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ON PROXIMITY SPACES AND TOPOLOGICAL HYPER NEARRINGS

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Abstract. In 1934 the concept of algebraic hyperstructures was first introduced by a French mathematician, Marty. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the result of this composition is a set. In this paper, we prove some results in topological hyper nearring. Then we present a proximity relation on an arbitrary hyper nearring and show that every hyper nearring with a topology that is induced by this proximity is a topological hyper nearring. In the following, we prove that every topological hyper nearring can be a proximity space.

1. Introduction

In 1934, the concept of hypergroups was first introduced by a French mathematician, Marty [22]. In the following, it was studied and extended by many researchers, namely, Corsini [3], Corsini and Leoreanu [4], Davvaz [6–8], Frenni [12], Koskas [20], Mittas [23], Vougiouklis, and others. The topological hyper nearring notion is defined and studied by Borhani and Davvaz in [2].

In the 1950's, Efremovič [10, 11], a Russian mathematician, gave the definition of proximity space, which he called infinitesimal space in a series of his papers. He axiomatically characterized the proximity relation A is near B for subsets A and B of any set X. The set X, together with this relation, was called an infinitesimal (proximity) space. Defining the closure of a subset A of X to be the collection of all points of X near A, Efremovič [10,11] showed that a topology can be introduced in a proximity space.

In this paper, we study some remarks on topological hyper nearring, then we

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define a proximity relation on hyper nearring and, we will prove that every hyper nearring with a topology that is induced by this proximity is a topological hyper nearring. In the following, we show that every topological hyper nearring is a proximity space.

2. Preliminaries

In this section, we recall some basic classical definitions of topology from [21] and definitions related to hyperstructures that are used in what follows.

Definition 1. [6] A *hyper nearring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a quasi canonical hypergroup, i.e., in $(R, +)$ the following conditions hold:

(i) $x + (y + z) = (x + y) + z$ for all $x, y, z \in R$;
(ii) There is $0 \in R$ such that $x + 0 = 0 + x = x$, for all $x \in R$;
(iii) For any $x \in R$ there exists one and only one $x' \in R$ such that $0 \in x + x'$ (we shall write $-x$ for x' and we call it the opposite of x);
(iv) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

If A and B are two non-empty subsets of R and $x \in R$, then we define:

$$A + B = \bigcup_{\substack{a \in A \\ b \in B}} a + b, \ x + A = \{x\} + A and A + x = A + \{x\}.$$

(2) $(R, \cdot)$ is a semigroup respect to the multiplication, having 0 as a left absorbing element, i.e., $x \cdot 0 = 0$ for all $x \in R$. But, in general, $0 \cdot x \neq 0$ for some $x \in R$.

(3) The multiplication is left distributive with respect to the hyperoperation $+$, i.e., $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

Note that for all $x, y \in R$, we have $-(-x) = x$, $0 = -0$, $-(x + y) = -y - x$ and $x(-y) = -xy$. Let R and S be two hyper nearrings. The map $f : R \to S$ is called a homomorphism if for all $x, y \in R$, the following conditions hold: $f(x+y) = f(x) + f(y)$, $f(x \cdot y) = f(x) \cdot f(y)$ and $f(0) = 0$. It is easy to see that if f is a homomorphism, then $f(-x) = -f(x)$, for all $x \in R$. A nonempty subset H of a hyper nearring R is called a *subhyper nearring* if $(H, +)$ is a subhypergroup of $(R, +)$, i.e., (1) $a, b \in H$ implies $a + b \subseteq H$; (2) $a \in H$ implies $-a \in H$; and (3) $(H, \cdot)$ is a subsemigroup of $(R, \cdot)$. A subhypergroup A of the hypergroup $(R, +)$ is called *normal* if for all $x \in R$, we have $x + A - x \subseteq A$. Let H be a normal hyper R-subgroup of hyper nearrring R. In [14], Heidari et al. defined the relation

$$x \sim y (\operatorname{mod} H) \text{ if and only if } (x - y) \cap H \neq \emptyset, \text{ for all } x, y \in H.$$

This relation is a regular equivalence relation on R. Let $\rho(x)$ be the equivalence class of the element $x \in H$ and denote the quotient set by R/H. Define the hyperoperation $\oplus$ and multiplication $\odot$ on R/H by

$$\begin{aligned}\rho(a) \oplus \rho(b) &= \{\rho(c) \ : \ c \in \rho(a) + \rho(b)\},\\ \rho(a) \odot \rho(b) &= \rho(a \cdot b),\end{aligned}$$

for all $a, b \in R$. Let $(R, +, \cdot)$ be a hyper nearring and τ a topology on R. Then, we consider a topology τ^* on $\mathcal{P}^*(R)$ which is generated by $\mathcal{B} = \{S_V \ : \ V \in \tau\}$, where $S_V = \{U \in \mathcal{P}^*(R) \ : \ U \subseteq V, U \in \tau\}$, $V \in \tau$. In the following we consider the product topology on $R \times R$ and the topology τ^* on $\mathcal{P}^*(R)$ [2].

Definition 2. [2] Let $(R, +, \cdot)$ be a hyper nearring and (R, τ) be a topological space. Then, the system $(R, +, \cdot, \tau)$ is called a *topological hyper nearring* if

(1) the mapping $(x, y) \mapsto x + y$, from $R \times R$ to $\mathcal{P}^*(R)$,
(2) the mapping $x \mapsto -x$, from R to R,
(3) the mapping $(x, y) \mapsto x.y$, from $R \times R$ to R,

are continuous.

Example 1. [2] The hyper nearring $R = (\{0, a, b, c\}, +, \cdot)$ defined as follows:

$+$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{0, a\}$	$\{b\}$	$\{c\}$
b	$\{b\}$	$\{b\}$	$\{0, a, c\}$	$\{b, c\}$
c	$\{c\}$	$\{c\}$	$\{b, c\}$	$\{0, a, b\}$

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	a	b	c
b	0	a	b	c
c	0	a	b	c

Let $\tau = \{\varnothing, R, \{0, a\}\}$. Then$(R, +, \cdot, \tau)$ is a topological hyper nearrring.

Lemma 1. *[2] Let $(R, +, \cdot, \tau)$ be a topological hyper nearring. If U is an open set and a complete part of R, then for every $c \in R$, $c + U$ and $U + c$ are open sets.*

Definition 3. [24] A binary relation δ on $P(X)$ is called a *proximity* on X if and only if δ satisfies the following conditions:

(P1) $A\delta B$ implies $B\delta A$,
(P2) $A\delta B$ implies $A \neq \emptyset$,
(P3) $A \cap B \neq \emptyset$ implies $A\delta B$,
(P4) $A\delta(B \cup C)$ if and only if $A\delta B$ or $A\delta C$,
(P5) $A \not\delta B$ implies there exists $E \subseteq X$ such that $A \not\delta E$ and $B \not\delta E^c$.

The pair (X, δ) is called a *proximity space*. If the sets $A, B \subseteq X$ are δ-related, then we write $A\delta B$, otherwise we write $A \not\delta B$.

Example 2. Let $A, B \subseteq X$ and $A\delta B$ if and only if $A \neq \emptyset$ and $B \neq \emptyset$. Then δ is a *proximity* on X.

The following theorem shows a proximity relation δ on X induces a topology on X.

Theorem 1. *[24] If a subset A of a proximity space (X, δ) is defined to be closed if and only if $x\delta A$ implies $x \in A$, then the collection of complements of all closed sets so defined yields a topology $\tau = \tau(\delta)$ on X.*

3. Some results on topological hyper nearrings

In this section, we present some results and properties in topological hyper nearring.

Lemma 2. *Let $(R, +, \cdot, \tau)$ be a topological hyper nearring. Then, $0 \in \bigcup_{R \neq U \in \tau} U$.*

Proof. If $0 \notin \bigcup_{R \neq U \in \tau} U$, then for every $R \neq U \in \tau$, $0 \notin U$. Let $U \in \tau$, $U \neq \emptyset$ and $0 \neq x \in U$. By the continuity of the mapping $+$, there exist neighborhoods $V_1, V_2 \in \tau$ of x and 0, respectively, such that $V_1 + V_2 \subseteq U$. Hence, we conclude that $V_2 = R$ and $V_1 + R \subseteq U$. Hence, we have $0 \in x + (-x) \subseteq V_1 + R \subseteq U$ and it is a contradiction. Therefore, we have $0 \in \bigcup_{R \neq U \in \tau} U$. □

Lemma 3. *Let $(R, +, \cdot, \tau)$ be a topological hyper nearring such that every open subset of it is a complete part of R. Let $\mathcal{U}$ be the system of all neighborhoods of 0, then for any subset A of R,*

$$\overline{A} = \bigcap_{U \in \mathcal{U}} (A + U).$$

Proof. Suppose that $x \in \overline{A}$ and $U \in \mathcal{U}$. $x - U$ is an open neighborhood of x, hence we have $x - U \cap A \neq \emptyset$. Thus there exists $a \in A$ such that $a \in x - U$. So, $x \in a + U \subseteq A + U$, for all $U \in \mathcal{U}$. Therefore, $\overline{A} \subseteq \bigcap_{U \in \mathcal{U}} (A + U)$. Now, let $x \in A + U$, for every $U \in \mathcal{U}$ and let V be a neighborhood of x. $x - V$ is a neighborhood of 0, hence $x \in A + (x - V)$. So, there exist $a \in A$ and $t \in x - V$ such that $x \in a + t$. Thus $a \in x - t \subseteq x + V - x = V$. Then $A \cap V \neq \emptyset$ and this proves that $x \in \overline{A}$ and $\bigcap_{U \in \mathcal{U}} (A + U) \subseteq \overline{A}$. Therefore, $\overline{A} = \bigcap_{U \in \mathcal{U}} (A + U)$. □

Corollary 1. *Let $(R, +, \cdot, \tau)$ be a topological hyper nearring such that every open subset of it is a complete part of R and let $\mathcal{U}$ be the system of all neighborhoods of 0. Then,*

(i) $\overline{\{0\}} = \bigcap_{U \in \mathcal{U}} U$;

(ii) *For every open set V and every closed set F such that $V \cap \overline{\{0\}} \neq \emptyset$ and $F \cap \overline{\{0\}} \neq \emptyset$, we have $\overline{\{0\}} \subseteq V$ and $\overline{\{0\}} \subseteq F$;*

(iii) *$\{0\}$ is dense in R if and only if R has trivial topology $\{\emptyset, R\}$.*

Proof. (i) It follows immediately from of Lemma 3.

(ii) Let V be open, $V \cap \overline{\{0\}} \neq \emptyset$ and $t \in V \cap \overline{\{0\}}$. V is a neighborhood of t and $t \in \overline{\{0\}}$, thus V is a neighborhood of 0 and by (i), $\overline{\{0\}} \subseteq V$. Now, suppose that

F is a closed subset and $F \cap \overline{\{0\}} \neq \emptyset$. Then, $\overline{\{0\}} \nsubseteq F^c$. F^c is open thus we have $\overline{\{0\}} \cap F^c = \emptyset$. Consequently, we get $\overline{\{0\}} \subseteq F$.

(iii) Let $\{0\}$ is dense in R and U be nonempty and open in R. Then, $R = \overline{\{0\}}$ and by (ii) $\overline{\{0\}} \subseteq U$. Therefore, $R = U$. □

Lemma 4. *Let $(R, +, \cdot, \tau)$ be a topological hyper nearring such that every open subset of it is a complete part of R. Then $\{0\}$ is open if and only if τ is discrete.*

Proof. It is straightforward. □

Theorem 2. *Let $(R, +, \cdot, \tau)$ be a topological hyper nearring such that every open subset of it is a complete part and H be a normal subhyper group of it. Then R/H is discrete if and only if H is open.*

Proof. Suppose that R/H is discrete and π is the natural mapping $x \mapsto \pi(x) = H + x$ of R onto R/H. Then, the identity, $\pi(0)$ of R/H is an isolated point. So, $\pi^{-1}(\pi(0)) = H$ is open of R. Now, if H is open, since π is open, it follows that $\pi(H)$ is open. Hence the identity $\pi(H)$ of R/H is an isolated point. Therefore, we conclude that R/H is discrete. □

Theorem 3. *Let $(R, +, \cdot, \tau)$ be a topological hyper nearring such that every open subset of it is a complete part. Then, the following conditions are equivalent:*

(1) *R is a T_0- space;*
(2) *$\{0\}$ is closed.*

Proof. (1⇒2) Let R be a T_0- space and let $x \in \overline{\{0\}}$. We prove that $x = 0$. If $x \neq 0$, then by (1) there exists an open neighborhood U containing only 0 or x, but since $x \in \overline{\{0\}}$, hence U is a neighborhood of 0, such that $x \notin U$. So, $x \in -U + x$. By Lemma 1, $-U + x$ is an open neighborhood of x, such that $0 \notin -U + x$ (Because if $0 \in -U + x$, then there exists $u \in U$ such that $0 \in -u + x$. So, $x = u + 0 \in U$), this is a contradiction. Thus, $x = 0$ and it follows that 0 is closed.

(2⇒1) Let $\{0\}$ be closed and $x, y \in R, x \neq y$. We show that there exist an open neighborhood U containing only x or y. If $y = 0$, since $\{0\}$ is closed and $x \neq 0$, then x is an interior point of $R \setminus \{0\}$. Hence, there exists a neighborhood U of x such that $0 \notin U$. Now, if $x \neq 0, y \neq 0$ and $x \neq y$, then $0 \notin x - y$. Consequently, by the previous part, for every $t \in x - y$ there exists a neighborhood U_t of t, such that $0 \notin U_t$. We consider $U = \bigcup_{t \in x-y} U_t$. Then $x - y \subseteq U$ and $0 \notin U$.Thus $U + y$ is a neighborhood of x such that $y \notin U + y$ (since $0 \notin U$). Therefore, R is a T_0-space. □

Let (X, τ) be a topological space. If f is a arbitrary mapping from X onto Y, then consider the family $\tau_f = \{U \ : \ U \subseteq Y, f^{-1}(U) \in \tau\}$. Obviously τ_f is a topology on Y.

Theorem 4. *[25] Let $f : (X, \tau) \to (Y, \tau')$ be a continuous function. Then $\tau' \leq \tau_f$.*

Lemma 5. *Let $f : R \to R'$ be a homomorphism of hyper nearrings. Then for every subset $A \subseteq R$, $f^{-1}(f(A)) = kerf + A$.*

Proof. Let $A \subseteq R$ and $t \in f^{-1}(f(A))$. Then $f(t) \in f(A)$ and it follows that there exists $a \in A$ such that $f(t) = f(a)$. Thus $0 \in f(t) - f(a) = f(t - a)$. Hence there exists $x \in t - a$ such that $f(x) = 0$. Then $x \in kerf$. Thus $t \in x + a \subseteq kerf + A$ and this shows that $f^{-1}(f(A)) \subseteq kerf + A$. It is obvious that $kerf + A \subseteq f^{-1}(f(A))$. Therefore, $f^{-1}(f(A)) = kerf + A$. □

Theorem 5. *Let $(R, +, \cdot, \tau)$ and $(R', +', \cdot', \tau')$ be two topological hyper nearring such that every open subset of them is a complete part and f from R onto R' be a homomorphism. Then (R', τ_f) is a topological hyper nearring.*

Proof. We should show that $+'$, $\cdot'$ and inverse operation are continuous on (R', τ'). Suppose that $x', y' \in R'$ and $x' +' y' \subseteq U' \in \tau_f$. Since f is onto, then there exist $x, y \in R'$ such that $f(x) = x'$ and $f(y) = y'$. Hence $f(x + y) = f(x) +' f(y) = x' +' y' \subseteq U'$. So, $x + y \subseteq f^{-1}(U') \in \tau$(since $U' \in \tau_f$). Since $+$ is continuous, then there exist neighborhoods $U_x \in \tau$ and $U_y \in \tau$ of elements x and y, respectively, such that $U_x + U_y \subseteq f^{-1}(U')$. By Lemmas 1 and 5, $f^{-1}(f(U_x)) = kerf +' U_x \in \tau$ and $f^{-1}(f(U_y)) \in \tau$. Hence $f(U_x) \in \tau_f$ and $f(U_y) \in \tau_f$. Therefore, we obtain

$$f(U_x) +' f(U_y) = f(U_x + U_y) \subseteq f(f^{-1}(U')) = U'.$$

This completes the proof. □

Theorem 6. *Let f from (R, τ) onto (R', τ') be a homomorphism of topological hyper nearrings. Then $f : (R, \tau) \to (R', \tau_f)$ is continuous and open.*

Proof. If $U \in \tau_f$, by the definition of τ_f, $f^{-1}(U) \in \tau$. Thus, f is continuous.
Now, let U be an open subset in R. Then by Theorem 5 $f^{-1}(f(U)) = kerf + U$ is open in (R, τ). Thus by the definition of τ_f, $f(U) \in \tau_f$. This means $f(U)$ is open in R'. Therefore, f is open. □

Let R be a topological hyper nearring, H be normal hyper R-subgroup of R and π be natural mapping of R onto R/H by $x \mapsto \pi(x) = H + x$. Then, by Theorem 3.30 [2] $(R/H, \tau_\pi)$ is a topological hyper nearring. It is called the quotient space of topological hyper nearring R that we showed τ_π by $\overline{\tau}$ in [2].

Theorem 7. *Let R be a T_0-topological hyper nearring such that every open subset of it is a complete part of R and H be a discrete subhypergroup of R. Then H is closed.*

Proof. Let $x \in \overline{H}$. Since H is a discrete subhypergroup of R, then $0 \in H$ and there exists an open neighborhood V of 0 such that $V \cap H = \{0\}$. By Lemma 1, $x - V$ is an open neighborhood of x. Therefore, $x - V \cap H \neq \emptyset$ (because $x \in \overline{H}$). Hence there exists $h \in H$ such that $h \in x - V$ and $h \in x - v$, for some $v \in V$. Thus $v \in -h + x \subseteq V \cap \overline{H} \subseteq \overline{V \cap H}$ (let $t \in V \cap \overline{H}$ and U_t is a neighborhood of t. $U_t \cap V$ is an open neighborhood of t and since $t \in \overline{H}$, then $(U_t \cap V) \cap H \neq \emptyset$

and $U_t \cap (V \cap H) \neq \emptyset$. It follows that $t \in \overline{V \cap H}$ and $V \cap \overline{H} \subseteq \overline{V \cap H}$). Thus $v \in \overline{V \cap H} = \overline{\{0\}} = \{0\}$(by Theorem 3) and it follows that $x = h \in H$ and H is closed. □

Theorem 8. *Let R be a topological hyper nearring and H a dense subhypergroup of R. If V is a neighborhood of 0 in H, then $\overline{V}$ is a neighborhood of 0 of R.*

Proof. Since V is a neighborhood of 0 in H, it follows that there exists an open neighborhood U of 0 in R such that $U \cap H \subseteq V$. Hence, we obtain $U = U \cap G = U \cap \overline{H} \subseteq \overline{U \cap H} \subseteq \overline{V}$. Therefore, 0 is an interior point $\overline{V}$ and $\overline{V}$ is open in R. □

4. Topological hyper nearring derived from a proximity space

In this section, we define a proximity relation on an arbitrary hyper nearring and prove that every hyper nearring with topology whose is induced by this proximity relation is a topological hyper nearring. Also, we show that every topological hyper nearring is a proximity space.

Theorem 9. *Let $(R, +, \cdot)$ be a hyper nearring, N be a normal subhypergroup of R and $A, B \subseteq R$. We define $A\delta B$ if and only if there exist $a \in A$ and $b \in B$ such that $-b + a \subseteq N$, then (R, δ) is a proximity space.*

Proof. (P_1) Suppose that $A\delta B$. Then, there exist $a \in A$ and $b \in B$ such that $-b + a \subseteq N$. So, we get $-a + b \subseteq -N = N$. Therefore, $B\delta A$.

(P_2) It is obvious.

(P_3) Let there exists $x \in A \cap B \neq \emptyset$. Then $-x + x \subseteq -x + N + x \subseteq N$. So, we conclude that $A\delta B$.

(P_4) It is straightforward.

(P_5) Let $A \not\delta B$ and $E := B + N$. If $A\delta E = B + N$, then there exist $a \in A$ and $b \in B$ such that $-(b + N) + a \subseteq N$. Therefore, $-N - b + a \subseteq N$ and this implies that $-b + a \subseteq N + N \subseteq N$. Thus, $A\delta B$ and it is a contradiction. Hence $A \not\delta E$. Also, $B \not\delta E^c$. If $B\delta E^c$, then there exist $b \in B$ and $x \in (B + N)^c$ such that $-x + b \subseteq N$. Therefore, $x \in b + N \subseteq B + N$ and it is a contradiction. □

Theorem 10. *In the proximity space (R, δ) that $(R, +, \cdot)$ is a hyper nearring and δ is defined relation in Theorem 9, the set $\beta = \{x + N \ : \ x \in R\}$ is a base for the topology $\tau = \tau(\delta)$.*

Proof. Let U be an open subset of R and let $y \in U$. We should show that $y + N \subseteq U$. Let $t \notin U$, then $t \in U^c$ and $t\delta U^c$ (since U^c is closed). $-y + t \subseteq -y + y + N \subseteq -y + N + y \subseteq N$. Hence $t\delta y$ and by $(P4)$, $y\delta U^c$. Thus $y \in U^c$ and it is a contradiction. This implies that β is a base for the topology $\tau(\delta)$. □

Lemma 6. *The normal subhypergroup N of R is a clopen set in the topology $\tau(\delta)$ is defined in Theorem 10.*

Proof. By Theorem 10, N is open. Now, let $x\delta N$, for $x \in R$. Then there exists $n \in N$ such that $-n + x \subseteq N$. Therefore $x \in n - n + x \subseteq n + N = N$. Thus N is a closed subset in R. □

Theorem 11. *Let $(R, +, \cdot)$ be a hyper nearring, the normal subhypergroup N be a complete part of R and the relation δ is defined in Theorem 9. Then the system $(R, +, \cdot, \tau(\delta))$ is a topological hyper nearring.*

Proof. We should show that $+$, $\cdot$ and inverse operation are continuous. Suppose that U is an open subset of R such that $x + y \subseteq U$, for $x, y \in R$. Then by Theorem 10, there exists $t \in R$ such that $x + y \subseteq t + N \subseteq U$. Therefore, $x + N$ and $y + N$ are neighborhoods of x and y such that $(x + N) + (y + N) = x + y + N \subseteq t+N+N = t+N \subseteq U$. Thus $+$ is continuous on R. Now, Suppose that U is an open neighborhood of $-x$. By Theorem 10, there exists $t \in R$ such that $-x \in t+N \subseteq U$. Therefore, $x \in -N - t = -t + N$. Hence $-t + N$ is a neighborhoods of x and $-(-t + N) = -N + t = N + t = t + N \subseteq U$. This proves that inverse operation is continuous. Now, we show that $\cdot$ is continuous. Suppose that U is an open subset of R such that $x \cdot y \in U$, for $x, y \in R$. Then there exist $t \in R$ such that $x \cdot y \in t + N \subseteq U$(by Theorem10). $x + N$ and $y + N$ are neighborhoods of x and y such that $(x + N) \cdot (y + N) \subseteq x \cdot y + N$ (N is a complete part of R, then $x \cdot y + N$ is a complete part of R. Hence $(x + N) \cdot (y + N) \subseteq x \cdot y + N$). So, $(x + N) \cdot (y + N) \subseteq x \cdot y + N \subseteq t + N + N = t + N \subseteq U$. Thus $\cdot$ is continuous on R. □

Example 3. Let $R = \{0, a, b\}$ be a set with a hyperoperation $+$ and a binary operation $\cdot$ as follows:

$+$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{0\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$

$\cdot$	0	a	b
0	0	a	b
a	0	a	b
b	0	a	b

Then, $(R, +, \cdot)$ is a hyper nearring. We consider a normal subhyperring $N = \{0, a\}$ of R and define:

$A\delta B$ if and only if there exist $a \in A$ and $b \in B$ such that $-b + a \subseteq N$.

Therefore, $\tau(\delta) = \{\varnothing, \{0, a, b\}, \{0, a\}, \{b\}\}$. Simply, we can show that $(R, +, \cdot, \tau(\delta))$ is a topological hyper nearring.

The following theorem, show that every topological hyper nearring is a proximity space.

Theorem 12. *Let $(R, +, \cdot, \tau)$ be a topological hyper nearring such that every open subset of it is a complete part of R. Then there exists a proximity relation δ such that (R, δ) is a proximity space.*

Proof. Let $\mathcal{U}$ be the system of symmetric neighborhoods at 0, for every $A, B \subseteq R$ and $V \in \mathcal{U}$. We define

$A\delta B$ if and only if $A \cap B + V \neq \emptyset$.

Now, we show that δ is a proximity relation.

(P_1) Suppose that $A\delta B$. Then, there exist $a \in A$ and $b \in B$ such that $a \in b+V$. Hence $b \in a - V = a + V \subseteq A + V$. Therefore, $B\delta A$.

(P_2) It is obvious.

(P_3) Let $A \cap B \neq \emptyset$. Then, there exists $x \in A \cap B$. Therefore, $x \in A \cap B + V \neq \emptyset$. Thus $A\delta B$.

(P_4) It is straightforward..

(P_5) Let $A \not\delta B$ and $E := B + V$. If $A\delta B + V$, then $A \cap (B + V) + V \neq \emptyset$. Therefore $A \cap B + V \neq \emptyset$ (since V is a complete part of R, then $V + V \subseteq V$) and this proves that $A\delta B$, that it is a contradiction. Hence $A \not\delta E$. Also, if $B\delta E^c$, it follows that $B \cap (B + V)^c + V \neq \emptyset$. Hence there exist $b \in B$, $x \in (B + V)^c$ and $v \in V$ such that $b \in x + v$. Thus $x \in b - v \subseteq B + V$ and it is a contradiction. Therefore, $B \not\delta E^c$. □

5. Conclusion

In this paper we expressed the relationship between two important subjects: algebraic hyperstructures and topology. We studied several characteristics of topological hyper nearrings and in the following, we related them to proximity spaces.

References

[1] Ameri, R., Topological (transposition) hypergroups, *Italian Journal of Pure and Applied Mathematics*, (13) (2003), 181-186.

[2] Borhani-Nejad, S., Davvaz, B., Topological hyper nearrings, submitted.

[3] Corsini, P., Prolegomena of Hypergroup Theory, Second ed., Aviani Editore, Tricesimo, Italy, 1993.

[4] Corsini, P., Leoreanu, V., Applications of Hypergroup Theory, Kluwer Academic Publishers, 2003.

[5] Dasic, V., Hypernearrings, *Fourth Int. Congress on Algebraic Hyperstructures and Applications (AHA 1990)*, World Scientific, (1991), 75-85.

[6] Davvaz, B., Hypernearrings and weak hypernearrings, *11th Algebra Seminar of Iranian Math. Soc. Isfahan University of Technology*, Isfahan, October 27-29, (1999), 68-78.

[7] Davvaz, B., Polygroup Theory and Related Systems, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

[8] Davvaz, B., Semihypergroup Theory, Elsevier, 2016.

[9] Davvaz, B., Leoreanu-Fotea, V., Hyperring Theory and Applications, International Academic Press, 115, Palm Harber, USA, 2007.

[10] Efremovč, V.A., The geometry of proximity, *Mat. Sb.*, 31 (1952), 189-200.

[11] Efremovič, V.A., Infinitesimal spaces, *Doklady Akademii Nauk SSSR (N.S.)*, (in Russian), 76 (1951), 341-343.

[12] Freni, D., A note on the core of a hypergroup and the transitive closure β^* of β, *Riv. Mat. Pura Appl.*, 8 (1991), 153-156.

[13] Gontineac, V.M., On hypernear-rings and H-hypergroups, *Fifth Int. Congress on Algebraic Hyperstructures and Applications (AHA 1993)*, Hadronic Press, Inc., USA, (1994), 171-179.

[14] Heidari, D., Davvaz, B., Modarres, S.M.A., Topological hypergroups in the sense of Marty, *Comm. algebra*, 42 (2014), 4712-4721.

[15] Heidari, D., Davvaz, B., Modarres, S.M.S., Topological polygroups, *Bull. Malays. Math. Sci. Soc.*, 3942 (2016), 707-721.

[16] Hošková-Mayerová, S., Topological hypergroupoids, *Comput. Math. Appl.*, 64(9) (2012), 2845-2849.

[17] Jancic-Rašovic S., Cristea, I., Division hypernear-rings, *An. Stiint. Univ. "Ovidius" Constanta Ser. Mat.*, 26(3) (2018), 109-126.

[18] Jancic-Rašovic S., Cristea, I., Hypernear-rings with a defect of distributivity, *Filomat*, 32(4) (2018), 1133-1149.

[19] Kim, K.H., Davvaz, B., Roh, E.H., On Hyper R-subgroups of hypernear-rings, *Scientiae Mathematicae Japonicae*, Online (2007), 649-656.

[20] Koskas, M., Groupoides, demi-hypergroupes et hypergroupes, *J. Math. Pure Appl.*, 49(9) (1970), 155-192.

[21] Munkres, J.R., Topology, Prentice Hall, Upper Saddle River, NJ, 2000.

[22] Marty, F., Sur une generalization de la notion de group, *8th congress Math. Scandinaves, Stockholm*, (1934), 45-49.

[23] Mittas, J., Hypergroups canoniques, *Math. Balkanica*, 2 (1972), 165-179.

[24] Naimpally S.A., Warrack, B.D., Proximity Spaces, *Cambridge Tract*, 1970.

[25] Ursul, M., Topological rings satisfying compactness conditions, Springer Science Business Media (2002), vol 549.