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SOME IDENTITIES INVOLVING (p,q)-FIBONACCI AND LUCAS QUATERNIONS

Gamaliel CERDA-MORALES

Instituto de Matemáticas, P. Universidad Católica de Valparaíso, Blanco Viel 596, Cerro Barón, Valparaíso, CHILE

Abstract. In this study, we firstly examined the Horadam quaternions defined and studied by Halici and Karataş in [4]. Then, we used the Binet's formula to show some properties of the (p,q)-Fibonacci and Lucas quaternions. We also give some important identities including these quaternions.

1. Introduction

Fibonacci and Lucas quaternions cover a wide range of interest in modern mathematics as they appear in the comprehensive works of [2, 4–6]. The Fibonacci quaternion $Q_{F,n}$ is the n-th term of the sequence where each term is the sum of the two previous terms beginning with the initial values $Q_{F,0} = i + j + 2k$ and $Q_{F,1} = 1 + i + 2j + 3k$. The well-known Fibonacci quaternion numbers are defined as

$$Q_{F,n} = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}, \ n \geq 0, \tag{1}$$

where $i^2 = j^2 = k^2 = ijk = -1$. Similarly, Lucas quaternions are defined as $Q_{L,n} = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}$ for $n \geq 0$, where F_n and L_n are n-th Fibonacci and Lucas number, respectively.

Ipek [8] studied the (p,q)-Fibonacci quaternions $Q_{\mathcal{F},n}$ which is defined as

$$Q_{\mathcal{F},n} = pQ_{\mathcal{F},n-1} + qQ_{\mathcal{F},n-2}, \ n \geq 2 \tag{2}$$

with initial conditions $Q_{\mathcal{F},0} = i + pj + (p^2+q)k$, $Q_{\mathcal{F},1} = 1 + pi + (p^2+q)j + (p^3+2pq)k$ and $p^2 + 4q > 0$. Note that the (p,q)-Fibonacci numbers are defined by $\mathcal{F}_n = p\mathcal{F}_{n-1} + q\mathcal{F}_{n-2}$, $\mathcal{F}_0 = 0$ and $\mathcal{F}_1 = 1$. Then, if $p = q = 1$, we get the

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gamaliel.cerda.m@mail.pucv.cl
0000-0003-3164-4434.

classical Fibonacci quaternion $Q_{F,n}$. If $p = 2q = 2$, we get the Pell quaternion $Q_{P,n} = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3}$, where P_n is the n-th Pell number.

Another important sequence is the (p, q)-Lucas sequence. This sequence is defined by the recurrence relation

$$\mathcal{L}_n = p\mathcal{L}_{n-1} + q\mathcal{L}_{n-2},\ \mathcal{L}_0 = 2,\ \mathcal{L}_1 = p. \tag{3}$$

The well-known Binet's formulas for (p, q)-Fibonacci and Lucas quaternion, see [8], are given by

$$Q_{\mathcal{F},n} = \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{\alpha - \beta} \text{ and } Q_{\mathcal{L},n} = \underline{\alpha}\alpha^n + \underline{\beta}\beta^n, \tag{4}$$

respectively. Here, α, β are roots of the characteristic equation $t^2 - pt - q = 0$, and $\underline{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$ and $\underline{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$. We note that $\alpha + \beta = p$, $\alpha\beta = -q$ and $\alpha - \beta = \sqrt{p^2 + 4q}$.

The generalized of Fibonacci quaternion $Q_{w,n}$ is defined by Halici and Karataş in [4] as

$$Q_{w,0} = a + bi + (pb + qa)j + ((p^2 + q)b + pqa)k,$$

$$Q_{w,1} = b + (pb + qa)i + ((p^2 + q)b + pqa)j + ((p^3 + 2pq)b + q(p^2 + q)a)$$

and $Q_{w,n} = pQ_{w,n-1} + qQ_{w,n-2}$, for $n \geq 2$ which is called as the generalized Fibonacci quaternions. So, each term of the generalized Fibonacci sequence $\{Q_{w,n}\}_{n\geq 0}$ is called generalized Fibonacci quaternion.

The Binet formula for generalized Fibonacci quaternion $Q_{w,n}$, see [4], is given by

$$Q_{w,n} = \frac{A\underline{\alpha}\alpha^n - B\underline{\beta}\beta^n}{\alpha - \beta}, \tag{5}$$

where $A = b - a\beta$, $B = b - a\alpha$, α, β are roots of the characteristic equation $t^2 - pt - q = 0$, and $\underline{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$ and $\underline{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$. If $a = 0$ and $b = 1$, we get the classical (p, q)-Fibonacci quaternion $Q_{\mathcal{F},n}$. If $a = 2$ and $b = p$, we get the (p, q)-Lucas quaternion $Q_{\mathcal{L},n}$.

In this paper, we study some properties of the (p, q)-Fibonacci quaternions, (p, q)-Lucas quaternions and the generalized Fibonacci quaternions.

2. Main results

There are three well-known identities for generalized Fibonacci numbers, namely, Catalan's, Cassini's, and d'Ocagne's identities. The proofs of these identities are based on Binet formulas. We can obtain these types of identities for generalized Fibonacci quaternions using the Binet formula for $Q_{w,n}$. Then, we require $\underline{\alpha}\underline{\beta}$ and $\underline{\beta}\underline{\alpha}$. These products are given in the following lemma.

Lemma 1. *We have*

$$\underline{\alpha}\underline{\beta} = Q_{\mathcal{L},0} - [q] - q\Delta\omega, \tag{6}$$

and

$$\underline{\beta}\underline{\alpha} = Q_{\mathcal{L},0} - [q] + q\Delta\omega, \tag{7}$$

where $\omega = qi + pj - k$, $[q] = 1 - q + q^2 - q^3$ *and* $\Delta = \alpha - \beta$.

Proof. From the definitions of $\underline{\alpha}$ and $\underline{\beta}$, and using $i^2 = j^2 = k^2 = -1$ and $ijk = -1$, we have

$$\begin{aligned}\underline{\alpha}\underline{\beta} &= 2 + (\alpha+\beta)i + (\alpha^2+\beta^2)j + (\alpha^3+\beta^3)k \\ &\quad - (1 + \alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3) + \alpha^2\beta^2(\beta-\alpha)i + \alpha\beta(\alpha^2-\beta^2)j + \alpha\beta(\beta-\alpha)k \\ &= 2 + pi + (p^2+2q)j + (p^3+3pq)k - (1 - q + q^2 - q^3) - q\Delta(qi + pj - k) \\ &= Q_{\mathcal{L},0} - [q] - q\Delta\omega,\end{aligned}$$

where $[q] = 1 - q + q^2 - q^3$ and $\omega = qi + pj - k$, and the final equation gives Eq. (6). The other identity can be computed similarly. □

The Lemma 1 gives us the following useful identity:

$$\underline{\alpha}\underline{\beta} + \underline{\beta}\underline{\alpha} = 2(Q_{\mathcal{L},0} - [q]). \tag{8}$$

The following theorem gives Catalan's identities for generalized Fibonacci quaternions.

Theorem 2. *For any integers* m *and* n *with* $m \geq n$, *we have*

$$Q_{w,m}^2 - Q_{w,m+n}Q_{w,m-n} = -AB(-q)^m\mathcal{F}_{-n}\left((Q_{\mathcal{L},0} - [q])\mathcal{F}_n - q\omega\mathcal{L}_n\right), \tag{9}$$

where $A = b - a\beta$, $B = b - a\alpha$, *and* $\mathcal{F}_n$, $\mathcal{L}_n$ *are the* n*-th* (p,q)*-Fibonacci and* (p,q)*-Lucas numbers, respectively.*

Proof. From the Binet formula for generalized Fibonacci quaternions $Q_{w,n}$ in (5) and $\Delta^2 = p^2 + 4q$, we have

$$\begin{aligned}&\Delta^2\left(Q_{w,m}^2 - Q_{w,m+n}Q_{w,m-n}\right) \\ &\quad = \left(A\underline{\alpha}\alpha^m - B\underline{\beta}\beta^m\right)^2 - \left(A\underline{\alpha}\alpha^{m+n} - B\underline{\beta}\beta^{m+n}\right)\left(A\underline{\alpha}\alpha^{m-n} - B\underline{\beta}\beta^{m-n}\right) \\ &\quad = AB(-q)^{m-n}\left(\underline{\alpha}\underline{\beta}\alpha^{2n} + \underline{\beta}\underline{\alpha}\beta^{2n} - (-q)^n\left(\underline{\alpha}\underline{\beta} + \underline{\beta}\underline{\alpha}\right)\right).\end{aligned}$$

We require Eqs. (6) and (7). Using this equations, we obtain

$$\begin{aligned}&Q_{w,m}^2 - Q_{w,m+n}Q_{w,m-n} \\ &\quad = \frac{AB(-q)^{m-n}}{\Delta^2}\left((Q_{\mathcal{L},0} - [q])(\alpha^{2n} + \beta^{2n} - 2(-q)^n) - q\Delta\omega(\alpha^{2n} - \beta^{2n})\right) \\ &\quad = \frac{AB(-q)^{m-n}}{\Delta^2}\left((Q_{\mathcal{L},0} - [q])(\mathcal{L}_{2n} - 2(-q)^n) - q\Delta^2\omega\mathcal{F}_{2n}\right).\end{aligned}$$

Using the identity $\Delta^2\mathcal{F}_n^2 = \mathcal{L}_{2n} - 2(-q)^n$ gives

$$Q_{w,m}^2 - Q_{w,m+n}Q_{w,m-n} = AB(-q)^{m-n}\left((Q_{\mathcal{L},0} - [q])\mathcal{F}_n^2 - q\omega\mathcal{F}_{2n}\right),$$

where $\mathcal{L}_n$, $\mathcal{F}_n$ are the n-th (p,q)-Lucas and (p,q)-Fibonacci numbers, respectively. With the help of the identities $\mathcal{F}_{2n} = \mathcal{F}_n\mathcal{L}_n$ and $\mathcal{F}_{-n} = -(-q)^n\mathcal{F}_n$, we have Eq. (9). The proof is completed. $\square$

Taking $n = 1$ in this theorem and using $\mathcal{F}_{-1} = \frac{1}{q}$, we obtain Cassini's identities for generalized Fibonacci quaternions. This result gives another version of the Corollary 3.6 in [10].

Corollary 3. *For any integer m, we have*

$$Q_{w,m}^2 - Q_{w,m+1}Q_{w,m-1} = AB(-q)^{m-1}\left(Q_{\mathcal{L},0} - [q] - pq\omega\right), \tag{10}$$

where $A = b - a\beta$, $B = b - a\alpha$ and $[q] = 1 - q + q^2 - q^3$.

The following theorem gives d'Ocagne's identities for generalized Fibonacci quaternions.

Theorem 4. *For any integers n and m with $n \geq m$, we have*

$$Q_{w,n}Q_{w,m+1} - Q_{w,n+1}Q_{w,m} = (-q)^m AB\left((Q_{\mathcal{L},0} - [q])\mathcal{F}_{n-m} - q\omega\mathcal{L}_{n-m}\right), \tag{11}$$

$\mathcal{F}_n$, $\mathcal{L}_n$ are the n-th (p,q)-Fibonacci and (p,q)-Lucas numbers, respectively.

Proof. Using the Binet formula for the generalized Fibonacci quaternions gives

$$\begin{aligned}
\Delta^2&(Q_{w,n}Q_{w,m+1} - Q_{w,n+1}Q_{w,m}) \\
&= \left(A\underline{\alpha}\alpha^n - B\underline{\beta}\beta^n\right)\left(A\underline{\alpha}\alpha^{m+1} - B\underline{\beta}\beta^{m+1}\right) \\
&\quad - \left(A\underline{\alpha}\alpha^{n+1} - B\underline{\beta}\beta^{n+1}\right)\left(A\underline{\alpha}\alpha^m - B\underline{\beta}\beta^m\right) \\
&= \Delta(-q)^m AB\left(\underline{\alpha}\underline{\beta}\alpha^{n-m} - \underline{\beta}\underline{\alpha}\beta^{n-m}\right).
\end{aligned}$$

We require the Eqs. (6) and (7). Substituting these into the previous equation, we have

$$\begin{aligned}
&Q_{w,n}Q_{w,m+1} - Q_{w,n+1}Q_{w,m} \\
&= \frac{1}{\Delta}(-q)^m AB\left((Q_{\mathcal{L},0} - [q])(\alpha^{n-m} - \beta^{n-m}) - q\Delta\omega(\alpha^{n-m} + \beta^{n-m})\right) \\
&= (-q)^m AB\left((Q_{\mathcal{L},0} - [q])\mathcal{F}_{n-m} - q\omega\mathcal{L}_{n-m}\right).
\end{aligned}$$

The second identity in the above equality, can be proved using $\mathcal{L}_{n-m} = \alpha^{n-m} + \beta^{n-m}$ and $\Delta\mathcal{F}_{n-m} = \alpha^{n-m} - \beta^{n-m}$. This proof is completed. $\square$

In particular, writing $m = n - 1$ in this theorem and using the identity $\mathcal{L}_1 = p$, we obtain Cassini's identities for generalized Fibonacci quaternions. Now, taking $m = n$ in this theorem and using the initial conditions $\mathcal{F}_0 = 0$ and $\mathcal{L}_0 = 2$, we obtain the next identity.

Corollary 5. *For any integer n, we have*

$$Q_{w,n}Q_{w,n+1} - Q_{w,n+1}Q_{w,n} = 2(-q)^{n+1}AB\omega, \tag{12}$$

where $A = b - a\beta$, $B = b - a\alpha$ and $\omega = qi + pj - k$.

Theorem 6. *For any integers n, r and s, we have*

$$Q_{\mathcal{L},n+r}Q_{\mathcal{F},n+s} - Q_{\mathcal{L},n+s}Q_{\mathcal{F},n+r} = 2(-q)^{n+r}\mathcal{F}_{s-r}(Q_{\mathcal{L},0} - [q]). \tag{13}$$

Proof. The Binet formulas for the (p,q)-Lucas and (p,q)-Fibonacci quaternions give

$$\begin{aligned}
\Delta(Q_{\mathcal{L},n+r}&Q_{\mathcal{F},n+s} - Q_{\mathcal{L},n+s}Q_{\mathcal{F},n+r}) \\
&= \left(\underline{\alpha}\alpha^{n+r} + \underline{\beta}\beta^{n+r}\right)\left(\underline{\alpha}\alpha^{n+s} - \underline{\beta}\beta^{n+s}\right) \\
&\quad - \left(\underline{\alpha}\alpha^{n+s} + \underline{\beta}\beta^{n+s}\right)\left(\underline{\alpha}\alpha^{n+r} - \underline{\beta}\beta^{n+r}\right) \\
&= (\alpha\beta)^n(\alpha^s\beta^r - \alpha^r\beta^s)(\underline{\alpha}\underline{\beta} + \underline{\beta}\underline{\alpha}).
\end{aligned}$$

Using initial condition $Q_{\mathcal{L},0}$, we have

$$Q_{\mathcal{L},n+r}Q_{\mathcal{F},n+s} - Q_{\mathcal{L},n+s}Q_{\mathcal{F},n+r} = 2(-q)^{n+r}\mathcal{F}_{s-r}(Q_{\mathcal{L},0} - [q]).$$

□

After deriving these famous identities, we present some other identities for the generalized Fibonacci quaternions. In particular, when using the Binet formulas to obtain identities for the (p,q)-Fibonacci and (p,q)-Lucas quaternions, we require $\underline{\alpha}^2$ and $\underline{\beta}^2$. These products are given in the next lemma.

Lemma 7. *We have*

$$\underline{\alpha}^2 = (Q_{\mathcal{L},0} - r_{p,q}) + \Delta(Q_{\mathcal{F},0} - s_{p,q}), \tag{14}$$

and

$$\underline{\beta}^2 = (Q_{\mathcal{L},0} - r_{p,q}) - \Delta(Q_{\mathcal{F},0} - s_{p,q}), \tag{15}$$

where $\Delta = \alpha - \beta$, $r_{p,q} = 1 + \frac{p}{2}(\mathcal{F}_2 + \mathcal{F}_4 + \mathcal{F}_6) + q(\mathcal{F}_1 + \mathcal{F}_3 + \mathcal{F}_5)$, $s_{p,q} = \frac{1}{2}(\mathcal{F}_2 + \mathcal{F}_4 + \mathcal{F}_6)$ *and* $\mathcal{F}_n$ *is the* n*-th* (p,q)*-Fibonacci number.*

Proof. From the definitions of $\underline{\alpha}$ and $\underline{\beta}$, and using $i^2 = j^2 = k^2 = -1$, $ijk = -1$ and $\alpha^n = \mathcal{F}_n\alpha + q\mathcal{F}_{n-1}$ for $n \geq 1$, we have

$$\begin{aligned}
\underline{\alpha}^2 &= 2(1 + \alpha i + \alpha^2 j + \alpha^3 k) - (1 + \alpha^2 + \alpha^4 + \alpha^6) \\
&= 2 + pi + (p^2 + 2q)j + (p^3 + 3pq)k + \Delta(i + pj + (p^2 + q)k) \\
&\quad - (1 + (\mathcal{F}_2\alpha + q\mathcal{F}_1) + (\mathcal{F}_4\alpha + q\mathcal{F}_3) + (\mathcal{F}_6\alpha + q\mathcal{F}_5)) \\
&= (Q_{\mathcal{L},0} - r_{p,q}) + \Delta(Q_{\mathcal{F},0} - s_{p,q}),
\end{aligned}$$

where $r_{p,q} = 1 + \frac{p}{2}(\mathcal{F}_2 + \mathcal{F}_4 + \mathcal{F}_6) + q(\mathcal{F}_1 + \mathcal{F}_3 + \mathcal{F}_5)$ and $s_{p,q} = \frac{1}{2}(\mathcal{F}_2 + \mathcal{F}_4 + \mathcal{F}_6)$ and the final equation gives Eq. (14). The other can be computed similarly. □

We present some interesting identities for (p,q)-Fibonacci, (p,q)-Lucas quaternions and generalized Fibonacci quaternions. A similar identity can be seen in Theorem 3.11 in [10].

Theorem 8. *For any integer n, we have*

$$Q_{\mathcal{L},n}^2 - Q_{\mathcal{F},n}^2 = \begin{pmatrix} \frac{\Delta^2-1}{\Delta^2}(Q_{\mathcal{L},0} - r_{p,q})\mathcal{L}_{2n} + (Q_{\mathcal{F},0} - s_{p,q})\mathcal{F}_{2n} \\ +2\frac{(\Delta^2+1)(-q)^n}{\Delta^2}(Q_{\mathcal{L},0} - [q]). \end{pmatrix} \tag{16}$$

Proof. Using the Binet formulas for the (p,q)-Fibonacci and (p,q)-Lucas quaternions, we obtain

$$\begin{aligned}\Delta^2(Q_{\mathcal{L},n}^2 - Q_{\mathcal{F},n}^2) &= \Delta^2\left(\underline{\alpha}\alpha^n + \underline{\beta}\beta^n\right)^2 - \left(\underline{\alpha}\alpha^n - \underline{\beta}\beta^n\right)^2 \\ &= (\Delta^2-1)(\underline{\alpha}^2\alpha^{2n} + \underline{\beta}^2\beta^{2n}) + (\Delta^2+1)(\alpha\beta)^n(\underline{\alpha}\underline{\beta} + \underline{\beta}\underline{\alpha}).\end{aligned}$$

Substituting Eqs. (6) and (7) into the last equation, we have

$$\Delta^2(Q_{\mathcal{L},n}^2 - Q_{\mathcal{F},n}^2) = (\Delta^2-1)(\underline{\alpha}^2\alpha^{2n} + \underline{\beta}^2\beta^{2n}) + 2(\Delta^2+1)(\alpha\beta)^n(Q_{\mathcal{L},0} - [q]). \tag{17}$$

Then, using Eqs. (14) and (15), we obtain

$$\underline{\alpha}^2\alpha^{2n} + \underline{\beta}^2\beta^{2n} = (\alpha^{2n} + \beta^{2n})(Q_{\mathcal{L},0} - r_{p,q}) + \Delta(Q_{\mathcal{F},0} - s_{p,q})(\alpha^{2n} - \beta^{2n}). \tag{18}$$

Substituting Eq. (18) into Eq. (17) gives Eq. (16). □

Corollary 9. *For any integers n and m with $m \geq n$, we have*

$$Q_{\mathcal{F},n}Q_{w,m} - Q_{w,m}Q_{\mathcal{F},n} = 2(-q)^{n+1}\omega W_{m-n}, \tag{19}$$

where $\omega = qi + pj - k$ and $W_n = \frac{A\alpha^n - B\beta^n}{\alpha-\beta}$ is the n-th generalized Fibonacci number.

Proof. The Binet formulas for the (p,q)-Fibonacci and generalized Fibonacci quaternions give

$$\begin{aligned}&\Delta^2(Q_{\mathcal{F},n}Q_{w,m} - Q_{w,m}Q_{\mathcal{F},n}) \\ &\quad= \left(\underline{\alpha}\alpha^n - \underline{\beta}\beta^n\right)\left(A\underline{\alpha}\alpha^m - B\underline{\beta}\beta^m\right) \\ &\qquad - \left(A\underline{\alpha}\alpha^m - B\underline{\beta}\beta^m\right)\left(\underline{\alpha}\alpha^n - \underline{\beta}\beta^n\right) \\ &\quad= (A\alpha^m\beta^n - B\alpha^n\beta^m)(\underline{\alpha}\underline{\beta} - \underline{\beta}\underline{\alpha}).\end{aligned}$$

Using Eqs. (6) and (7), we have

$$\begin{aligned}Q_{\mathcal{F},n}Q_{w,m} - Q_{w,m}Q_{\mathcal{F},n} &= (\alpha\beta)^n(A\alpha^{m-n} - B\beta^{m-n})(\underline{\alpha}\underline{\beta} - \underline{\beta}\underline{\alpha}) \\ &= 2(-q)^{n+1}\omega W_{m-n},\end{aligned}$$

where $\omega = qi + pj - k$ and W_n is the n-th generalized Fibonacci number defined by $W_n = \frac{A\alpha^n - B\beta^n}{\alpha-\beta}$. □

Taking $m = n$ in this corollary and using $W_0 = a$, we obtain the next identity.

Corollary 10. *For any integer n, we have*

$$Q_{\mathcal{F},n}Q_{w,n} - Q_{w,n}Q_{\mathcal{F},n} = 2(-q)^{n+1}a\omega, \tag{20}$$

where $A = b - a\beta$, $B = b - a\alpha$ and $\omega = qi + pj - k$.

3. Conclusion

Sequences of numbers have been studied over several years, including the well known Horadam sequence and, consequently, on the Horadam quaternions studied in [4]. In this paper we have also contributed for the study of (p,q)-Fibonacci and Lucas quaternion sequence, deducing some of their identities using the Binet-style formula of Horadam quaternions. It is our intention to continue the study of this type of sequences, exploring some their applications in the science domain. For example, a new type of sequences in the complex algebra with the use of these numbers and their combinatorial properties.

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