Turk. J. Math. Comput. Sci.
12(2)(2020) 86–91
© MatDer
DOI : 10.47000/tjmcs.765297

On Quasi-Conformally Flat Para-Sasakian Manifolds

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Received: 06-07-2020 • Accepted: 03-09-2020

Abstract. In this paper, we present some new results on invariant submanifolds of a para-Sasakian manifold under the quasi-conformally flatness condition. Firstly, we examine flatness of quasi-conformal curvature tensor on para-Sasakian manifolds. We prove that a quasi-conformally flat para-Sasakian manifold is an η-Einstein manifold. Also, we give some results on the sectional curvature of such manifolds. Secondly, we consider the invariant submanifolds of a quasi-conformally flat para-Sasakian manifold. We prove that a totally umbilical submanifold of a para-Sasakian manifold is invariant. In addition, we investigate curvature properties of such submanifolds and we show that a totally umbilical invariant submanifold of a quasi-conformally flat para-Sasakian manifold is an η-Einstein manifold. Finally, we work on the sectional curvature properties of an invariant submanifold of a quasi-conformally flat para-Sasakian manifold.

2010 AMS Classification: 53C15, 53C25, 53D10

Keywords: Para-Sasakian manifold, Invariant submanifold, Quasi-conformal curvature tensor.

1. Introduction

Sasakian manifolds were defined by S.Sasaki in the 1960s [9]. This definition, which is the tensorial approach, has become very interested by differential geometer in the following years. The fundamental idea of Sasaki was based on some facts from almost complex and Hermitian geometry. Sasakian manifolds are seen as a one-dimensional analogue of Kähler manifolds. For a comprehensive introduction to Sasakian manifolds, we refer to readers [3, 14]. Similar to almost contact structure, para-contact manifolds were defined in [7] with considering the para-complex manifolds. Para-Sasakian manifolds were defined by T. Adati and K. Matsumoto [1]. Many researcher have studied on para-Sasakian manifolds in [2, 4–6].

The curvature tensors have essential applications in global differential geometry. The conformal curvature tensor gives us the flatness and symmetry conditions a manifold under conformal transformations. The other tensor is given by Yano [15] as it is named concircular curvature tensor. A concircular transformation is a special transformation that is not conformal. This tensor is invariant under this type of transformations and gives us information about the constancy of the curvature of a Riemannian manifold. Quasi-conformal curvature tensor is a linear combination of the conformal curvature tensor and the concircular curvature tensor [15]. Many researchers have studied on these curvature tensors for Riemannian manifolds with different structures [8, 10–13].

The presented paper is on invariant submanifolds of para-Sasakian manifolds. We consider the quasi-conformal curvature tensor on a para-Sasakian manifold. Our aim is to obtain some conditions for totally geodesic and totally

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umbilical invariant submanifolds of quasi-conformally flat para-Sasakian manifolds. Firstly, we prove that a quasi conformally flat para-Sasakian manifold is η-Einstein. We examine some curvature relations on invariant submanifolds. Also, we prove that a totally umbilical invariant submanifold of a quasi-conformally flat para-Sasakian manifold is an η-Einstein manifold.

2. Preliminaries

Let M be a $(2n+1)-$dimensional smooth manifold. (φ, ξ, η) is called an almost para-contact structure on M if we have

$$\varphi^2 X = X - \eta(X)\xi \ , \ \varphi(\xi) = 0, \quad , \ \eta \circ \varphi = 0 \ , \ \eta(\xi) = 1,$$

where φ is a $(1,1)$ tensor field, ξ is a vector field and η is a $1-$ form on M. The tensor field φ induces an almost para-complex structure on the distribution $D = ker\eta$. There are two distributions $D\pm$ corresponding to the eigenvalues ± 1 and they are in same dimension m. The rank of φ is $2m$. A manifold with an almost para-contact structure is called an almost para-contact manifold [16].

Let g be a pseudo-Riemann metric on M. g is called compatible metric if $g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y)$, and associated metric if $d\eta(X,Y) = g(\varphi X, Y)$ for all $X, Y \in \Gamma(TM)$. Any compatible metric g with a given almost para-contact structure is necessarily of signature $(m+1, m)$. Also by taking $Y = \xi$, we get $\eta(X) = g(X, \xi)$. The $2-$form $\Phi(X,Y) = d\eta(X,Y)$ is called second fundamental form. For a C^∞ function f on a para-contact metric manifold M, a complex structure $\mathcal{J}$ is defined on $M \times \mathbb{R}$ by $\mathcal{J}(X, f\frac{d}{dt}) = (\varphi X + f\xi, \eta(X)\xi)$. If $\mathcal{J}$ is integrable i.e Nijenhuis tensor vanishes then almost para-contact structure (φ, ξ, η) is called normal. Like contact manifolds this condition is restricted to $N_\varphi + 2d\eta \otimes \xi = 0$. Normal para-contact manifolds are called para-Sasakian manifolds.

Theorem 2.1 ([4]). *Let $(M, \varphi, \xi, \eta, g)$ be an almost para-contact metric manifold. M is a para-Sasakian manifold if and only if*

$$(\nabla_X \varphi)\, Y = -g(X,Y)\xi + \eta(Y)X$$

for all $X, Y \in \Gamma(TM)$, where ∇ is Levi-Civita connection on M.

By setting $Y = \xi$, from above theorem on M we have

$$\nabla_X \xi = -\varphi X$$

for all $X \in \Gamma(TM)$.

Let X, Y be two arbitrary vector fields on M. For the convention Riemannian curvature; $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ we have the following the properties;

$$\begin{aligned} R(X,Y)\xi &= \eta(X)Y - \eta(Y)X \\ R(X,\xi)\xi &= -X + \eta(X)\xi \end{aligned} \tag{2.1}$$

Also, Ricci curvature of M is given by

$$Ric(X,\xi) = -2n\eta(X). \tag{2.2}$$

A para-Sasakian manifold M is said to be $\eta-$Einstein if its Ricci tensor satisfies

$$Ric(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y)$$

for all $X, Y \in \Gamma(TM)$ and constants α, β. It is clear that if a para-Sasakian manifold is $\eta-$Einstein then $Ric(\xi,\xi) = \alpha + \beta = -2n$, thus $Q\xi = -2n\xi$.

Quasi-conformal curvature tensor of a para-Sasakian manifold is given by

$$\begin{aligned} \widetilde{C}(X,Y)Z \;=\; & aR(X,Y)Z \\ & +b\{-(g(X,Z)QY + g(Y,Z)QX - Ric(X,Z)Y + Ric(Y,Z)X)\} \\ & K\{(g(Y,Z)X - g(X,Z)Y)\} \end{aligned} \tag{2.3}$$

for all $X, Y, Z \in \Gamma(TM)$, where a, b are constant, $K = -\frac{r}{2n+1}[\frac{a}{2n} + 2b]$, r is scalar curvature of M and Q is Ricci operator which is given by $Ric(X,Y) = g(QX,Y)$ [15]. A Riemannian manifold is called quasi-conformally flat if $\widetilde{C} = 0$.

3. Quasi-Conformally Flat Para-Sasakian Manifold

Let M be a quasi-conformally flat para-Sasakian manifold. Then, for all $X, Y, Z, T \in \Gamma(TM)$ using (2.3), we get

$$\begin{aligned} R(X,Y,Z,T) &= \frac{r}{a(2n+1)}[\frac{a}{2n}+2b]\{(g(Y,Z)g(X,T)-g(X,Z)g(Y,T)\} \\ &-\frac{b}{a}\{-g(X,Z)Ric(Y,T)+g(Y,Z)Ric(X,T)-Ric(X,Z)g(Y,T)+Ric(Y,Z)g(X,T)\}. \end{aligned} \tag{3.1}$$

Taking $Y = Z = \xi$ in (3.1) and by using equation (2.1) and (2.2), we have

$$Ric(X,T) = \alpha g(X,T) + \beta\eta(X)\eta(T)$$

which provides M is $\eta-$Einstein for $\alpha = \frac{2nb+K+a}{b}$ and $\beta = -\frac{4nb+K+a}{b}$. Thus, we state following theorem.

Theorem 3.1. *A quasi-conformally flat para-Sasakian manifold is $\eta-$Einstein.*

From above theorem and consider (3.1), the Riemannian curvature a quasi-conformally flat para-Sasakian manifold is given by

$$\begin{aligned} R(X,Y,Z,T) &= \frac{-4nb+K+a}{a}(g(Y,Z)g(X,T)-g(X,Z)g(Y,T)) \\ &+\frac{4nb+K+2a}{a}\{-g(X,T)\eta(Y)\eta(X)+g(Y,T)\eta(X)\eta(Z) \\ &-g(Y,Z)\eta(X)\eta(T)+g(X,Z)\eta(Y)\eta(T)\} \end{aligned}$$

for all $X, Y, Z, T \in \Gamma(TM)$. It is easily to see that if M is Einstein with constant scalar curvature, then M is a reel space form with constant curvature $c = \frac{-2\alpha b+K}{a}$. Also for unit and mutually orthogonal vector fields X, Y we get following.

Proposition 3.2. *Let M be a quasi-conformally flat para-Sasakian manifold. The sectional curvature of M is given by*

$$k(X,Y) = \frac{-4nb}{a} + \frac{r}{2n+1}(\frac{1}{2n}+2\frac{b}{a}) + 1$$

where X, Y are mutually orthogonal and unit vector fields.

Also with same approach we get $\phi-$sectional curvature of a quasi-conformally flat para-Sasakian manifold M as $k(X,\phi X) = \frac{-4nb}{a} + \frac{r}{2n+1}(\frac{1}{2n}+2\frac{b}{a}) + 1$ which is equal to $k(X,Y)$. In addition we can obtain $\xi-$sectional curvature of quasi-conformally flat para-Sasakian manifold M. By taking $T = X$ and $Y = Z = \xi$ in (3.1) for a unit vector field X is orthogonal to ξ we get $k(X,\xi) = 1$.

4. Invariant Submanifolds of Quasi-Conformally Flat Para-Sasakian Manifolds

Now, let N be an m-dimensional submanifold isometrically immersed in a para-Sasakian manifold M^{2n+1}. Denote by TN and $TN^{\perp}$ the tangent bundle of N and the normal bundle to N, respectively, ξ be tangent to N. Then for $X, Y \in \Gamma(TN)$, the Gauss formula is given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

where $h(X,Y)$ denote the second fundamental form and, $\bar{\nabla}, \nabla$ is the Levi-Cevita connections on M and N, respectively. On the other hand the Wiengarten formula is given by

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V$$

where A_V is the shape operator is related to V. In this notation $\bar{\nabla}, \nabla$ and $\nabla^{\perp}$ are; the Riemannian, induced Riemannian and induced normal connections in M, N and the normal bundle $TN^{\perp}$ of N respectively.

The shape operator and the second fundamental form has following relation:

$$\bar{g}(h(X,Y),V) = \bar{g}(A_V X, Y).$$

For X, Y, Z, T are tangent to N, we have a well known relation from Gauss equation;

$$\begin{aligned} \overline{g}(\overline{R}(X,Y,Z,T) &= R(X,Y,Z,T) \\ &+ g(h(Y,T),h(X,Y)) - g(h(X,T),h(Y,Z)). \end{aligned} \tag{4.1}$$

A submanifold is called totally geodesic if the second fundamental form vanishes. Also we recall a submanifold by totally umbilical if $h(X,Y) = g(X,Y)H$, where $X,Y \in \Gamma(TM)$ and H is the mean curvature of N is given by $H = \frac{1}{dimM} traceh.$

Definition 4.1. Let M^{2n+1} be a para-Sasakian manifold, N be an $m-$ dimensional submanifold of M and ξ be tangent to N. If $\varphi T_P N \subset T_P N$ for all $p \in N$, then N is called invariant submanifold.

Then we have a decomposition $TM = TN \oplus TN^{\perp} = \mathcal{D} \oplus sp\{\xi\} \oplus TN^{\perp}$, where $\mathcal{D}$ is called by invariant distribution. On an invariant submanifold of para-Sasakian manifolds we have following relations:

$$h(X,\varphi Y) = h(\varphi X, Y)$$

$$h(\varphi X,\varphi Y) = h(X,Y) \tag{4.2}$$

$$h(X,\xi) = 0 \tag{4.3}$$

where $X,Y \in \Gamma(TN)$.

Theorem 4.2. *A totally umbilical submanifold of a para-Sasakian manifold is invariant.*

Proof. Let N be a totally umbilical submanifold. Then for $X \in \Gamma(TN)$, we have

$$h(X,\xi) = g(X,\xi)H.$$

By replacing X by φX we have $g(\varphi X,\xi) = 0$. From (4.3) this shows us $\varphi X \in \Gamma(TN)$, which complete proof. □

Let N be an invariant submanifold of a quasi-conformally flat para-Sasakian manifold M. For brevity of computations we state

$$\begin{aligned} \mathcal{P}(X,Y,Z,T) &= g(Y,Z)g(X,T) - g(X,Z)g(Y,T) \qquad (4.4)\\ \mathcal{Q}(X,Y,Z,T) &= g(X,T)\eta(Y)\eta(Z) + g(Y,Z)\eta(X)\eta(T) - g(Y,T)\eta(X)\eta(Z) - g(X,Z)\eta(Y)\eta(T)\\ \mathcal{S}(X,Y,Z,T) &= -\ g(h(Y,T),h(X,Z)) + g(h(X,T),h(Y,Z)). \end{aligned}$$

Then from (3.1) and (4.1), the Riemannian curvature of N is given by

$$R(X,Y,Z,T) = \frac{-4nb+K+a}{a}\mathcal{P}(X,Y,Z,T) + \frac{4nb+K+2a}{a}\mathcal{Q}(X,Y,Z,T) + \mathcal{S}(X,Y,Z,T) \tag{4.5}$$

for all X,Y,Z,T vector fields are tangent to N.

Theorem 4.3. *A totally umbilical invariant submanifold of a quasi-conformally flat para-Sasakian manifold is a generalized reel space form.*

Proof. Let N be totally umbilical submanifold. Then for $X \in \Gamma(TN)$ and using (4.3), we have

$$g(X,\xi)H = 0$$

which provides that either $H = 0$ or $\eta(X) = 0$. Assume that $H \neq 0$. Since $\eta(X) = 0$ then $\mathcal{Q}(X,Y,Z,T) = 0$. On the other hand, we get

$$\mathcal{S}(X,Y,Z,T) = -g(H,H)\mathcal{P}(X,Y,Z,T).$$

Thus from (4.5), we get

$$R(X,Y,Z,T) = (\frac{-4nb+K+a}{a} + g(H,H))(g(Y,Z)g(X,T) - g(X,Z)g(Y,T)).$$

This gives us N is a generalized reel space form. □

By the following corollary we obtain a condition for the flatness of N.

Corollary 4.4. *Let N be a totally umbilical invariant submanifold of a quasi-conformally flat para-Sasakian manifold. Then, N is flat if and only if*

$$\|H\|^2 = \frac{-4nb+K+a}{a}.$$

Let take $X, Y, Z \in \Gamma(TN)$ and $U \in \Gamma(TN^{\perp})$. Then, from (3.1) we obtain

$$\begin{aligned} g(\bar{R}(X,Y)Z,U) &= \frac{-4nb+K+a}{a}(g(Y,Z)g(X,U)-g(X,Z)g(X,U)) \\ &+\frac{4nb+K+2a}{a}(-g(X,U)\eta(Y)\eta(X)+g(Y,U)\eta(X)\eta(Z) \\ &-g(Y,Z)\eta(X)\eta(U)+g(X,Z)\eta(Y)\eta(U)) \end{aligned}$$

and since $\eta(U)=0$ then we obtain $g(\bar{R}(X,Y)Z,U)=0$. Thus we state following result.

Corollary 4.5. *Let N be a totally umbilical invariant submanifold of a quasi-conformally flat para-Sasakian manifold. Then $\bar{R}(X,Y)Z$ is tangent to N.*

The sectional curvature of plane section which is spanned by unit and mutually orthogonal horizontal vector fields X, Y on submanifold N is given by

$$k(X,Y) = \frac{-4nb+K+a}{a} + g(h(Y,Y),h(X,X)) - g(h(X,Y),h(Y,X)).$$

The φ-sectional curvature of N which means the sectional curvature of a plane section is spanned by unit horizontal vector fields X and φX, using (4.2), we have

$$k(X,\varphi X) = \frac{-4nb+K+a}{a} + \|h(X,X)\|^2 - \|h(X,\varphi X)\|^2 .$$

Also the ξ-sectional curvature of N is given by

$$k(X,\xi)=1.$$

Let $\{E_1, ...E_m, \xi\}$ be an orthonormal basis of N. Using (4.5), setting $Y=Z=E_i$ and taking sum on i, we have

$$Ric_N(X,T) = \sum_{i=1}^{m+1}\{\frac{-4nb+K+a}{a}\mathcal{P}(X,E_i,E_i,T) + \frac{4nb+K+2a}{a}\mathcal{Q}(X,E_i,E_i,T) + \mathcal{S}(X,E_i,E_i,T)\}$$

where $Ric_N(X,T)$ is the Ricci curvature of N. From (4.4) we get

$$\begin{aligned} \sum_{i=1}^{m+1}\mathcal{P}(X,E_i,E_i,T) &= (m-1)g(X,T), \\ \sum_{i=1}^{m+1}\mathcal{Q}(X,E_i,E_i,T) &= g(X,T)-\eta(X)\eta(T), \\ \sum_{i=1}^{m+1}\mathcal{S}(X,E_i,E_i,T) &= \sum_{i=1}^{m+1}\{g(h(E_i,T),h(X,E_i)) - g(h(X,T),h(E_i,E_i))\}. \end{aligned}$$

Consequently, we have

$$\begin{aligned} Ric_N(X,T) &= (\frac{-4nb+K+a}{a}(m-1)+\frac{4nb+K+2a}{a})g(X,T) - (\frac{4nb+K+2a}{a})\eta(X)\eta(T) \\ &+\sum_{i=1}^{m+1}\{g(h(E_i,T),h(X,E_i)) - g(h(X,T),h(E_i,E_i))\}. \end{aligned}$$

Thus immediately one can seen the following result.

Theorem 4.6. *A totally geodesic invariant submanifold of a quasi-conformally flat para-Sasakian manifold is $\eta-$Einstein.*

Suppose that N is totally umbilical. Then we get

$$\begin{aligned} \sum_{i=1}^{m+1}\{g(h(E_i,T),h(X,E_i)) - g(h(X,T),h(E_i,E_i))\} &= \sum_{i=1}^{m+1}\{g(E_i,T)g(E_i,X)g(H,H)\} \\ &+g(X,T)g(H,H) \\ &= 2g(H,H)g(X,W). \end{aligned}$$

Then, we have

$$\begin{aligned} Ric_N(X,T) &= (\frac{-4nb+K+a}{a}(m-1)+\frac{4nb+K+2a}{a}+2g(H,H))g(X,T) \\ &\quad -(\frac{4nb+K+2a}{a})\eta(X)\eta(T) \end{aligned}$$

and we can state following result.

Theorem 4.7. *A totally umbilical invariant submanifold of a quasi-conformally flat para-Sasakian manifold is* $\eta-$*Einstein.*

Corollary 4.8. *Let N be a totally umbilical invariant submanifold of quasi-conformally flat para-Sasakian manifold. If N is minimal then it is an* η*- Einstein manifold.*

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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