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A New Approach on Roman Graphs

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Abstract. Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. A Roman dominating function (RDF) on a graph G is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v such that $f(v) = 2$. The weight of f is $\omega(f) = \Sigma_{v \in V} f(v)$. The minimum weight of an RDF on G, $\gamma_R(G)$, is called the Roman domination number of G. $\gamma_R(G) \leq 2\gamma(G)$ where $\gamma(G)$ denotes the domination number of G. A graph G is called a Roman graph whenever $\gamma_R(G) = 2\gamma(G)$. On the other hand, the differential of X is defined as $\partial(X) = |B(X)| - |X|$ and the differential of a graph G, written $\partial(G)$, is equal to $max\{\partial(X) : X \subseteq V\}$. By using differential we provide a sufficient and necessary condition for the graphs to be Roman. We also modify the proof of a result on Roman trees. Finally we characterize the large family of trees T such that $\partial(T) = n - \gamma(T) - 2$.

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1. Introduction

Graph protection involves the placement of mobile guards on the vertices of a graph to protect its vertices and edges against single or sequences of attacks and has its historical roots in the time of the ancient Roman Empire. The modern study of graph protection was initiated in the late twentieth century by the appearance of four publications in quick succession that referred to the military strategy of Emperor Constantine (Constantine The Great, 274-337 AD). The seminal paper is Ian Stewart's "Defend the Roman Empire!" in Scientific American, December 1999 [16], which contains a reply to C. S. ReVelle's "Can you protect the Roman Empire?", Johns Hopkins Magazine, April 1997 [14], and which is based on ReVelle and K. E. Rosing's "Defendens Imperium Romanum: A Classical Problem in Military Strategy" in American Mathematical Monthly, August-September 2000 [15]. ReVelle's work [14] in turn is a response to the paper " Graphing' an Optimal Grand Strategy" by J. Arquilla and H. Fredricksen [2], which appeared in Military Operations Research in 1995 and which is the oldest reference we could find that places the strategy of Emperor Constantine in a mathematical setting.

According to ancient history-some say mythology-Rome was founded by Romulus and Remus in 760-750 BC on the banks of the Tiber in central Italy. It was a country town whose power gradually grew until it was the centre of a large empire. In the third century AD Rome dominated not only Europe, but also North Africa and the Near East. The Roman army at that time was strong enough to use a forward defense strategy, deploying an adequate number of legions to secure on-site every region throughout the empire. However, the Roman Empire's power was greatly

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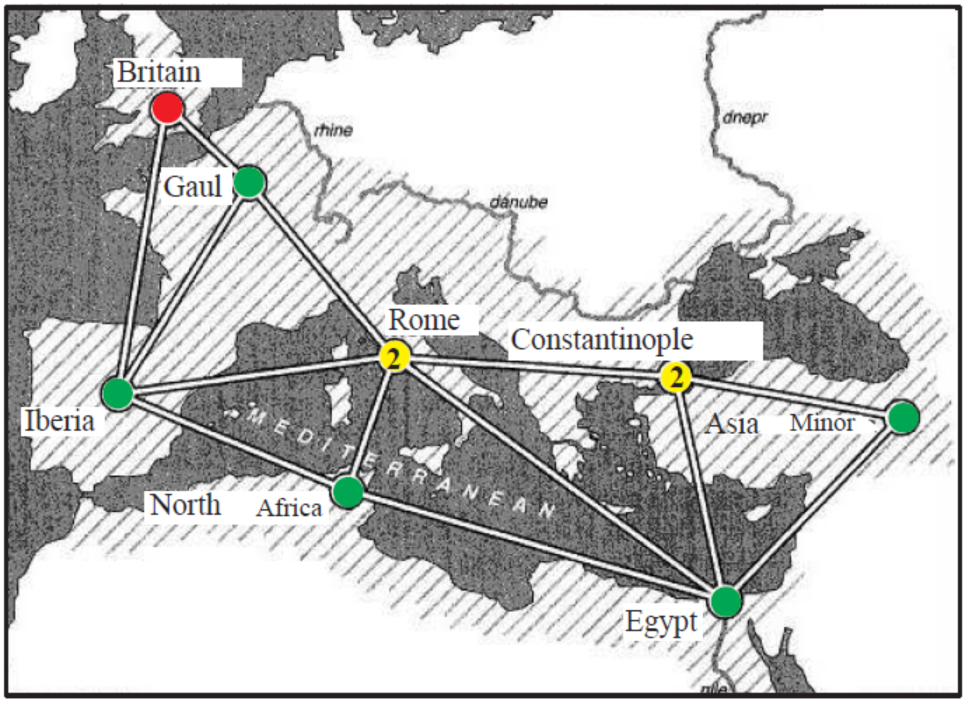


Figure 1. The Roman Empire, fourth century AD

reduced over the following hundred years. By the fourth century AD only twenty-five legions of the Roman army were available, which made a forward defense strategy no longer feasible.

According to E. N. Luttwak, The Grand Strategy of the Roman Empire, as cited in [15], to cope with the reducing power of the Empire, Constantine devised a new strategy called a defense in depth strategy, which used local troops to disrupt invasion. He deployed mobile Field Armies (FAs), units of forces consisting of roughly six legions powerful enough to secure any one of the regions of the Roman Empire, to stop the intruding enemy, or to suppress insurrection. By the fourth century AD there were only four FAs available for deployment, whereas there were eight regions to be defended (Britain, Gaul, Iberia, Rome, North Africa, Constantinople, Egypt and Asia Minor) in the empire. See Figure 1. An FA was considered capable of deploying to protect an adjacent region only if it moved from a region where there was at least one other FA to help launch it. The challenge that Constantine faced was to position four FAs in the eight regions of the empire. Consider a region to be secured if it has one or more FAs stationed in it already, and securable if an FA can reach it in one step. Constantine decided to place two FAs in Rome and another two FAs in Constantinople, making all regions either secured or securable -with the exception of Britain, which could only be secured after at least four movements of FAs. It is mentioned in [2,15,16] that Constantine's "defense in depth" strategy was adopted during World War II by General Douglas MacArthur. When conducting military operations in the Pacific theatre he pursued a strategy of "island-hopping" -moving troops from one island to a nearby one, but only when he could leave behind a large enough garrison to keep the first island secure. The efficiency of Constantine's strategy under different criteria, and ways in which it can be improved, were also discussed in these three articles. It should be noted that this history is entirely from the article [10].

Let $G = (V, E)$ be a simple undirected graph with set of vertices $V = V(G)$ and set of edges $E = E(G)$. We refer the reader to [17] for any terminology and notation not given here. We denote *minimum degree* of a graph G with $\delta(G)$ and *maximum degree* with $\Delta(G)$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u : uv \in E(G)\}$ and *closed neighborhood* of a vertex $v \in V$ is the set $N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$. The *closed neighborhood* of a set $S \subseteq V$ is the set $N[S] = N(S) \bigcup S$. Let E_v be the set of edges incident with v in G, that is, $E_v = \{uv \in E(G) : u \in N(v)\}$. We denote the *degree of* v by $d_G(v) = |E_v|$. A *leaf* or *pendant vertex* of G is a vertex with degree one, a *support vertex* is a vertex adjacent to a leaf, a *strong support vertex* is a support vertex adjacent to at least two leaves, and an *end-support vertex* is a support vertex, all of whose neighbors with the exception of at most one are leaves. The set of all leaves adjacent to a vertex v is denoted by $L(v)$. The *distance* $d_G(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest uv-path in G. The *diameter* of a graph G, denoted by $diam(G)$, is the greatest distance between two vertices of G. Given a set $S \subseteq V$, the *private neighborhood* $pn[v,S]$ of $v \in S$ is defined by $pn[v,S] = N[v] - N[S - \{v\}]$, equivalently, $pn[v,S] = \{u \in V : N[u] \cap S = \{v\}\}$. Each vertex in $pn[v,S]$ is called a private neighbor of v. From the definition of $pn[v,S]$, it is possible that $v \in pn[v,S]$.

The *external private neighborhood* $epn(v,S)$ of v with respect to S consists of those private neighbors of v in $V - S$. Thus $epn(v,S) = pn[v,S] \cap (V - S)$.

A set $S \subseteq V$ is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set $S \subseteq V$ is called a $\gamma(G)$-set if $|S| = \gamma(G)$. A set $S \subseteq V$ is an *independent dominating set* if $N[S] = V$ and the induced subgraph by S has no edge. The *independent domination number* $\gamma_i(G)$ is the minimum cardinality of an independent dominating set of G. Let G be a graph with no isolated vertex. A set $S \subseteq V$ is a *total dominating set* if $N(S) = V$. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G [8].

A graph G has property EPN if for every $\gamma(G)$-set S and for every $v \in S$, $epn(v,S) \neq \emptyset$. We call a tree with property EPN, an EPN-tree [11].

Let $G = (V,E)$ be a graph, $X \subseteq V$ and $B(X)$ be the set of vertices in $V - X$ that have a neighbor in the set X. If $X \subseteq V \neq \emptyset$, we define $C(X) = V - (X \cup B(X))$. We define the *differential* of a set X to be $\partial(X) = |B(X)| - |X|$ [11], and the *differential of a graph* G to be equal to $\partial(G)$ =max $\{\partial(X) : X \subseteq V\}$. A set D satisfying $\partial(D) = \partial(G)$ is called a ∂-set or differential set. A graph G is said to be a *dominant differential* if it contains a ∂-set which is also a dominating set, [3, 4]. Some examples of dominant differential graphs are complete graphs, stars, wheels, paths P_{3k}, P_{3k+2}, cycles C_{3k} and C_{3k+2}. An enclaveless number(or B-differential) of a graph $G = (V,E)$ is $\Psi(G) = max\{|B(X)| : X \subseteq V\}$.

A *rooted tree* is a tree in which one vertex has been designated the root. For a vertex v in a rooted tree T, let $Ch(v)$ denote the *set of children* of v, $D(v)$ denotes the *set of descendants* of v, and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. We denote a *star* with a central vertex and r leaves as $K_{1,r}$. A *double star* is a tree with exactly two vertices of degree at least two. We present a double star with respectively, r and t leaves attached to its support vertices by notation $DS_{r,t}$. The *subdivision graph* $S(G)$ of a graph G is that graph obtained from G by replacing each edge uv of G by a vertex w and edges uw and vw. A *healthy spider* is the subdivision graph of a star $K_{1,r}$ for $r \geq 2$. A *wounded spider* $S_{r,t}$ is a graph obtained from a star $K_{1,r}$ by subdividing t edges exactly once, where $1 \leq t \leq r - 1$. In a wounded spider $S_{r,t}$, the vertex of degree r is called the head vertex and the vertices that are placed at the distance of two from head vertex are called the foot vertices.

For a graph $G = (V,E)$, let $f : V \to \{0,1,2\}$ be a function, and let $f = (V_0, V_1, V_2)$ be the ordered partition of V induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$. A Roman dominating function (or just an RDF) on graph G is a function $f : V \to \{0,1,2\}$ such that if $v \in V_0$ for some $v \in V$, then there exists a vertex $w \in N(v)$ such that $f(w) = 2$. The weight of a Roman dominating function is the sum $w_f = \sum_{v \in V(G)} f(v)$, and the minimum weight of w_f of a Roman dominating function f on G is called Roman domination number of G. We denote this number with $\gamma_R(G)$. A Roman dominating function on G with weight $\gamma_R(G)$ is called a γ_R-function of G. For more on the Roman domination number see for example [3].

Let $f : V \to \{0,1,2,3\}$ be a function, and let $f = (V_0, V_1, V_2, V_3)$ be the ordered partition of V induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$. A *double Roman dominating function* (or just a DRDF) on graph G is a function $f : V \to \{0,1,2,3\}$ such that the following conditions are met:

(a) if $f(v) = 0$, then vertex v must have at least two neighbors in V_2 or one neighbor in V_3.

(b) if $f(v) = 1$, then vertex v must have at least one neighbor in $V_2 \bigcup V_3$.

The weight of a double Roman dominating function is the sum $w_f = \sum_{v \in V(G)} f(v)$, and the minimum weight of w_f for every double Roman dominating function f on G is called *double Roman domination number* of G. We denote this number with $\gamma_{dR}(G)$. A double Roman dominating function of G with weight $\gamma_{dR}(G)$ is called a γ_{dR}-function of G. Beeler et al. [4] have studied double Roman domination of graphs and Mojdeh et al. [13] have studied the double Roman trees. For a double Roman dominating function f if $V \setminus V_0$ is an independent set, then f is an independent double Roman dominating function [12], and if $V \setminus V_0$ has no isolated vertex, then f is a total double Roman dominating function [7].

A graph G is said to be a *Roman graph* if $\gamma_R(G) = 2\gamma(G)$. Henning [9] has studied the Roman trees. He specified exactly the family of Roman trees. But finding Roman graphs in general is still an open question. In the second section of this paper, we present a necessary and sufficient condition for a general graph to be Roman. Also, Lewis [11] introduced a necessary condition for the general graphs to be dominant differential. This necessary condition states: "If G does not have property EPN, then $\partial(G) \geq n - 2\gamma(G) + 1$". Also, he determined the family of trees T with the property $\partial(T) = n - \gamma(T) - 1$. In the third section of this paper, we specify exactly the family of dominant differential trees. Then in the fourth section, we determine a family of trees T such that $\partial(T) = n - \gamma(T) - 2$.

The following results are useful for the proofs of our main theorems.

Theorem A ([5].) A graph is dominant differential if and only if $\partial(G) = n - 2\gamma(G)$.

Theorem B ([3].) If G is a graph of order n, then $\gamma_R(G) = n - \partial(G)$.

Theorem C ([5].) For any graph G of order n without isolated vertices,

$$n - 2\gamma(G) \leq \partial(G) \leq n - \gamma(G) - 1.$$

Theorem D ([6].) For any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

Theorem E ([13].) Let G be a graph without isolated vertices. Then $\gamma_{dR}(G) \leq 2n - \partial(G) - \Psi(G)$. This bound is sharp.

Theorem F ([4].) If T is a non-trivial tree, then $\gamma_{dR}(T) \geq 2\gamma(T) + 1$.

Theorem G ([13].) For any tree T of order $n \geq 2$, $\gamma_{dR}(T) = 2\gamma(T) + 1$ if and only if $\partial(T) = n - \gamma(T) - 1$.

Theorem H ([11].) For any graph G of order n, $\Psi(G) = n - \gamma(G)$.

Theorem I ([6].) If T is a tree of order $n \geq 2$, then $\gamma_R(T) = \gamma(T) + 2$ if and only if **(i)** T is a healthy spider or **(ii)** T is a pair of wounded spiders T_1 and T_2, with a single edge joining $v \in V(T_1)$ and $w \in V(T_2)$, subject to the following conditions:

(1) if either T is a P_2, then neither vertex in P_2 is joined to the head vertex of the other tree.

(2) v and w are not both foot vertices.

2. A Necessary and Sufficient Condition for Roman Graphs

In this section we investigate a necessary and sufficient condition to identify Roman graphs. Recall that a graph consisting of one central vertex c and d neighbors that in turn have no further neighbors other than c is also known as a star $S_d = K_{1,d}$. We say S_d is a big star if $d \geq 2$.

Theorem 2.1. *A connected graph G is a Roman graph if and only if $V(G)$ can be partitioned into sets $X_1, \ldots, X_k, Y_1, \ldots, Y_k$, such that*

(i) $G[X_i]$ is a big star for each $i = 1, \ldots, k$, and $G[Y_j] \cong K_2$ for each $j = 1, \ldots, k'$,

(ii) $k + k' = \gamma(G)$, and

(iii)$\Sigma_{i=1}^{k}(|X_i| - 2) = \partial(G)$.

Proof. Let G be a Roman graph. Then by definition, we have $\gamma_R(G) = 2\gamma(G)$. Thus by Theorem B, we deduced $\partial(G) = n - \gamma_R(G) = n - 2\gamma(G)$. Now we suppose that $D = \{x_1, ..., x_k, y_1, ..., y_{k'}\}$ is a $\gamma(G)$-set such that x_i's and y_j's are central vertices of big stars X_i and edges Y_j respectively that do not have a common vertex. We put $\mathcal{S} = \{X_1, ..., X_k, Y_1, ..., Y_{k'}\}$. Therefore, $k+k' = \gamma(G)$ and $\sum_{Z\in\mathcal{S}}(|Z|-2) = |X_1|+...+|X_k|+|Y_1|+...+|Y_{k'}|-2-...-2 = n-2(k+k') = n-2\gamma(G) = \partial(G)$. Conversely, if G can be partitioned into a set of it's subgraphs, such as $\mathcal{S} = \{X_1, ..., X_k, Y_1, ..., Y_{k'}\}$, so that for any i, j; X_i's and Y_j's are the big stars and the edges respectively that do not have a common vertex and the same time we have, $k+k' = \gamma(G)$ and $\sum_{Z\in\mathcal{S}}(|Z|-2) = \partial(G)$ then we have $\partial(G) = \sum_{Z\in\mathcal{S}}(|Z|-2) = |X_1|+...+|X_k|+|Y_1|+...+|Y_{k'}|-2-...-2 = n - 2(k + k') = n - 2\gamma(G)$. Thus, by Theorem B, $\gamma_R(G) = n - \partial(G) = n - (n - 2\gamma(G)) = 2\gamma(G)$. □

In [9], Henning introduced a large family $\mathcal{T}$ of trees T so that $\gamma_R(T) = 2\gamma(T)$. We now modify the proof of Roman trees that has been given by Henning [9].

Theorem 2.2. *Let T be a tree. Then T is a Roman tree if and only if $V(T)$ can be partitioned into sets $X_1, \ldots, X_k$, such that*

(i) $G[X_i]$ is a big star for each $i = 1, \ldots, k$, or

(ii) $V(T)$ can be partitioned into sets $X_1, \ldots, X_k, Y_1, \ldots, Y_{k'}$ such that $G[X_i]$ is a big star for each $i = 1, \ldots, k$, and $G[Y_j] \cong K_2$ for each $j = 1, \ldots, k'$, where

1. no vertex of Y_js is adjacent to the center of any X_i and a vertex of another Y_l,

2. for each X_is there is at most one Y_j for which one of vertex of X_i is adjacent to one of vertex of Y_j.

Proof. Let T be tree. If T can be partitioned to a set $\mathcal{S}$ consists of k big stars $X_1, ..., X_k$, then we put $D = \{x_1, ..., x_k\}$ which $x_1, ..., x_k$ are central vertices for X_i's respectively. Now assigning label 2 to the vertices $x_1, ..., x_k$ and use the Theorem B, we have $\gamma_R(T) = 2k = 2|D| = 2\gamma(T)$. But if T can be partitioned to a set $\mathcal{S}$ consists of k big stars $X_1, ..., X_k$ and k' edges $Y_1, ..., Y_{k'}$ such that no vertex of Y_i's are connected to the central vertices of X_j's or vertices in another Y_l and each X_j is connected to at most one edge Y_i, then we put $D = \{x_1, ..., x_k, y_1, ..., y_{k'}\}$ where $x_1, ..., x_k, y_1, ..., y_{k'}$ are central vertices for X_i's and Y_j's respectively. Now assigning label 2 to the vertices $x_1, ..., x_k, y_1, ..., y_{k'}$ and use the Theorem B, we have $\gamma_R(T) = 2(k + k') = 2|D| = 2\gamma(T)$.

Conversely, if T is a Roman tree, then by Theorem 2.1, T can be partitioned into a set of it's subgraphs, such as $\mathcal{S} = \{X_1, ..., X_k, Y_1, ..., Y_{k'}\}$, in which for any i, j; X_i's and Y_j's are the big stars and the edges respectively without any common vertex and $k + k' = \gamma(T)$, and also $\sum_{Z\in\mathcal{S}}(|Z| - 2) = \partial(T)$. If $k' = 0$, then the result is obtained. Otherwise, we have $k' \geq 1$. On the other hand, since $k + k' = \gamma(T)$, and also $\sum_{Z\in\mathcal{S}}(|Z| - 2) = \partial(T)$ we must have the set $\mathcal{S}$ consists of k big stars $X_1, ..., X_k$ and k' edges $Y_1, ..., Y_{k'}$ such that no vertex of Y_js is adjacent to the central vertices of X_j's or vertices in another Y_l and for each X_is there is at most one Y_j for which one of vertex of X_i is adjacent to one of vertex of Y_j. □

3. Dominant Differential Trees

Lewis in [11] has posed the following open problem.

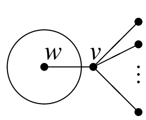
Problem. Characterize the dominant differential graphs, in particular, characterize the dominant differential trees. According to the Theorems A and C we must find trees of T such that $\partial(T) = n - 2\gamma(T)$. We would like to characterize the dominant differential trees.

For a vertex v in a (rooted) tree T, we let $Ch(v)$ and $De(v)$ denote the set of children and descendants, respectively. We denote the set of support vertices of T by $S(T)$. In the paper [9], Henning describes a procedure to build Roman trees. For this purpose, he defines two families of trees as follows. Let F_1^* denote the family of all rooted trees such that every leaf different from the root is at distance 2 from the root and all, except possibly one, child of the root is a strong support vertex. Let F_2^* denote the family of all rooted trees such that every leaf is at distance 2 from the root and all but two children of the root are strong support vertices. For a tree T we let $V_S(T) = \{v \in V(T) : v \in S(T) \text{ and } \gamma_{dR}(T - v) \geq \gamma_{dR}(T)\}$. Note that every strong support vertex of T belongs to $V_S(T)$. Let $\mathcal{T}$ be the family of unlabelled trees T that can be obtained from a sequence $T_1, ..., T_j$ ($j \geq 1$) of trees such that T_1 is a star $K_{1,r}$ for $r \geq 1$, and if $j \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the three operations $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$.

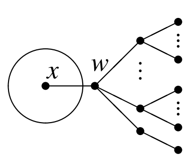
Operation $\mathcal{T}_1$. Assume $w \in V_S(T_i)$. Then the tree T_{i+1} is obtained from T_i by adding a star $K_{1,s}$ for $s \geq 2$ with central vertex v and adding the edge vw.

Operation $\mathcal{T}_2$. Assume $x \in V(T_i)$. Then the tree T_{i+1} is obtained from T_i by adding a tree T from the family F_1^* by adding the edge xw, where w is a leaf of T if $T = P_3$ or w is the central vertex of T if $T \neq P_3$.

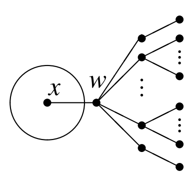
Operation $\mathcal{T}_3$. Assume $x \in V_S(T_i)$. Then the tree T_{i+1} is obtained from T_i by adding a tree T from the family F_2^* and adding the edge xw, where w denotes the central vertex of T.



$\mathcal{T}_1$: Figure 1



$\mathcal{T}_2$: Figure 2



$\mathcal{T}_3$: Figure 3

Theorem J ([9].) A tree T is a Roman tree if and only if $T \in \mathcal{T}$.

Theorem 3.1. *A tree T is dominant differential if and only if $T \in \mathcal{T}$.*

Proof. If T is a dominant differential tree then by Theorem A, we have $\partial(T) = n - 2\gamma(T)$. On the other hand, by Theorem B, $\gamma_R(T) = n - \partial(T)$. Thus, we conclude, $\gamma_R(T) = n - (n - 2\gamma(T)) = 2\gamma(T)$. Hence, T is a Roman tree. Now by Theorem J, $T \in \mathcal{T}$. Conversely, if $T \in \mathcal{T}$, then by Theorem J, T is a Roman tree. Thus, we have $\gamma_R(T) = 2\gamma(T)$. Now by Theorem B, $\partial(T) = n - \gamma_R(T) = n - 2\gamma(T)$. Finally, by Theorem A we conclude, T is a dominant differential tree. □

4. Characterization of Trees with $\partial(T) = n - \gamma(T) - 2$

Lewis in his thesis [11], entitled Differential of graphs, showed that for any graph G, we have $\partial(G) \leq n - \gamma(G) - 1$. Also, he determined the family of trees T such that $\partial(T) = n - \gamma(T) - 1$. On the other hand, Mojdeh et al. in the paper [13], showed that for any graph G, $\gamma_{dR}(G) \leq 2n - \psi(G) - \partial(G)$. They also showed a necessary condition for trees T such that $\partial(T) = n - \gamma(T) - 2$. In this section, we identify the family of trees T with the property $\partial(T) = n - \gamma(T) - 2$. For this, we introduce the families of trees.

Let $\mathcal{T}$ be a family of trees, each of which is either a non-trivial star or a wounded spider. In 2004, Cockayne et al. [6] identified, the family of trees T with $\gamma_{dR}(T) = 2\gamma(T) + 2$.(Theorem I)

In [1], Ahangar et al. showed the whole family of trees with $\gamma_{dR}(T) = 2\gamma(T) + 2$ is the same as the trees represented in the Theorem I. To this end, they introduced eight families of trees as follows with more details:

Let $\mathcal{T}_0$ be the class consisting of the path P_2 and all wounded spiders different from a path P_4 whose head vertex has a unique leaf. Since $\mathcal{T}_0 \subseteq \mathcal{T}$, we let $\mathcal{H} = \mathcal{T} - \mathcal{T}_0$. Let c denote either the unique leaf adjacent to the head of wounded spiders in class $\mathcal{T}_0$ or a vertex of the path P_2. Then they introduced the following families of trees.

1. $\mathcal{T}_1$ is the family of trees T obtained from a tree $T' \in \mathcal{T}$ by adding a star $K_{1,r}(r \geq 2)$ and joining a leaf of $K_{1,r}$ to a vertex of T'.

2. $\mathcal{T}_2$ is the family of trees T obtained from a tree $T' \in \mathcal{T}_0$ by adding a double star $DS_{1,q}$ $(q \geq 2)$ and joining the support vertex of degree 2 in $DS_{1,q}$ to the head vertex or a support vertex of T'.

3. $\mathcal{T}_3$ is the family of trees T obtained from a tree $T' \in \mathcal{H}$ by adding a double star $DS_{1,q}$ $(q \geq 2)$ and joining the support vertex of degree 2 in $DS_{1,q}$ $(q \geq 2)$ to a vertex of T' different from leaves at distance 2 of the head vertex in T'.

4. $\mathcal{T}_4$ is the family of trees T obtained from a tree $T' \in \mathcal{H}$ by adding P_4 (resp. $K_{1,r}(r \geq 2)$) and joining a support vertex of P_4 (resp. the center of $K_{1,r}$) to a vertex of T'.

5. $\mathcal{T}_5$ is the family of trees T obtained from a tree $T' \in \mathcal{T}_0$ by adding P_4 (resp. $K_{1,r}(r \geq 2)$), and joining a support vertex of P_4 (resp. the center of $K_{1,r}$) to a vertex of T' different from c.

6. $\mathcal{T}_6$ is the family of trees T obtained from a tree $T' \in \mathcal{T}_0 - P_2$ by adding a corona P_3oK_1 and joining an end-support vertex of P_3oK_1 to a support vertex adjacent to the head of T'. 7. $\mathcal{T}_7$ is the family of trees T obtained from a tree $T' \in \bigcup_{i=1}^{6} \mathcal{T}_i \bigcup \{P_6oK_1\}$ by adding $r \geq 1$ copies of P_2 and joining a vertex of each copy of P_2 to a strong support vertex or a support vertex adjacent to an end-support vertex or a support vertex adjacent to a vertex of degree 2 of T'.

8. $\mathcal{T}_8$ is the family of trees T obtained from a tree $T' \in \mathcal{T}_0 - P_2$ by adding the healthy spider and joining the head of healthy spider to c.

Theorem K ([1].) Let T be a tree of order $n \geq 5$. Then $\gamma_{dR}(T) = 2\gamma(T) + 2$ if and only if $T \in \bigcup_{i=1}^{8} \mathcal{T}_i$ or T is a healthy spider or $T = P_6oK_1$.

Theorem 4.1. *Let T be a tree of order $n \geq 5$. Then $\partial(T) = n - \gamma(T) - 2$ if and only if $T \in \bigcup_{i=1}^{8} \mathcal{T}_i$ or T is a healthy spider or $T = P_6oK_1$.*

Proof. If $\partial(T) = n - \gamma(T) - 2$ then by Theorems E and H, we have $\gamma_{dR}(T) \leq 2n - \psi(T) - \partial(T) = 2n - (n - \gamma(T)) - (n - \gamma(T) - 2) = 2\gamma(T) + 2$. On the other hand, by Theorem F, $2\gamma(T) + 1 \leq \gamma_{dR}(T)$. But $2\gamma(T) + 1 \neq \gamma_{dR}(T)$ because otherwise by Theorem G, we obtain $\partial(T) = n - \gamma(T) - 1$ which is contradiction. Thus, we conclude $2\gamma(T) + 2 = \gamma_{dR}(T)$.

Conversely, we show that, if $T \in \bigcup_{i=1}^{8} \mathcal{T}_i$ or T is a healthy spider or $T = P_6oK_1$, then $\partial(T) = n - \gamma(T) - 2$. To this end, by Theorem K $\gamma_{dR}(T) = 2\gamma(T) + 2$. Thus, by Theorems E and K, $\partial(T) \leq 2n - \gamma_{dR}(T) - \psi(T) = 2n - (2\gamma(T) + 2) - (n - \gamma(T)) = n - \gamma(T) - 2$. So, it is enough to show that $\partial(T) \geq n - \gamma(T) - 2$. So, we have to find a set $X \subseteq V(T)$, such that $\partial(X) \geq n - \gamma(T) - 2$. If T is a healthy spider, then we suppose $T = S(K_{1,r})$ and v is a central vertex of T. Now we put $X = \{v\}$. Thus, we have $\partial(X) = |B(X)| - |X| = r - 1$, $n = 2r + 1$ and $\gamma(T) = r$. Hence, $\partial(X) = \gamma(T) - 1 = n - \gamma(T) - 2$. So in this case, $\partial(G) \geq \partial(X) \geq n - \gamma(T) - 2$. If $T = P_6oK_1$, then we suppose that x and y are two vertices at the distance of 3 from each other in tree T. Thus, clearly $n = 12$, $\gamma(T) = 6$. Now we put $X = \{x, y\}$. Hence, $\partial(X) = |B(X)| - |X| = 6 - 2 = 4 = 12 - 6 - 2 = n - \gamma(T) - 2$. Therefore, $\partial(T) \geq n - \gamma(T) - 2$. Hence, let $T \in \bigcup_{i=1}^{8} \mathcal{T}_i$. Now there are the following cases.

Case 1. $T \in \mathcal{T}_1$.

Then T is obtained from a tree $T' \in \mathcal{T}$ with the central vertex x by adding a star $K_{1,r}$ $(r \geq 2)$ centered at y and joining a leaf of $K_{1,r}$ to a vertex of a T'. If T' is a star $K_{1,t}$, then clearly $\gamma(T) = 2$ and $n = r + t + 1 + 1$. We put $X = \{x, y\}$. Thus, $\partial(X) = |B(X)| - |X| = r + t - 2 = r + t + 2 - 2 - 2 = n - \gamma(T) - 2$. If T' is a wounded spider $S_{t,j}$ $(1 \leq j \leq t - 1)$, then clearly, $n = r + 1 + t + 1 + j$ and $\gamma(T) = t + 1$. Thus, we have $r + t = n - 2 - j$. Now we put $X = \{x, y\}$. Hence, $\partial(X) = |B(X)| - |X| = t + r - 2 = n - 2 - j - 2$. But clearly, $\gamma(T) \geq j + 2$. So, we have $\partial(X) = |B(X)| - |X| = t + r - 2 = n - 2 - j - 2 \geq n - \gamma(T) - 2$.

Case 2. $T \in \mathcal{T}_2$.

Then T is obtained from a tree $T' \in \mathcal{T}_0$ by adding a double star $DS_{1,q}$ ($q \geq 2$) and joining the support vertex of degree 2 in $DS_{1,q}$ to the head vertex or a support vertex of T'. So, we have two cases that are as follows. Let $T' = P_2 = xy$. We suppose that z is the support vertex of degree $q + 1$ in $DS_{1,q}$. We put $X = \{y, z\}$. Then, clearly $\gamma(T) = 3$ and $n = 2+q+1+2 = q+5 = q+3+2 = q+\gamma(T)+2$. Hence, $\partial(X) = |B(X)|-|X| = 2+q-2 = q = n-\gamma(T)-2$. Now let $T' = S_{r,t}$ is a wounded spider different from a path P_4 whose head vertex has a unique leaf. We suppose that z is the support vertex of degree $q + 1$ in $DS_{1,q}$ and x is head vertex of T'. Now we put $X = \{x, z\}$. Then clearly, $n = 3 + t + r + 1 + q = 4 + t + r + q$ and $\gamma(T) = 5$. Thus we have $\partial(X) = |B(X)| - |X| = q + 1 + t - 2 = q + t - 1$. Hence, $\partial(X) = n - 4 - r - 1 = n - 5 - r$. But clearly, $\gamma(T) \geq r + 3$. Therefore, we have $\partial(T) \geq n - \gamma(T) - 2$.

With the same method, we can follow that if $T \in \bigcup_{i=3}^{8} \mathcal{T}_i$, then we have $\partial(T) \geq n - \gamma(T) - 2$. □

Theorem L ([11]). A tree T has $\partial(T) = n - \gamma(T) - 1$ if and only if T is nontrivial wounded spider.

Theorem M ([4]). In a double Roman dominating function of weight $\gamma_{dR}(G)$, no vertex needs to be assigned the value 1.

Theorem 4.2. *If T is a tree of order n, then $\gamma_{dR}(T) = 2\gamma(T) + 2$ if and only if*
(1) T does not have a vertex of degree $n - \gamma(T)$.
(2) T has a vertex of degree $n - \gamma(T) - 1$ or T has two vertices x and y such that $|N[x] \bigcup N[y]| = n - \gamma + 2$.

Proof. $\Leftarrow$: Let two conditions (1) and (2) hold. By Theorem F, we have $\gamma_{dR}(T) \geq 2\gamma(T) + 1$. By Theorems G and L, if $\gamma_{dR}(T) = 2\gamma(T) + 1$, then T is a wounded spider and thus, T has a vertex of degree $n - \gamma(T)$, is a contradiction with the condition (1). Therefore, $\gamma_{dR}(T) > 2\gamma(T) + 1$. Now if T has a vertex x of degree $n - \gamma(T) - 1$, then we put $D = \{x\}$. Hence, we have $\partial(T) \geq \partial(D) = (n - \gamma(T) - 1) - 1 = n - \gamma(T) - 2$. On the other hand, by Theorems E and H, $\gamma_{dR}(T) \leq 2n - \psi(T) - \partial(T) \leq 2n - (n - \gamma(T)) - (n - \gamma(T) - 2) = 2\gamma(T) + 2$. Now we deduce $\gamma_{dR}(T) = 2\gamma(T) + 2$. Similarly, if there are two vertices, x and y such that $|N[x] \bigcup N[y]| = n - \gamma + 2$, then we put $D = \{x, y\}$. Hence, we have $\partial(T) \geq \partial(D) = n - \gamma(T) - 2$. So, by Theorems E and H, $\gamma_{dR}(T) \leq 2n - \psi(T) - \partial(T) \leq 2n - (n - \gamma(T)) - (n - \gamma(T) - 2) = 2\gamma(T) + 2$. Therefore we have also $\gamma_{dR}(T) = 2\gamma(T) + 2$.

$\Rightarrow$: Conversely, let $\gamma_{dR}(T) = 2\gamma(T) + 2$. First of all it is simply verifiable the condition (1) holds. Now by Theorem M, let $f = (V_0, V_2, V_3)$ be a γ_{dR}-function on T. Thus, $\gamma_{dR}(T) = 2|V_2| + 3|V_3| = 2(|V_2| + |V_3|) + |V_3| \geq 2\gamma(T) + |V_3|$. Since, $\gamma_{dR}(T) = 2\gamma(T)+2$, $2\gamma(T)+2 = 2(|V_2|+|V_3|)+|V_3|$. Thus $|V_3|$ is even and $0 \leq |V_3| \leq 2$, that is $|V_3| = 0$ or $|V_3| = 2$. Now if $|V_3| = 2$, then we say $V_3 = \{x, y\}$. So we have $|V_2| = \gamma(T) - 2$ and $|V_2| + |V_3| = \gamma(T)$. This shows $V_2 \bigcup V_3$ is a $\gamma(T)$-set of T. Thus the vertices with label 2 are not adjacent to the x or y, and we have $|N[x] \bigcup N[y]| = n-(\gamma(T)-2) = n-\gamma(T)+2$. If $|V_3| = 0$, then $|V_2| = \gamma(T) + 1$. According to the definition of DRDF of a graph, $\gamma(T) + 1 = |V_2| > |V_0| \geq \gamma(T)$. So, we have $|V_0| = \gamma(T)$. On the other hand, no two vertices of the labels 2 and no two vertices of the labels 0 adjacent to each other because otherwise, we find a dominating set with cardinality at most $\gamma(T) - 1$ which is a contradiction. So, the tree T is a V_0, V_2-bigraph. It is clearly that, for all $S \subseteq V_0$, we have $|N(S)| \geq |S|$. Therefore, by Hall's Theorem [17], T has a matching that saturates V_0. Hence, T has a vertex $x \in V_2$ such that $deg(x) = n - (\gamma(T) + 1)$ or T have two vertices $x, y \in V_0 \bigcup V_2$ such that $|N[x] \bigcup N[y]| = n - \gamma(T) + 2$. □

Discussions. In the introduction section, we expressed the definition of Total double Roman domination in graphs [7] and independent double Roman domination in Graphs [12].

In Theorem 4.2 we discuss the relationship between $\gamma_{dR}(T)$ and $\gamma(T)$. Therefore we have the following problems.

1. Is there a such relationship between Total double Roman domination number and total domination number of trees?

2. The problem 1, may be raised between independent double Roman domination number and independent domination number of trees.

We discussed on the dominant differential trees versus domination number and order of trees. Therefore we may have the problems as follows:

3. Let $\gamma_t(T)$ be the total domination of T. What is the relationship between $\partial(T)$, $\gamma_t(T)$ and order of T?

4. Let $\gamma_i(T)$ be the independent domination number of T. What is the relationship between $\partial(T)$, $\gamma_i(T)$ and order of T?

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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