1,\dots,n}$, such that $a_{i_0i_0} \neq 0$ and $a_{ii_0} = 0$, for all $i \neq i_0$. Moreover, the corresponding weight function is $\sigma(x) = a_{i_0i_0}x_{i_0}$.*

Since an evolution algebra is not a baric algebra, in general, using Theorem 4.3, we can give the baric property diagram. Let us do this for the above-given chains $E_i^{[s,t]}$, $i = 0, \dots, 8$.

Denote by $\mathcal{T}_b^{(i)}$ the baric property duration of the CEA $E_i^{[s,t]}$, $i = 0, \dots, 8$.

Theorem 4.4.

(i) (There is no non-baric property transition) *The algebras $E_i^{[s,t]}$, $i = 0, 1, 2, 5, 6, 7, 8$, are not baric for any time $(s,t) \in \mathcal{T}$;*

(ii) (There is no baric property transition) *The algebra $E_3^{[s,t]}$ is baric for any time $(s,t) \in \mathcal{T}$;*

(iii) (There is baric property transition) *The CEA $E_4^{[s,t]}$ has baric property transition with baric property duration set as the following*

$$\mathcal{T}_b^{(4)} = \{(s,t) \in \mathcal{T} : s \leq t < a\}.$$

Proof. By Theorem 4.3, a two-dimensional evolution algebra $E^{[s,t]}$ is baric if and only if $a_{11}^{[s,t]} \neq 0$, $a_{21}^{[s,t]} = 0$ or $a_{22}^{[s,t]} \neq 0$, $a_{12}^{[s,t]} = 0$. The assertions of the theorem are results of the meticulous checking of these conditions. □

4.2. Absolute nilpotent elements transition

Recall that the element x of an algebra A is called an *absolute nilpotent* if $x^2 = 0$.

Let $E = \mathbb{R}^n$ be an evolution algebra over the field $\mathbb{R}$ with structural constant coefficients matrix $\mathcal{M} = (a_{ij})$. Then for arbitrary $x = \sum_i x_i e_i$ and $y = \sum_i y_i e_i \in \mathbb{R}^n$, we have

$$xy = \sum_j \left(\sum_i a_{ij} x_i y_i \right) e_j, \qquad x^2 = \sum_j \left(\sum_i a_{ij} x_i^2 \right) e_j.$$

For an n-dimensional evolution algebra $\mathbb{R}^n$ consider the operator $V \colon \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto V(x) = x'$, defined as

$$x'_j = \sum_{i=1}^{n} a_{ij} x_i^2, \qquad j = 1, \ldots, n. \tag{4.1}$$

This operator is called an *evolution operator* [16].

We have $V(x) = x^2$, hence the equation $V(x) = x^2 = 0$ is given by the following system

$$\sum_i a_{ij} x_i^2 = 0, \qquad j = 1, \ldots, n. \tag{4.2}$$

In this section, we shall solve system (4.2) for $E_i^{[s,t]}$, $i = 0, \ldots, 8$.

For a CEA $E_i^{[s,t]}$ with matrix $\mathcal{M}_i^{[s,t]}$ denote

$$\mathcal{T}_{nil}^{(i)} = \{(s,t) \in \mathcal{T} : E_i^{[s,t]} \text{ has a unique absolute nilpotent}\}, \quad \mathcal{T}_{nil}^{0} = \mathcal{T} \setminus \mathcal{T}_{nil}.$$

The following theorem answers the problem of the existence of "uniqueness of absolute nilpotent element" property transition.

Theorem 4.5.

(1) *There CEAs* $E_i^{[s,t]}$, $i = 0, 3, 4, 5, 6, 7, 8$, *have infinitely many of absolute nilpotent elements for any time* $(s,t) \in \mathcal{T}$.

(2) *The CEAs* $E_i^{[s,t]}$, $i = 1, 2$, *have "uniqueness of absolute nilpotent element" property transition with the property duration sets as the following*

$$\mathcal{T}_{nil}^{(1)} = \{(s,t) \in \mathcal{T} : \rho(s)\phi(s) > 0\},$$
$$\mathcal{T}_{nil}^{(2)} = \{(s,t) \in \mathcal{T} : s \leq t < a,\ \sigma(s) > 0\}.$$

Proof. The proof consists of the simple examination of the solutions of system (4.2) for each $E_i^{[s,t]}$, $i = 0, \ldots, 8$. □

4.3. Idempotent elements transition

A element x of an algebra $\mathcal{A}$ is called *idempotent* if $x^2 = x$. The idempotents of an evolution algebra are especially significant because they are the fixed points of the evolution operator V (4.1), i.e. $V(x) = x$. We denote by $\mathcal{I}d(E)$ the set of idempotent elements of an algebra E. Using (4.1) the equation $x^2 = x$ can be written as

$$x_j = \sum_{i=1}^{n} a_{ij} x_i^2, \qquad j = 1, \ldots, n. \tag{4.3}$$

The extensive analysis of the solutions of system (4.3) is very hard. We shall solve this problem for the CEAs $E_i^{[s,t]}$, $i = 0, \ldots, 8$.

The following theorem provides the time-dynamics of the idempotent elements for the algebras $E_i^{[s,t]}$, $i = 0, \ldots, 8$.

Theorem 4.6.

(1) *The algebras* $E_i^{[s,t]}$, $i = 0, 5, 6, 7, 8$, *have a unique idempotent* $(0,0)$ *in any time* $(s,t) \in \mathfrak{T}$.

(2) *The algebra* $E_1^{[s,t]}$ *has two idempotents* $(0,0)$, $(0, \frac{\phi(s)}{\phi(t)})$ *for all* $(s,t) \in \{(s,t) : s \le t < a\}$.

(3) *The algebra* $E_2^{[s,t]}$ *has two idempotents* $(0,0)$, $(0,1)$ *in any time* $(s,t) \in \mathfrak{T}$.

(4) *The algebra* $E_3^{[s,t]}$ *has two idempotents* $(0,0)$, $(\frac{f(t)\phi(s)}{\phi^2(t)}, \frac{\phi(s)}{\phi(t)})$ *in any time* $(s,t) \in \mathfrak{T}$.

(5) *The algebra* $E_4^{[s,t]}$ *has two idempotents* $(0,0)$, $(g(t), 1)$ *for all* $(s,t) \in \{(s,t) : s \le t < a\}$.

Proof. The proof contains a precise analysis of the solutions of system (4.3) for each $E_i^{[s,t]}$, $i = 0, \ldots, 8$. □

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