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Large deviation principle for reflected diffusion process fractional Brownian motion

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Abstract

In this paper we establish a large deviation principle for solution of perturbed reflected stochastic differential equations driven by a fractional Brownian motion B_t^H with Hurst index $H \in (0;1)$.
The key is to prove a uniform Freidlin-Wentzell estimates of solution.

Keywords: Fractional Brownian motion, Large deviation principle, Contraction principle, Reflected stochastic differential equation, Skorohod problem.
2010 MSC: 60F10, 60G22, 60H20, 60H40.

1. Introduction

There are different methods to show that the diffusion process satisfies the principle of large deviations (LDP), for which several authors have determined the rate function in different spaces. In addition, in the case of the large deviation principle for a standart Brownian motion, many autors had established the LDP for perturbed diffusion processes see among others L. Bo and T. Zhang ([3]) , H. Doss and P. Priouret ([8]). Regarding a fractional Brownian motion, Y. Inahama ([10]) proved, in the framework of the rough trajectory theory that the process εB_t^H obeys a large deviation principle for $H \in (\frac{1}{4};\frac{1}{2})$ because the integral only verifies the Young theorem if $H \in (\frac{1}{4};\frac{1}{2})$. Other authors have established large deviation for local times of fractional Brownian motion, X. Chen, W. V. Li, J. Rosinski and Q. Shao ([4]), M. M. Meerschaert and E. X. Y. Nane ([11]), and Z. Chen and W. Wang ([17]). The novelty of our work is to extend the work of R.A. Doney

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and T. Zhang ([5]) a new approch via the principle of contraction,in the framework of stochastic differential equations directed by a fractional Brownian motion (fBm) of Hurst parameter $H \in (0;1)$,by determining the rate function of the dual of Schwartz space. The rest of this paper is organized as follows. Sections 2 contains some definitions and theorems of the fBm and LDP which we need for our results. Section 3 contain our main results.

2. Preliminaries

Let $B^H = \{B^H_t, t \in [0;T]\}$ be a fractional Brownian motion (fBm) of Hurst parameter $H \in (0;1)$ with covariance function:

$$\mathcal{R}(t,s) = E(B^H_t B^H_s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}) = \int_0^t \int_0^s \phi(r,u)dudr,$$

$\phi(t,s) = \frac{\partial^2 \mathcal{R}(t,s)}{\partial t \partial s} = H(2H-1)|t-s|^{2H-2}$.

Consider the perturbed reflected stochastic differential equations:

$$X^{H,\varepsilon}_t = x + \int_0^t b(X^{H,\varepsilon}_s)ds + \varepsilon \int_0^t \sigma(X^{H,\varepsilon}_s)dB^H_s + \alpha \sup_{0 \le r \le t}(X^{H,\varepsilon}_r), \;\; s,t \in [0;T] \tag{1}$$

$$Y^{H,\varepsilon}_t = y + \int_0^t b(Y^{H,\varepsilon}_s)ds + \varepsilon \int_0^t \sigma(Y^{H,\varepsilon}_s)dB^H_s + L^\varepsilon_t, \;\; s,t \in [0;T] \tag{2}$$

defined on noise probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mathbb{P})$, where

1. $\alpha \in [0;1]$;
2. x and $y \in \mathcal{S}'(\mathbb{R})$ are deterministic;
3. b and $\sigma : [0;T] \times \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ are measurable functions such that the integrals are defined as white noise integral (see [2],[1] and [15]) and they are bounded lipschitz continuous;
4. L^ε_t is non-decreasing such that

$$L^\varepsilon_t = \begin{cases} 0 \; if \; t = 0 \\ \int_0^t 1_{\{Y_r = 0\}}(Y_r)dL^\varepsilon_r \; if \; t \in [0;T] \end{cases} \tag{3}$$

⋆ $\mathcal{S}'(\mathbb{R})$ is a space of tempered distriution, called dual space of Schwartz space.

Consider a white noise space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mathbb{P})$ and denote $\langle .,. \rangle$ the scalar product and $|.|$ the norm in $\mathcal{S}'(\mathbb{R})$. It well know that $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$.

Definition 2.1. *For $\omega \in \mathcal{S}'(\mathbb{R})$, a process $\langle \omega, f \rangle_\phi = \int_0^t f(r)dB^H_r$ is a gaussian with covariance (see [1]), $\langle f, f \rangle_\phi = |f|^2_\phi = \int_0^t \int_0^s f(r)f(u)\phi(r,u)dudr$ and $L^2_\phi(\mathbb{R}) = \{f \in \mathcal{S}'(\mathbb{R}), \int_0^t \int_0^s f(r)f(u)\phi(r,u)dudr < +\infty\}$.*

Definition 2.2. *The family $(X^\varepsilon_t)_{\varepsilon>0}$ of probability $\mathbb{P}^\varepsilon$ is said to satisfy a large deviation principle if there exists a rate function I defined on $L^2_\phi(\mathbb{R})$ and a speed ε tending to 0 such that:*

1. *$0 \le I(x) \le +\infty$;*
2. *I is lower semicontinuous that is, for all $a \in \mathbb{R}$, $\{x : I(x) \le a\}$ is a closed of $L^2_\phi(\mathbb{R})$;*
3. *for all $a \in \mathbb{R}$, $\{x : I(x) \le a\}$ is a compact of $L^2_\phi(\mathbb{R})$, in which case I is a good rate function;*
4. *for any closed set $C \subset L^2_\phi(\mathbb{R})$,*

$$\lim_{\varepsilon \to 0} \sup \varepsilon \log \mathbb{P}^\varepsilon(X^\varepsilon_t \in C) \le -\inf_{x \in C} I(x)$$

5. *for any open set* $O \subset L^2_\phi(\mathbb{R})$,

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}^\varepsilon(X^\varepsilon_t \in O) \geq -\inf_{x\in O} I(x)$$

Theorem 2.3. ***(Contraction principle**, [5])*
Let $E_1,\ E_2 \subset L^2_\phi(\mathbb{R})$ *and* $g : E_1 \to E_2$ *is a continuous function. If the family* $(X^\varepsilon_t)_{\varepsilon>0}$ *satisfies a large deviation principle of a rate function* I *then the family* $g((X^\varepsilon_t)_{\varepsilon>0})$ *satisfies the LDP on* E_2 *of a rate function* J *defined by:*

$$J(y) = \inf\{I(x) : x \in E_1,\ y = g(x)\}\ ,\ \textit{for each } y \in E_2.$$

3. Main Results

In this section of our results, we first present the asymptotic behavior study of the solution process $Y^{H,\varepsilon}_t$ (2) and lastly that of $X^{H,\varepsilon}_t$ (1) . Before giving our main theorem for this part, we first present the LDP theorem for fBm εB^H_t with probability measure $\mathbb{P}^{H,\varepsilon}_\phi$.
Thus this theorem is as follow:

Theorem 3.1. *(see [6])*
The family $(\varepsilon B^H_t)_{(\varepsilon>0)}$ *satisfies the large deviation principle of speed* ε^2 *with a rate function given by:*

$$I(f) = \begin{cases} \frac{1}{2}|f|^2_\phi = \frac{1}{2}\int_0^t\int_0^s f(r)f(u)\phi(r,u)dudr \ if\ f \in L^2_\phi(\mathbb{R}) \\ +\infty\ otherwise\ . \end{cases}$$

1. $I(f)$ *is lower semicontinuous and* $\{f : I(f) \leq a\}$ *is a compact subset of* $L^2_\phi(\mathbb{R})$,
2. *For all closed set* $C \subset L^2_\phi(\mathbb{R})$,

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}^{H,\varepsilon}_\phi(\varepsilon B^H_t \in C) \leq -\frac{1}{2}|f|^2_\phi$$

3. *For any open set* $O \subset L^2_\phi(\mathbb{R})$,

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}^{H,\varepsilon}_\phi(\varepsilon B^H_t \in O) \geq -\frac{1}{2}|f|^2_\phi\ .$$

3.1. Large deviation principle for reflected fractional diffusion process

We will prove the LDP for solution of the perturbed stochastic differential equation (2).
For $\varphi \in L^2_\phi(\mathbb{R})$, define an operators $\Gamma : L^2_\phi(\mathbb{R}) \to L^2_\phi(\mathbb{R})$ by

$$\Gamma\varphi = \varphi - \inf_{0\leq s\leq t}(\varphi(s) \wedge 0) \qquad for\ t \in [0;T]. \tag{4}$$

verifying the following inequality:

$$\sup_{0\leq r\leq t} |\Gamma\varphi_1(r) - \Gamma\varphi_2(r)| \leq 2 \sup_{0\leq r\leq t} |\varphi_1(r) - \varphi_2(r)|$$

By the reflection principle (see [3] and [5]), the solution of (2) is given by

$$\begin{cases} Y^{H,\varepsilon}_t = \Gamma Z^{H,\varepsilon}_t \\ L^\varepsilon_t = Y^{H,\varepsilon}_t - Z^{H,\varepsilon}_t = \Gamma Z^{H,\varepsilon}_t - Z^{H,\varepsilon}_t \end{cases} \tag{5}$$

where $Z^{H,\varepsilon}$ is a solution of the following stochastic differential equation:

$$Z^{H,\varepsilon}_t = y + \int_0^t b(\Gamma Z^{H,\varepsilon}_r)dr + \int_0^t \sigma(\Gamma Z^{H,\varepsilon}_r)dB^H_r,\ s,t \in [0;T] \tag{6}$$

and we denote the probability law of $Y_t^{H,\varepsilon}$ by $\mathbb{Q}^{H,\varepsilon} = \mathbb{P}_\phi^{H,\varepsilon} o G^{-1}$ such that $G(f)$ is the only solution of the ordinary differential equation:

$$G(f_t) = y + \int_0^t b(G(f_r))dr + \int_0^t \sigma(G(f_r)) f_r \phi(r,s)dr + \eta_t, \;\; s,t \in [0;T] \tag{7}$$

where $\eta_t = \int_0^t \chi_{\{G(f_r)=0\}} d\eta_r$ is an increasing continuous function.
$G(f)$ can also be written as

$$\begin{cases} G(f) = \Gamma\varphi(f_t) \\ \eta_t = G(f) - \varphi(f_t) = \Gamma\varphi(f_t) - \varphi(f_t) \; . \end{cases} \tag{8}$$

Where $\varphi(f)$ is a solution of the following stochastic equation:

$$\varphi(f_t) = y + \int_0^t b(\Gamma\varphi(f_r))dr + \int_0^t \sigma(\Gamma\varphi(f_r)) f_r \phi(r,s)dr, \;\; s,t \in [0;T] \tag{9}$$

and f is the function induced by the LDP of the fBm.

Lemma 3.2.
Let σ be a bounded lipschitz function and f be bounded and continuous function. Then there exists $K > 0$ and $N > 0$ such that

$$\begin{aligned} |f(t)\phi(t,s)| &\leq K \\ |\sigma(h(t))\phi(t,s)| &\leq N \quad \textit{for all } s,t \in [0,T]. \end{aligned}$$

Proof.
f is a bounded function, so there exists δ such that $|f| \leq \delta$. We have for $s,t \in [0;T]$

$$\begin{aligned} |f(t)\phi(s,t)| &= |f(t)||\phi(s,t)| \\ &= |f| H(2H-1)|t-s|^{2H-2} \\ &\leq \delta H|(2H-1)|T^{2H} = K. \end{aligned}$$

σ is bounded, so there exists M such that $|\sigma(h_t)| \leq M \;\forall\; h \in L_\phi^2(\mathbb{R})$, we have for $s,t \in [0;T]$

$$\begin{aligned} |\sigma(h_t)\phi(s,t)| &= |\sigma(h_t)||\phi(s,t)| \\ &= |\sigma| H(2H-1)|t-s|^{2H-2} \\ &\leq MH|(2H-1)|T^{2H} = N. \end{aligned}$$

□

Lemma 3.3. *$G(f)$ and η are continuous on the compact set*
$\{J(G(f),\eta) \leq a, G(f), \eta \in L_\phi^2(\mathbb{R})\}$ *for any $a \geq 0$.*

Proof. Let's show $G(f)$ and η are continuous:

For $G(f)$, Let $G(f_1) = \Gamma\varphi(f_1)$ and $G(f_2) = \Gamma\varphi(f_2)$ with

$$\begin{aligned}
\varphi(f_t) &= y + \int_0^t b(\Gamma\varphi(f_r))dr + \int_0^t \sigma(\Gamma\varphi(f_r))f_r\phi(r,s)dr, \\
|G(f_1(t)) - G(f_2(t))| &= |\Gamma\varphi(f_1(t)) - \Gamma\varphi(f_2(t))| \\
\sup_{0\le r\le t}|G(f_1(r)) - G(f_2(r))| &= \sup_{0\le r\le t}|\Gamma\varphi(f_1(r)) - \Gamma\varphi(f_2(r))| \\
&\le 2\sup_{0\le r\le t}|\varphi(f_1(r)) - \varphi(f_2(r))| \\
\varphi(f_1(t)) - \varphi(f_2(t)) &= \int_0^t [b(\Gamma\varphi(f_1(r))) - b(\Gamma\varphi(f_2(r)))]dr \\
&+ \int_0^t \sigma(\Gamma\varphi(f_1(r)))f_1(r)\phi(r,s)dr - \int_0^t \sigma(\Gamma\varphi(f_2(r)))f_2(r)\phi(r,s)dr \\
&= \int_0^t [b(\Gamma\varphi(f_1(r))) - b(\Gamma\varphi(f_2(r)))]dr + \int_0^t \sigma(\Gamma\varphi(f_1(r)))f_1(r)\phi(r,s)dr \\
&- \int_0^t \sigma(\Gamma\varphi(f_2(r)))f_1(r)\phi(r,s)dr + \int_0^t \sigma(\Gamma\varphi(f_2(r)))f_1(r)\phi(r,s)dr \\
&- \int_0^t \sigma(\Gamma\varphi(f_2(r)))f_2(r)\phi(r,s)dr \\
&= \int_0^t [b(\Gamma\varphi(f_1(r))) - b(\Gamma\varphi(f_2(r)))]dr \\
&+ \int_0^t [\sigma(\Gamma\varphi(f_1(r))) - \sigma(\Gamma\varphi(f_2(r)))]f_1(r)\phi(r,s)dr \\
&+ \int_0^t \sigma(\Gamma\varphi(f_2(r)))\phi(r,s)[f_1(r) - f_2(r)]dr
\end{aligned}$$

$$\begin{aligned}
|\varphi(f_1(t)) - \varphi(f_2(t))| &\le \int_0^t |b(\Gamma\varphi(f_1(r))) - b(\Gamma\varphi(f_2(r)))|dr \\
&+ \int_0^t |\sigma(\Gamma\varphi(f_1(r))) - \sigma(\Gamma\varphi(f_2(r)))||f_1(r)\phi(r,s)|dr \\
&+ \int_0^t |\sigma(\Gamma\varphi(f_2(r)))\phi(r,s)||f_1(r) - f_2(r)|dr \\
&\le L\int_0^t |\Gamma\varphi(f_1(r)) - \Gamma\varphi(f_2(r))|dr + LK\int_0^t |\Gamma\varphi(f_1(r)) - \Gamma\varphi(f_2(r))|dr \\
&+ N\int_0^t |f_1(r) - f_2(r)|dr \\
&\le L(1+K)\int_0^t |\Gamma\varphi(f_1(r)) - \Gamma\varphi(f_2(r))|dr + N\delta T \\
\sup_{0\le r\le t}|\varphi(f_1(r)) - \varphi(f_2(r))| &\le L(1+K)\int_0^t \sup_{0\le r\le t}|\Gamma\varphi(f_1(r)) - \Gamma\varphi(f_2(r))|dr + N\delta T \\
&\le 2L(1+K)\int_0^t \sup_{0\le r\le t}|\varphi(f_1(r)) - \varphi(f_2(r))|dr + N\delta T \\
&\le 2L(1+K)\int_0^t \sup_{0\le r\le t}|\varphi(f_1(r)) - \varphi(f_2(r))|dr + N\delta T
\end{aligned}$$

$$
\begin{aligned}
\sup_{0\le r\le t}|G(f_1(r))-G(f_2(r))| &\le 2\sup_{0\le r\le t}|\varphi(f_1(r))-\varphi(f_2(r))|\\
&\le 4L(1+K)\int_0^t \sup_{0\le r\le t}|\varphi(f_1(r))-\varphi(f_2(r))|dr + N\delta T\\
|G(f_1(t))-G(f_2(t))| &\le N\delta T e^{4L(1+K)T}
\end{aligned}
$$

hence $G(f)$ is continuous.
For η,
accoding to (8) $\eta_t = \Gamma\varphi(f_t) - \varphi(f_t)$, so

$$
\begin{aligned}
\eta_1(t)-\eta_2(t) &= \Gamma\varphi(f_1(t)) - \varphi(f_1(t)) - \Gamma\varphi(f_2(t)) + \varphi(f_2(t))\\
|\eta_1(t)-\eta_2(t)| &\le |\Gamma\varphi(f_1(t)) - \Gamma\varphi(f_2(t))| + |\varphi(f_1(t)) - \varphi(f_2(t))|\\
\sup_{0\le r\le t}|\eta_1(r)-\eta_2(r)| &\le \sup_{0\le r\le t}|\Gamma\varphi(f_1(r)) - \Gamma\varphi(f_2(r))| + \sup_{0\le r\le t}|\varphi(f_1(r)) - \varphi(f_2(r))|\\
&\le 2\sup_{0\le r\le t}|\varphi(f_1(r)) - \varphi(f_2(r))| + \sup_{0\le r\le t}|\varphi(f_1(r)) - \varphi(f_2(r))|\\
&\le 3\sup_{0\le r\le t}|\varphi(f_1(r)) - \varphi(f_2(r))|\\
&\le 6L(1+K)\int_0^t \sup_{0\le r\le t}|\varphi(f_1(r))-\varphi(f_2(r))|dr + N\delta T (see\ *)\\
|\eta_1(t)-\eta_2(t)| &\le N\delta T e^{6L(1+K)T}\ ,
\end{aligned}
$$

hence η is continuous.

∎

Theorem 3.4. *The family $(Y_t^{H,\varepsilon}, L_t^{\varepsilon})_{(\varepsilon>0)}$ of the stochastic differential equation (2) satisfies the large deviation principle of the good rate function given by*

$$
J(g,\eta) = \begin{cases} \frac{1}{2}|\sigma^{-1}(g)[\dot g - b(g) - \chi_{\{g=0\}}(g)\dot\eta]|^2_{\phi^{-1}} \ if\ g = G(f) \in L^2_\phi(\mathbb{R})\ \eta \in L^2_\phi(\mathbb{R}) \\ +\infty\ otherwise \end{cases} \tag{10}
$$

(a) *$J(g,\eta)$ is lower semi-continuous and $\{g,\eta \in L^2_\phi(\mathbb{R}), J(g,\eta) \le a\}$ is a compact subset of $L^2_\phi(\mathbb{R})$,*

(b) *For all closed set C subset $L^2_\phi(\mathbb{R})$,*

$$
\lim_{\varepsilon\to 0}\sup \varepsilon^2 \log \mathbb{Q}^{H,\varepsilon}[(Y_t^{H,\varepsilon}, L_t^\varepsilon) \in C] \le -J(g,\eta),
$$

(c) *For any open set $O \subset L^2_\phi(\mathbb{R})$,*

$$
\lim_{\varepsilon\to 0}\inf \varepsilon^2 \log \mathbb{Q}^{H,\varepsilon}[(Y_t^{H,\varepsilon}, L_t^\varepsilon) \in O] \ge -J(g,\eta).
$$

Proof. Now let's show the upper and the lower bound by the contraction principle. $G(f)$ is continuous and the process εB_t^H of probability law $\mathbb{P}_\phi^{H,\varepsilon}$ has a LDP with a rate function $I(f) = \dfrac{1}{2}|f|^2_\phi$, according to the contraction principle we have for:

(b) Any open set O of $L^2_\phi(\mathbb{R})$, we have,

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\inf \varepsilon^2 \log \mathbb{Q}^{H,\varepsilon}[(Y_t^{H,\varepsilon}, L_t^\varepsilon)\in O] &= \lim_{\varepsilon\to 0}\inf \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon} o(G(f))^{-1}[(Y_t^{H,\varepsilon}, L_t^\varepsilon)\in O]\\
&= \lim_{\varepsilon\to 0}\inf \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[(G(f))^{-1}[(Y_t^{H,\varepsilon}, L_t^\varepsilon)\in O]]\\
&= \lim_{\varepsilon\to 0}\inf \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[(G(f))^{-1}[(Y_t^{H,\varepsilon}, L_t^\varepsilon)\in O]]\\
&= \lim_{\varepsilon\to 0}\inf \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[(G(f))^{-1}((Y_t^{H,\varepsilon}, L_t^\varepsilon))\in (G(f))^{-1}(O)]\\
&= \lim_{\varepsilon\to 0}\inf \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[\varepsilon B_t^H \in (G(f))^{-1}(O)]\\
&= \lim_{\varepsilon\to 0}\inf \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[\varepsilon B_t^H \in (G(f))^{-1}(O)]\\
&= \lim_{\varepsilon\to 0}\inf \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[\varepsilon B_t^H \in (G(f))^{-1}(O)]\\
&\geq -\inf_{f\in G(f)^{-1}(O)} I(f)\\
&= -\inf_{G(f)\in O}\{\inf I(f) = \frac{1}{2}|f|^2_\phi, f\in L^2_\phi(\mathbb{R}), G(f_t) = g_t\}\\
&= -J(g,\eta).
\end{aligned}
$$

(c) Any closed set C of $L^2_\phi(\mathbb{R})$

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\sup \varepsilon^2 \log \mathbb{Q}^{H,\varepsilon}[(Y_t^{H,\varepsilon}, L_t^\varepsilon)\in O] &= \lim_{\varepsilon\to 0}\sup \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon} o(G(f))^{-1}[(Y_t^{H,\varepsilon}, L_t^\varepsilon)\in C]\\
&= \lim_{\varepsilon\to 0}\sup \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[(G(f))^{-1}[(Y_t^{H,\varepsilon}, L_t^\varepsilon)\in C]]\\
&= \lim_{\varepsilon\to 0}\sup \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[(G(f))^{-1}[(Y_t^{H,\varepsilon}, L_t^\varepsilon)\in C]]\\
&= \lim_{\varepsilon\to 0}\sup \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[(G(f))^{-1}((Y_t^{H,\varepsilon}, L_t^\varepsilon))\in (G(f))^{-1}(C)]\\
&= \lim_{\varepsilon\to 0}\sup \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[\varepsilon B_t^H \in (G(f))^{-1}(C)]\\
&= \lim_{\varepsilon\to 0}\sup \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[\varepsilon B_t^H \in (G(f))^{-1}(C)]\\
&= \lim_{\varepsilon\to 0}\sup \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[\varepsilon B_t^H \in (G(f))^{-1}(C)]\\
&\leq -\inf_{f\in G(f)^{-1}(C)} I(f)\\
&= -\inf_{G(f)\in C}\{\inf I(f) = \frac{1}{2}|f|^2_\phi, f\in L^2_\phi(\mathbb{R}), G(f_t) = g_t\}\\
&= -J(g,\eta).
\end{aligned}
$$

Let's show that $J(g,\eta) = \dfrac{1}{2}|\sigma^{-1}(g_t)[\dot{g}_t - b(g_t) - \chi_{\{g_t=0\}}(g_t)\dot{\eta}_t]|^2_{\phi^{-1}}$
$G(f_t) = y + \int_0^t b(G(f_r))dr + \int_0^t \sigma(G(f_r))f_r\phi(r,s)dr + \eta_t.$
Let's put $G(f_t) = g_t$
$g_t = y + \int_0^t b(g_r)dr + \int_0^t \sigma(g_r)f_r\phi(r,s)dr + \int_0^t \chi_{\{g_r=0\}}(g_r)d\eta_r$
$\dot{g}_t = b(g_t) + \sigma(g_t)f_t\phi(t,s) + \chi_{\{g_t=0\}}(g_t)\dot{\eta}_t$

$f_t = \dfrac{1}{\sigma(g_t)\phi(t,s)}[\dot{g}_t - b(g_t) - \chi_{\{g_t=0\}}(g_t)\dot{\eta}_t]$. So,we show that

$$\begin{aligned}
J(g,\eta) &= \inf_{f\in G(f)^{-1}(C)} I(f) \\
&= \frac{1}{2}|\frac{1}{\sigma(g_t)\phi(t,s)}[\dot{g}_t - b(g_t) - \chi_{\{g_t=0\}}(g_t)\dot{\eta}_t]|_\phi^2 \\
&= \frac{1}{2}\int_0^t\int_0^s (\frac{1}{\sigma(g_r)\phi(r,u)}[\dot{g}_r - b(g_r) - \chi_{\{g_r=0\}}(g_r)\dot{\eta}_r]) \\
&\times (\frac{1}{\sigma(g_u)\phi(r,u)}[\dot{g}_u - b(g_u) - \chi_{\{g_u=0\}}(g_u)\dot{\eta}_u])\phi(r,u)dudr \\
&= \frac{1}{2}\int_0^t\int_0^s (\sigma^{-1}(g_r)[\dot{g}_r - b(g_r) - \chi_{\{g_r=0\}}(g_r)\dot{\eta}_r]) \\
&\times (\sigma^{-1}(g_u)[\dot{g}_u - b(g_u) - \chi_{\{g_u=0\}}(g_u)\dot{\eta}_u])\phi^{-1}(r,u)dudr \\
J(g,\eta) &= \frac{1}{2}|\sigma^{-1}(g_t)[\dot{g}_t - b(g_t) - \chi_{\{g_t=0\}}(g_t)\dot{\eta}_t]|_{\phi^{-1}}^2
\end{aligned}$$

hence, we have

$$J(g,\eta) = \frac{1}{2}|\sigma^{-1}(g_t)[\dot{g}_t - b(g_t) - \chi_{\{g_t=0\}}(g_t)\dot{\eta}_t]|_{\phi^{-1}}^2 \ .$$

(a) **Lower semicontinuous:**
Let g_ε, and $\eta_\varepsilon \in L^2_\phi(\mathbb{R})$ such that $g_\varepsilon \longrightarrow g, \eta_\varepsilon \longrightarrow \eta \in L^2_\phi(\mathbb{R})$.
So we have,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} J(g_\varepsilon,\eta_\varepsilon) \\
&= \lim_{\varepsilon\to 0}\frac{1}{2}|\sigma^{-1}(g_\varepsilon)[\dot{g}_\varepsilon - b(g_\varepsilon) - \chi_{\{g_\varepsilon=0\}}(g_\varepsilon)\dot{\eta}_\varepsilon]|_{\phi^{-1}}^2 \\
&= \frac{1}{2}\lim_{\varepsilon\to 0}[\int_0^t\int_0^s \sigma^{-1}(g_\varepsilon(r))[\dot{g}_\varepsilon(r) - b(g_\varepsilon(r)) - \chi_{\{g_\varepsilon(r)=0\}}(g_\varepsilon(r))\dot{\eta}_\varepsilon] \\
&\times \sigma^{-1}(g_\varepsilon(u))[\dot{g}_\varepsilon(u) - b(g_\varepsilon(u)) - \chi_{\{g_\varepsilon(u)=0\}}(g_\varepsilon(u))\dot{\eta}_\varepsilon]\phi^{-1}(r,u)dudr \\
&\geq \frac{1}{2}\int_0^t\int_0^s \lim_{\varepsilon\to 0}[\sigma^{-1}(g_\varepsilon(r))[\dot{g}_\varepsilon(r) - b(g_\varepsilon(r)) - \chi_{\{g_\varepsilon(r)=0\}}(g_\varepsilon(r))\dot{\eta}_\varepsilon] \\
&\times \sigma^{-1}(g_\varepsilon(u))[\dot{g}_\varepsilon(u) - b(g_\varepsilon(u)) - \chi_{\{g_\varepsilon(u)=0\}}(g_\varepsilon(u))\dot{\eta}_\varepsilon]\phi^{-1}(r,u)dudr \qquad (Fatou's\ lemma) \\
&= \frac{1}{2}\int_0^t\int_0^s \{\lim_{\varepsilon\to 0}\sigma^{-1}(g_\varepsilon(r))[\dot{g}_\varepsilon(r) - b(g_\varepsilon(r)) - \chi_{\{g_\varepsilon(r)=0\}}(g_\varepsilon(r))\dot{\eta}_\varepsilon]\} \\
&\times \{\lim_{\varepsilon\to 0}\sigma^{-1}(g_\varepsilon(u))[\dot{g}_\varepsilon(u) - b(g_\varepsilon(u)) - \chi_{\{g_\varepsilon(u)=0\}}(g_\varepsilon(u))\dot{\eta}_\varepsilon]\phi^{-1}(r,u)dudr\} \\
&= \frac{1}{2}\int_0^t\int_0^s \sigma^{-1}(g(r))[\dot{g}(r) - b(g(r)) - \chi_{\{g(r)=0\}}(g(r))\dot{\eta}]\} \\
&\times \sigma_H^{-1}(g(u))\dot{g}(u) - b(g(u)) - \chi_{\{g(u)=0\}}(g(u))\dot{\eta}]\phi^{-1}(r,u))dudr\} \\
&= \frac{1}{2}[|\sigma_H^{-1}(g)[\dot{g} - b(g) - \chi_{\{g=0\}}(g)\dot{\eta}]|_{\phi^{-1}}^2 \ .
\end{aligned}$$

So $J(g,\eta) \leq \lim_{\varepsilon\to 0} J(g_\varepsilon,\eta_\varepsilon)$, hence $J(g,\eta)$ is lower semicontinuous.

For compactness: $J(g,\eta) < +\infty$ for all $g,\eta \in L^2_\phi(\mathbb{R})$, so there exists $a > 0$ such that $J(g,\eta) \leq a$, he's in a closed-off place and we deduces that the set of level: $\{g,\eta \in L^2_\phi(\mathbb{R}), J(g,\eta) \leq a\}$ is compact subset of $L^2_\phi(\mathbb{R})$. We can finally conclude that $J(g,\eta)$ is a good rate function.

□

3.2. Large deviation principle for perturbed fractional diffusion process

Denote $\mu^{H,\varepsilon}$ the probability law of the process solution $X_t^{H,\varepsilon}$ (1) such that $\mu^{H,\varepsilon} = \mathbb{P}_\phi^{H,\varepsilon} o F^{-1}$ where F is a determinist function associated with f such that $F(f) = h$, the only solution of ordinary differential equation:

$$h(t) = x_0 + \int_0^t b(h_r)dr + \int_0^t \sigma(h_r) f_r \phi(r,s)dr + \alpha \sup_{0\le r\le t}(h_r), \;\; s,t \in [0;T] \tag{11}$$

Lemma 3.5. *For $\alpha \in (0;1)$, F is a continuous function.*

Proof. Denote $F(f_1) = h_1$ and $F(f_2) = h_2$ with
$F(f_t) = h(t) = x_0 + \int_0^t \sigma(h_r) f_r \phi(r,s)dr + \int_0^t b(h_r)dr + \alpha \sup_{0\le r\le t}(X_t)$

$$\begin{aligned}
h_2(t) - h_1(t) &= \int_0^t [\sigma(h_2(r))f_2(r) - \sigma(h_1(r))f_1(r)]\phi(r,s)dr + \int_0^t [b(h_2(r)) - b(h_1(r)]dr \\
&\quad + \alpha(\sup_{0\le r\le t}(h_2(r)) - \sup_{0\le r\le t}(h_1(r))) \\
&= \int_0^t [\sigma(h_2(r)) - \sigma(h_1(r))]f_2(r)\phi(r,s)dr + \int_0^t [f_2(r) - f_1(r)]\sigma(h_1(r))\phi(r,s)dr \\
&\quad + \int_0^t [b(h_2(r)) - b(h_1(r)]dr + \alpha(\sup_{0\le r\le t}(h_2(r)) - \sup_{0\le r\le t}(h_1(r))) \\
|h_2(t) - h_1(t)| &\le L\int_0^t |h_2(r) - h_1(r)||f_2(r)\phi(r,s)|dr + \int_0^t |f_2(r) - f_1(r)||\sigma(h_1(r))\phi(r,s)|dr \\
&\quad + L\int_0^t |h_2(r) - h_1(r)|dr + \alpha|\sup_{0\le r\le t}(h_2(r)) - \sup_{0\le r\le t}(h_1(r))|
\end{aligned}$$

$$\begin{aligned}
&\le LK\int_0^t |h_2(r) - h_1(r)|dr + \delta NT + L\int_0^t |h_2(r) - h_1(r)|ds + \alpha \sup_{0\le r\le t}|h_2(r) - h_1(r)| \\
&= L(K+1)\int_0^t |h_2(s) - h_1(s)|ds + \alpha \sup_{0\le r\le t}|h_2(r) - h_1(r)| + \delta NT \\
\sup_{0\le r\le t}|h_2(r) - h_1(r)| &\le L(K+1)\int_0^t \sup_{0\le r\le t}|h_2(r) - h_1(r)|dr + \alpha \sup_{0\le r\le t}|h_2(r) - h_1(r)| + \delta NT \\
(1-\alpha)\sup_{0\le r\le t}|h_2(r) - h_1(r)| &\le L(K+1)\int_0^t \sup_{0\le r\le t}|h_2(r) - h_1(r)|dr + \delta NT \\
\sup_{0\le r\le t}|h_2(r) - h_1(r)| &\le \frac{L(K+1)}{1-\alpha}\int_0^t \sup_{0\le r\le t}|h_2(r) - h_1(r)|dr + \frac{\delta NT}{1-\alpha} \\
|F(f_2) - F(f_1)| = |h_2 - h_1| &\le \frac{\delta NT}{1-\alpha} e^{(\frac{L(K+1)}{1-\alpha})T},
\end{aligned}$$

hence F is continuous if $\alpha \in (0;1)$. □

Theorem 3.6. *For all $\alpha \in (0;1)$ then the family $(X_t^{H,\varepsilon})_{(\varepsilon>0)}$ satisfies the large deviation principle of speed ε^2 with a rate function given by:*

$$J(h) = \begin{cases} \inf\{\inf I(f), f \in L_\phi^2(\mathbb{R}), F(f) = h\} \\ +\infty \;\; otherwise \end{cases} \tag{12}$$

(*i*) *$J(h)$ is lower semicontinuous and $\{h : J(h) \leq a\}$ is a compact subset of $L^2_\phi(\mathbb{R})$,*

(*ii*) *for any open set $O \subset L^2_\phi(\mathbb{R})$ and $h \in O$,*

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mu^{H,\varepsilon}(X_t^{H,\varepsilon} \in O) \geq -J(h).$$

(*iii*) *for any closed set $C \subset L^2_\phi(\mathbb{R})$ and $h \in C$,*

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mu^{H,\varepsilon}(X_t^{H,\varepsilon} \in C) \leq -J(h).$$

Proof. **For the lower semicontinuous** of $J(h)$ we have the same reasoning that theorem (3.4) for lower semicontinuous .
For compactness: $J(h) < +\infty$ for all $h \in L^2_\phi(\mathbb{R})$, so there exists $a \in \mathbb{R}$ such that $J(h) \leq a$, he's in a closed-off place and we deduces that the set of level: $\{h \in L^2_\phi(\mathbb{R}), J(h) \leq a\}$ is compact subset of $L^2_\phi(\mathbb{R})$.
For the upper bounded: $C \subset L^2_\phi(\mathbb{R})$, εB_t^H of probability measure $\mathbb{P}_\phi^{H,\varepsilon}$ has LDP with good rate function $I(f) = \frac{1}{2}|f|^2_\phi$ for $f \in L^2_\phi(\mathbb{R})$ and F is a contnuous function. So by contraction principle, we have

$$\begin{aligned}
\limsup_{\varepsilon \to 0} \varepsilon^2 \log \mu^{H,\varepsilon}[X_t^{H,\varepsilon} \in C] &= \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon} o F^{-1}[X_t^{H,\varepsilon} \in C] \\
&= \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[F^{-1}(X_t^{H,\varepsilon} \in C)] \\
&= \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[F^{-1}(X_t^{H,\varepsilon}) \in F^{-1}(C)] \\
&= \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[\varepsilon B^H \in F^{-1}(C)] \leq -\inf_{f \in F^{-1}(C)} I(f) \\
&= -\inf_{F(f) \in C} \{\inf I(f), f \in L^2_\phi(\mathbb{R}), F(f) = h\} = -J(h)\ .
\end{aligned}$$

So $\limsup_{\varepsilon \to 0} \varepsilon^2 \log \mu^{H,\varepsilon}[X_t^{H,\varepsilon} \in C] \leq -J(h) = -\inf_{h \in C} \{\inf I(f), f \in L^2_\phi(\mathbb{R}), F(f) = h\}$.
For the lower bounded: $O \subset L^2_\phi(\mathbb{R})$ and $I(f) = \frac{1}{2}|f|^2_\phi$ is a good rate function of the law $\mathbb{P}_\phi^{H,\varepsilon}$. Then by the contraction principle,

$$\begin{aligned}
\liminf_{\varepsilon \to 0} \varepsilon^2 \log \mu^{H,\varepsilon}[X_t^{H,\varepsilon} \in O] &= \liminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}^{H,\varepsilon} o F^{-1}[X_t^{H,\varepsilon} \in O] \\
&= \liminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[F^{-1}(X_t^{H,\varepsilon} \in O)] \\
&= \liminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[F^{-1}(X_t^{H,\varepsilon}) \in F^{-1}(O)] \\
&= \liminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}_\phi^{H,\varepsilon}[\varepsilon B_t^H \in F^{-1}(O)] \geq -\inf_{f \in F^{-1}(O)} I(f) \\
&= -\inf_{F(f) \in O} \{\inf I(f), f \in \mathcal{S}(\mathbb{R}), F(f) = h\} = -J(h)\ .
\end{aligned}$$

So $\liminf_{\varepsilon \to 0} \varepsilon^2 \log \mu^{H,\varepsilon}[X_t^{H,\varepsilon} \in O] \geq -J(h) = -\inf_{h \in O} \{\inf I(f), f \in L^2_\phi(\mathbb{R}), F(f) = h\}$. □

Conclusion 3.7. *In the present paper, we have established a large deviation principle for reflected diffusion process driven by a fBm for any Hurst parameter $H \in (0; 1)$. This construction is carried out in the tempered distribution space $\mathcal{S}'(\mathbb{R})$ using the method of Freidlin-Wentzell ([9]). So it would be very interesting to do this in a space larger than that considered here.*

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