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Research Article

Number pattern generalization process by provincial mathematics olympiad winner students

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Abstract

This research aims to the Number Generalization process based on APOS Framework (action, process, object, and schema). Furthermore, the research is explorative research with a qualitative approach. The subjects of the study were one of the junior high school students who won the provincial mathematics Olympiad in South Sulawesi, Indonesia. Subjects were given instruments that had been developed, namely the number patterns generalization test. Data collection of this study is the number patterns generalization test and in-depth interviews. The data analysis process was reducing, describing, validating, and concluding. The results showed that the students who won the Olympics in the action stage determine the next term, if given a sequence by using a number pattern. At the stage of the process, action interiorizing by calculating the value of the next term repeatedly and explaining the process of determining the next term. At the object stage, encapsulating the process by showing that a number pattern has certain characteristics, de-encapsulating by assessing the observed pattern, and checking the number pattern found. At the schema stage, explaining the generalization properties of number sequence patterns by linking the actions, processes, objects of a concept with other concepts, doing thematization by linking the existing concept patterns to the general sequence.

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Introduction

Pattern generalization is one of the mathematical content taught to students in secondary schools. Pattern generalization is an important aspect at all levels in mathematics and it is contained in every material that is often highlighted in teaching (Dindyal, 2007). Exploration patterns in secondary school are developing patterns. The developing pattern is closely related to number sequence and images. Mulligan & Mitchelmore (2009) state that patterns in mathematics is represented as the form of numerical, spatial, or logical relationships. Correspondingly, Tikekar (2009) in his research showed that when students are given a number pattern, then each student has a different way of generalizing patterns.

In terms of generalizing patterns, students can draw conclusions that depend on their understanding and what the steps can be taken to generalize them. This generalization process can be used in enriching cognitive structures in students' minds because there are linking activities of it (Firdaus et al. 2019b). This is reinforced by Barbosa & Vale (2015) which states that generalizations play an important role in mathematical activities, generalizations are considered as abilities inherent in general mathematical thinking. When students can make generalizations of patterns as students' steps in abstracting knowledge, these students are in structural understanding and by the objectives of mathematics learning. Barbosa, Vale, & Palhares (2012) also say that students still experience difficulties when determining patterns in a larger term. For students, the formation of patterns in larger term seems more complicated than finding the results of the closest term of existing patterns. This means that students can look for the closest term

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of the pattern by using a counting or recursive strategy, rather than looking for general rules or generalizing a pattern for a larger term.

Research by Lannin, Barker, & Townsend (2006) found that students seemed to have difficulty determining the number of stickers that had to have adhered from a series of cubes that were given stickers on each surface. Students see the geometry object in front of them and do not consider the context of the problem so when calculating the next pattern, it seems that students do not pay attention to the context of the problem is proposed. In generalizing patterns, each student has different abilities with others such as seeing and processing information from the problems found (Muzaini et al. 2019). Pound (2008) states that each student has different abilities seen in his interaction with the surrounding environment. The environment of this can come from peers, teachers, older people, as well as reading sources. In the end, each student will get a variety of information so that it becomes knowledge that becomes the ability to recognize patterns.

Researchers conducted a preliminary study in Grade 8. Students are asked to complete the pattern generalization test presented in Figure 1 to investigate the initial tendency in the process of pattern generalization. How do you determine the number of sticks in figure 1?

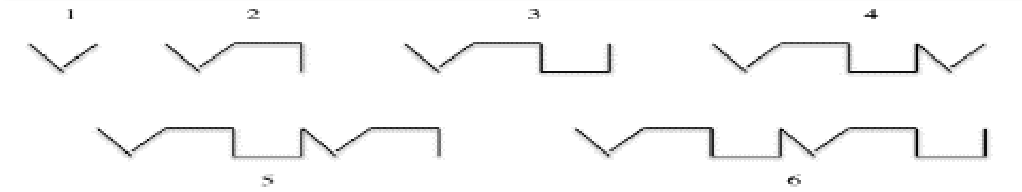


Figure 1.
Pattern Generalization Test

Researchers found a student who won the Provincial Mathematics Olympiad had different answers from other students. Provincial Mathematics Olympiad is a competition of students at the provincial level in solving mathematical problems. So it is not surprising that these students have different answers from other students. The student who won the Provincial Mathematics Olympiad generalizes a pattern that begins by identifying the problem given, and concludes that the problem is related to arithmetic sequences, recall the concepts based on previous knowledge related to arithmetic sequences, that is about the first term, and the difference of each term, then he uses the formula of the term-n of arithmetic sequences. Next, he looks for the 25 th term in question by substituting the value of the first term and the difference of each term so that finding the 25th term. In this case, the students generalize the number patterns.

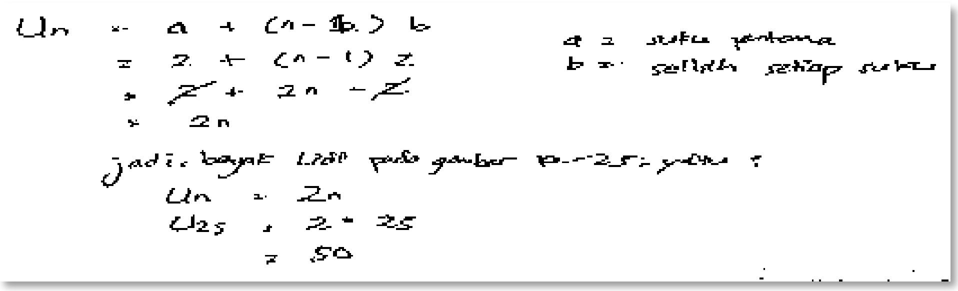


Figure 2.
The answers of Mathematics Olympiad Winner Students in Preliminary Studies

The way to find the answer to this problem is very interesting for further research. The strategy to solve the problem of pattern generalization is still the stage of action based on APOS theory, still determining the next term using a formula that has been studied previously. Therefore, efforts are needed to find out in more detail the profile flow patterns of generalization that occur for the student (Firdaus et al. 2019a). The results of the student's completion become the basis in building generalization patterns based on APOS theory.

The pattern generalization studied in this study is the result of student construction, and one of the tools for analyzing the profile of pattern generalization is to use APOS theory. In generalizing patterns, each student has different abilities with others such as seeing and processing information from the problem found.

Generalizing patterns that are analyzed using APOS theory which is a framework for explaining the construction process of understanding mathematical concepts or solving mathematical problems in individual minds. APOS theory can also be used as an analytical tool to describe the development of one's schema and explain how individuals build a mental understanding of a mathematical topic. APOS theory is based on the hypothesis that one's mathematical knowledge is a tendency to overcome situations that are mathematical problems by building actions, processes, and objects and arranging them in schemes to understand situations in solving problems (Dubinsky & McDonald, 2001). Furthermore, Dubinsky (2001) proposed a theory of mathematical concept construction through 4 (four) stages, namely action, process, object, and schema. These four stages are then called the APOS framework. APOS theory is a theory that explains a mathematical concept that is constructed in learning (Dubinsky, 2001). Asiala et al. (1997) explain that mental understanding is built by mental structures consisting of actions, processes, objects, and schemes. Based on the description APOS theory can be used as a tool to analyze mental activities carried out by students in building knowledge.

The research results described above indicate the diversity of research results regarding pattern generalization, but no one has examined the generalization of patterns based on APOS theory. This gap is very important because by knowing the generalization of student patterns based on APOS theory, a new theory is found. It can be used as a reference in creating models, strategies, approaches, and learning methods of pattern generalization based on APOS theory that can be implemented by the teacher in managing the learning process in heterogeneous classes.

Problem of Study

Based on the information that has been explained, the problem in this study is how the profile of provincial mathematics Olympiad winner on the number generalization process based on APOS framework (action, process, object, and schema)?

Method

Research Design

This research is explorative research with a qualitative approach (Moleong, 2012; Ikram et al. 2020; Muhtarom et al. 2019). In explorative research with a qualitative approach, researchers explore and conduct in-depth analyzes of the process of pattern generalization process students based on APOS theory. The subjects of this study were secondary school students who won the provincial mathematics Olympiad in South Sulawesi, Indonesia. This research is the same as research conducted by Maf'ulah, Juniati & Siswono (2017), Murtafiah, Sa'dijah, Chandra & Susiswo (2019) which is a qualitative case study with one research subject. Maf'ulah et al. (2017) examined the students of national science Olympiad winners, while Murtafiah et al. (2019) examines students who win national student creativity programs.

Participants/ Sample

Subjects were given the instrument that had been developed, namely the test of number pattern generalization. The problem test is modified from questions that have been developed by (Rivera, 2013), then validated by mathematic education experts. Data collection is done from written work and in-depth interviews. Based on the introduction study, this study selected a junior high school student who won the provincial Mathematical Olympiad in South Sulawesi, Indonesia.

Data Collection

The subject was asked to complete the task within the set time limit. Furthermore, researchers interviewed the subject of research in more depth based on students' responses to the task of generalizing given number patterns. Interviews are also used to obtain information that might not be obtained during written assignments because not everything students think is able to be written down. This might be revealed during the interview. Interview results documented with recorders. After the data is collected, then the data validity is checked. According to (Moleong, 2012), there are 4 (four) criteria to test the validity of the data, namely: 1) credibility, 2) transferability, 3) dependability, 4) confirmability. In this study two types of data validity criteria will be used, namely credibility and dependability.

Data Analysis

Data from the task of generalizing patterns and interview results are then analyzed using a qualitative approach. Data analysis in this study refers to the stages of qualitative data analysis according to (Miles & Huberman, 2014) Analyzing Data, Data Reduction Stage, Data Presentation Stage, Making Coding, and Conclusion Drawing Stage. The type of data analysis used in this study is the content analysis about the generalization of number patterns.

While the data analysis is by reducing the data of the test and interviews. Furthermore, it is explained, validated, and concluded. Data analysis of the process of generalizing number patterns is based on APOS theory. The indicators of the number pattern generalization process are presented in Table 1.

Table 1.
Indicators of the Number Pattern Generalization Process

The "APOS"	Descriptor Theory
Action	✓ Determine the next term, if given a sequence using a number pattern.
Process	✓ Explain how to determine the next term in a number pattern ✓ Explain the difference of sequences by spotting to the pattern of numbers ✓ Reflecting transformation steps procedurally.
Object	✓ Indicates that the pattern of the numbers has certain characteristics ✓ Explain the relevance of this problem to the problem understood previously ✓ Determine the concept of the next term conceptually.
Schema	✓ Explain the properties of patterns generalization by linking the actions, processes, objects of a concept with other concepts.

The test of number patterns generalization can be seen in Figure 3.

Given the sequence below:
- $\mathbf{-7, -2, 3, 8, 13 \cdots}$
- $\mathbf{-7, -4, -3, 0, 1, \cdots}$

  a) Write down the next term in the sequence!
  b) Write down the 7th term in the sequence!
  c) Write down the 20th term in the sequence!
- $\mathbf{11, 14, 17, 20, 23, 26, 29 \cdots}$
- $\mathbf{7, 12, 19, 24, 31, 36, 43 \cdots}$

  a) How do you determine the next term?
  b) How do you determine the 15th term?
  c) How do you determine the 33rd term?
- $\mathbf{15, 22, 29, 36, 43, 50 \cdots}$
- $\mathbf{33, 45, 60, 72, 87, 99, 114 \cdots}$

  a) Mention the characteristics of the sequence above!
  b) Mention the interrelationship of the pattern of sequences above in general!
  c) Determine the next term in the sequence above according to your understanding!

Figure 3. *Instrument Test for Number Patterns Generalization*

Based on these results, the researchers selected a secondary student who had won the provincial mathematics Olympiad. In the next step, the subjects were given a test. Then, the researcher conducted interviews based on the results of the subject's written work. Next, the researcher triangulated the time by giving a test. So, researchers get valid data about the generalization of student patterns based on APOS Framework.

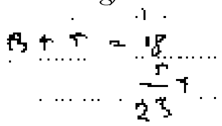
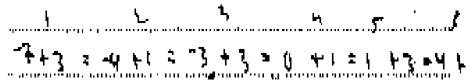
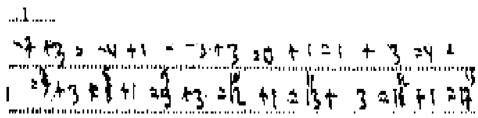
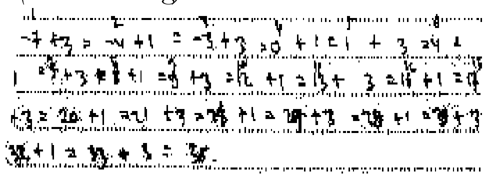
Results

The following is an explanation of the number pattern generalization of the subject which consists of the stages of action, process, objects, and schemes.

Generalization Student Patterns at the Action Stage

In the action stage, the subject solves the problem in the first and second sequences by determining the next term by finding the difference of consecutive terms then adding the difference with the last term in the sequence. Then problem in part b, subject in determining the value of the next term by repeating the same way before that is adding the difference with the last term in the sequence. In part c, subjects determine the next term using a formula from previous experience, namely the term n-th or $Un = a + (n-1)\,b$, then use the procedural method to ensure the answers found. This is evident in the task-based interview excerpts Table 2.

Table 2.
Excerpt of Interview in Question Number 1

Interview	Interview Transcript
Researcher	What do you understand from this problem?
Student	For number, the difference is 5 for each term, while the second sequence is the different show pattern, 3, 1, 3, 1. Then the number of terms in the first and second term is 5.
Researcher	So for the first sequence of part a, what is the next term?
Student	The next term is 18. 
Researcher	What about part b?
Student	(*While counting)* There are 23. 
Researcher	So, what about part c?
Student	*(While counting)* there are 83.
Researcher	How can you get the answer?
Student	For this one, I use the formula $Un = a + (n-1)\ b$.
Researcher	Now in the second sequence, what the next term?
Student	The next term is 4. 
Researcher	What is the 7th term?
Student	The 7th term is 5. 
Researcher	What about the 20th term?
Student	*(While counting)* the answer is 32. 

Generalization Student Patterns at the Process Stage

At the process stage, the subject interiorizes the action by explaining how to determine the next term procedurally by adding the difference in consecutive terms. In question no.2 the first and second sequence, the subject distinguishes a sequence by paying attention to the pattern of several terms by calculating the difference in each sequence, i.e in question no.2, the difference in he first sequence is 3 and the second sequence is 5, 7, 5, 7, etc. The subject reflects the transformation steps procedurally by determining the n-th term using the procedural method that is adding the difference of consecutive terms and using the formula $Un = a + (n-1)\ b$. Subjects coordinate by arranging patterns of each sequence by calculating the difference of each term, compiling information on the task to design a strategy. For finding the small terms, it is enough by adding the term manually, while to look for large terms, it must use formulas, then composing patterns to determine the next term by determining the difference. This is seen in the task-based interview excerpt in Table 3.

Table 3.
Excerpts of Interview Question Number 2

Interview	Interview Transcript
Researcher	How do you determine the next term?
Student	I added directly because it was known that the difference in each term was 3, so I just needed to add the 7th term, 29 with 3→ 29 + 3 = 32
Researcher	And how do you determine the 15th term?
Student	(*While counting silently)* I also added it manually because the 15th term can still be counted and is not too time-consuming.
Researcher	So what is the 15th term?
Student	*(While counting silently)* the results are 53.
Researcher	Okay, what about part c of the 33rd term?
Student	*(While counting)* the answer is 107. $Un = a + (n-1)\,b$ $= 11 + 32 \cdot 3$ $= 11 + 96$
Researcher	So, what is the 8th term?
Student	The 8th term is 48.
Researcher	How do you get it?
Student	I add it manually, it can be seen that the difference between the terms is not similar but regular that is 5, 7, 5, 7 etc. The difference between the 6th term and the 7th term is 7. So, the difference between the 7th and 8th term is 5 automatically.
Researcher	What about the 15th term?
Student	*(While counting)* 91. 7, 12, 19, 24, 31, 36, 43, ... 5 7 5 7 5 7 5 43 + 5 = 48 $U_{\frac{15+1}{2}} = a + (n-1)\,b$ $= 7 + (8-1)\,12$ $U_8 = 7 + 7 \cdot 12$ $= 7 + 84$ $= 91$ 12 × 7 = 84; 16 × 12 = 192; 192 + 7 = 199
Researcher	What about part c of the 33rd term?
Student	*(While counting)* the answer is 199 $\frac{33+1}{2}$ = 17 $U_{17} = 7 + (17-1)\,12$ $= 7 + 16 \cdot 12$ $= 7 + 192$ $= 199$

Generalization Student Patterns at the Object Stage

At the object stage, the subject shows that the pattern has certain characteristics, in the first part the difference of consecutive terms is always the same and it is 7, and the second part is different but regular, the pattern is 12, 15, 12, 15, etc. The subject relates this problem with previous, when the difference is similar, determining the next term with the formula $Un = a + (n-1)\,b$. The subject determines the value of the next term conceptually by explaining, if the sequences have just a small term, it can be calculated manually, but if it is a large-term it must use the formula. The subject relates the problem that was worked out before with the current problem, namely calculating the next term in the second sequence based on the results of the first sequence. The subject encapsulated by describing the characteristics of each sequence that is a similar difference or different but regular. The subject describes the pattern of sequences based on repeated differences to obtain the next term, then outlines the next term of the pattern found. The subject de-encapsulation by checking the pattern found, determining the first term, determining the number of terms in the sequence, determining the difference of each term in the sequence so that it is not wrong when answering questions. This is seen in Table 4.

Table 4.
Excerpts of Interview Question Number 3

Interview	**Interview Transcript**
Researcher	Mention the characteristics of the sequence!
Student	*(While thinking)* both sequences are arithmetic, the first of part a is having a similar difference between each term, it is 7 difference. Where the first term is 15. While part b is the difference between each term is different but regular 12, 15, 12, 15 and so on. And the first term is 33 and for part b we can divide again into 2 sequences, some for even and odd numbers for the first odd number is 33, while for even 45.
Researcher	Is any connection between this problem and the problem earlier?
Student	*(While thinking)* Yes, the connection is both problems are arithmetic when the difference is still similar, we can determine the term with the formula $Un = a\ (n + 1)\ b$.
Researcher	Now, what is the relation between the first part and the second part?
Student	Both of these patterns are arithmetic numbers, where the first part has the same difference, and the second part has a different but regular pattern
Researcher	Now for the first and second parts, how to determine the next term on the sequence according to your understanding!
Student	To determine the close term, it can be calculated manually. But for large terms, then I will use the formula that is $Un = a + (n - 1)\ b$.
Researcher	Now spot this matter. Determine the 20th term of sequence! $\overset{7}{15, 22}, \overset{7}{29}, \overset{7}{36}, \ldots$ U_{20} ... ?
Student	*(While counting)* 148. $U_{20} = 15 + (20-1)\,7$ $= 15 + 133$ $= 148$ 15, 22, 29, 36, 43, 50, 57, 64, 71, 78 +7 +7 +7 +7 +7 +7 +7 +7 +7 85, 92, 99, 106, 113, 120, 127, 134, 141, 148 +7 +7 +7 +7 +7 +7 +7 +7 +7
Researcher	Ok. If I make a problem where the difference is 2 times from the previous question, determine the 20th term based on the answer you just found! $\overset{7}{15, 22}, \overset{7}{29}, \overset{7}{36}, \ldots$ U_{20} ... ? $\underbrace{15, 29}_{14}, \underbrace{43}_{14}, \ldots$ U_{20} ... ?
Student	*(While counting)* 281. $U_1 = 15$ $U_2 = U_1 + 14 = 29 \rightarrow U_1 + (1 \times 14)$ $U_3 = U_2 + 14 \rightarrow U_1 + (2 \times 14)$ $= 29 + 14 = 43$ $15 + 28 = 43$ $U_4 = U_3 + 14 \rightarrow U_1 + (3 \times 14)$ $43 + 14 = 57$ $15 + 42 = 57$ $U_{20} = U_{19} + 14 \rightarrow U_1 + (19 \times 14)$ $15 + 266 = 281$
Researcher	How do you get it?

Interview	**Interview Transcript**
Student	*(While thinking)* From the previous problem when looking for the next term, each last term is added by the difference to get the results. So for this problem I see that U1 = 15 is the first term, then it affects the results of the next term if the difference is added. For example U2 = U1 + 14, then U3 = (U1 + 14) + 14, then U4 = ((U1 + 14) + 14) + 14, here I see that every increasing term, the difference 14 is also increasing. We can see that in U2, the difference of 14 appears once, then in U3, the difference of 14 is repeated or increased 2 times, etc.
Researcher	Why multiplied by 19?
Student	If you want to look for the 20th term, based on what I did earlier where the difference of 14 of the sequence repeated 19 times to look for the 20th term.
Researcher	It seems the difference in the term of the second sequence, try to find another way!
Student	*(While counting)* $U_2 = U_2 + 7(2-1)$ $U_3 = U_3 + 7(3-1)$ $U_4 = U_4 + 7(4-1)$ $U_{20} = U_{20} + 7(20-1)$ $= 148 + 7(19)$ $= 148 + 133$ $= 281$
Researcher	Tell me what you found!
Student	Let me say U2 in second sequence = U2 in first sequence + 7 (2-1)= 22 + 7 = 29, U3 in second sequence = U3 in first sequence + 7 (3-1)= 29+ 7 (2) = 43, U4 in first line + 7 (4-1) = 36 + 7 (3) = 57, U20 in second sequence= U20 in first sequence + 7 (20-1) = 148 + 7 (19) = 281. Same result with the previous one.
Researcher	How do you get 7?
Student	From the difference in the second part, I subtract the difference in the first part.

Generalization Student Patterns at the Scheme Stage

At the scheme stage, the subject explains the characteristics of pattern generalization by linking the actions, processes, objects of a concept with other concepts, that is if the difference of consecutive terms is the same or regular, then we can use the formula $Un = a + (n-1)\,b$, but if the difference of consecutive terms is different but still regular, we first calculate the difference of each odd and even term, then divide the sequence into 2 groups, after that, we can only use the formula $Un = a + (n-1)\,b$. The subject conducts thematization by linking the concept of the existing pattern with the general sequence, where each pattern has a difference between two terms, there is a sequence with a similar difference or not. Determine the pattern sequence by adding up the last term with the difference, or using the formula according to the sequence pattern, if looking for a large term. Subjects use the concept of existing sequence patterns and the concepts can be developed to find new concepts in completing certain sequence patterns. The subject relates the concept of pattern sequence, in general, to see the characteristic of the patterns, because the concepts of all types of pattern sequences are almost the same. The only contrast is the difference between each term, so from the difference between each term, we can determine the pattern of sequence. This is seen in Table 5.

Table 5.
Excerpts of Interviews

Interview	Interview Transcript
Researcher	Explain the relation of these three problems that you are working on?
Student	The relation of these three problems is that if the differences between each term are similar, then we can use the formula $Un = a + (n-1)\,b$, but if the differences between the term are different but regular, first we calculate the difference between each odd and even terms if they are the same, then we divide into 2 sequences of these terms, the even and odd terms, after that, we can only use the formula $Un = a + (n-1)\,b$ like the second part of the question.
Researcher	So how is the relation between the sequences that you have learned to the problems you are working on now?
Student	*(While thinking)* Every pattern of a sequence has a difference between 2 consecutive terms, there is a difference similar or not. Determining the sequence pattern by adding up the terms by the difference, a formula can be used according to the sequence pattern if asking for a large or far term. Another way is using the concept of existing sequence patterns and from the concepts can be developed to find new concepts in completing a particular sequence pattern.
Researcher	How do you relate the sequence you have learned with the problems you're working on now?
Student	*(While thinking)* In my opinion, by looking at the characteristics and types of sequence patterns, because the concepts of all the types of sequence patterns are almost the same. The only contrast is the difference between each of the term, so from the difference in each term we can determine the sequence pattern.

Discussion

The process of generalizing number patterns to students winning the provincial mathematics Olympiad is based on APOS theory through several stages, namely the action stage, the subject's mental activity in solving problems, that is, finding the difference of consecutive terms and continuing to add the difference with the last term in the number sequence (the first and second parts of a). The subject makes interiorization by repeating on how to determine the next term by finding the difference of the consecutive terms and then adding the difference with the last term in the sequence (first and second problems of part b). The subject uses a formula from previous experience, i.e. the n-th term formula or $Un = a + (n-1)\,b$ (the first and second questions of part c). In this case, the subject's way of thinking process is influenced by problem situations, where the subject initially uses procedural/manual steps/strategies, because the situation is still simple. However, when faced with complex situations, the subject begins to change his strategy by using the n-th term formula based on his knowledge. This is in line with the findings (Ed Dubinsky, 2001; DeVries, 2001) that someone can go through the stages of action if someone has been able to focus their mental processes to understand a given concept. This is reinforced by the opinion of Zazkis & Campbell (1996) which states, if students' understanding according to APOS theory is at an action stage, then the student only performs procedural activities.

The subject can find the pattern of each sequence after interiorizing some actions, after that it is coordinated to be able to predict the next picture, so it can be used as a basis for encapsulation. Changes in the transformation of a given problem (external) into the students' internal (mind) are called interiorization. This is by the opinion (Ed Dubinsky, 2001; DeVries, 2001) said that interiorization is a change from a procedural activity to be able to do the activity again in imagining some concepts that affect some conditions. In other words, if an action is repeated and a reflection of the action is carried out, then the actions have been interpreted into a process. In the interiorization of actions, not only coordinate some actions that have been carried out but also coordinated with the initial knowledge possessed by students. The initial knowledge possessed by students is like looking for differences from each number sequence.

At the stage of the process, the subject explains how to determine the next term procedurally by adding the difference of consecutive terms and using the formula $U_n = a + (n-1)\,b$. Then the subject distinguishes a sequence by observing the pattern of several terms then calculating the difference in each sequence, for example in question no.2, the first sequence is 3 and the second is 5, 7, 5, 7, etc. The subject reflects the transformation steps procedurally by determining the n-th term using the procedural method that is adding the difference of consecutive terms and using the formula $U_n = a + (n-1)\,b$. (Dubinsky et al. 2005) explain the process, which is a mental structure by carrying out the same operations, but entirely in the individual's mind, allowing individuals to imagine transformations without having to do each stage explicitly.

Zazkis & Campbell (1996) states, if students' understanding based on APOS theory is at the stage of the process, then the student has a procedural understanding. So, students whose understanding of sequences are at the stage of the process according to the APOS theory framework, in addition to having the ability to explain how to determine certain terms, then these students also know the concept of sequence numbers.

At the stage of the process, the subject coordinates by arranging the patterns of each row by calculating the difference of each term, compiling information on the task to design a strategy if the terms sought by a small term can then add procedurally, while to look for large terms using formulas, then arrange patterns to determine the next term by determining the difference of sequential terms in the row of numbers. At the stage of the process, the subject reverses by tracking/compiling previous knowledge related to the problem if the syllable asked by a small syllable can then add procedurally, while to search for large syllables using a formula.

At the object stage, the subject shows that the pattern has certain characteristics. In the first part the difference of consecutive terms is always similar and it is 7, and the second part is different but regular, the pattern is 12, 15, 12, 15, etc. The subject associates the problem with the problem that was previously understood, when a sequence with a similar difference, determining the next term with the formula $Un = a + (n-1)\,b$. If the difference is different but regular, determining the next term using the formula $Un = a + (n-1)\,b$ by first dividing the line into 2 groups namely odd and even terms. Subject Determine the concept of the next term by explaining if the term is small, it can be calculated manually/procedurally. But if you are looking for a large term, you will use the formula $Un = a + (n-1)\,b$. The subject relates the problem that was worked out before with the current problem, it is calculating the next term in the second sequence based on the results of the first sequence.

At the object stage, the subject knows a sequence as an arithmetic sequence based on the characteristics of the arithmetic sequence. The subject can also state the definition of an arithmetic sequence correctly, he can give an example and not an example of an arithmetic sequence and can state the relationship between one term with another term in a number sequence of. This means students whose understanding of sequences are at the object stage, then these students already have conceptual knowledge about number sequences. Zazkis & Campbell (1996) states, if students' understanding according to the APOS theory framework is at the object stage, then the student already has a conceptual understanding.

At the object stage, the subject encapsulates by describing the characteristics of each sequence, which has a similar/fixed difference and or different but regular. The subject describes the sequence pattern based on repeated differences to obtain the next term, then outlines the next term of the pattern found. The encapsulation process is when the subject can show that the sequence pattern has certain characteristics, a term has a relation to the next term in a particular category. Based on the characteristics of the number sequence, the subject can determine whether this sequence is in a particular category.

If a process can be transformed by an action, it means that the process has been encapsulated into an object (Ed Dubinsky, 2001; DeVries, 2001). Ed Dubinsky & McDonald (2001) said that new processes can also be constructed (formed) by coordinating existing processes, then the process becomes an own process that can be transformed by an action, then it is said to have been encapsulated become an object.

At the object stage, the subject de-encapsulates by checking the pattern found, by determining the first term, determining the number of terms in the sequence, determining the difference of each term in the sequence so that it is not wrong when answering questions or there is no miscalculation in the process.

At the schema stage, the subject explains the characteristics of pattern generalization by linking actions, processes, objects of a concept with other concepts, if the difference of consecutive terms is similar or fixed, then we can use the formula $Un = a + (n-1)\,b$, but if the difference of consecutive terms is different but still regular, we first calculate the difference of each odd and even term, then dividing the number sequence into 2 groups namely even and odd terms, after that we can only use the formula $U_n = a + (n-1)\,b$.

At the scheme stage, the subject has been able to construct coordination that links the actions, processes or separate objects to solve a number sequence generalization problem. The subject can construct coordination that links the actions, processes, and objects about the sequences. In this case, the subject has been able to coordinate actions, processes, and objects to form an initial scheme of number sequence.

At the scheme stage, the subject conducts thematization by linking the concept of the existing pattern with the general sequence, where each pattern has a difference between two consecutive terms, there is a similar difference or not. Determine the sequence pattern by adding up the last term by the difference, or using the formula according to sequence the pattern, if you are looking for a large number of term. Subjects use the concept of existing number sequence patterns and from the concepts can be developed to find new concepts in completing certain number sequence patterns. The subject relates the concept of the existing pattern with the sequence of numbers in general by looking at the characteristics and types of the number sequence patterns because the concepts of all types of patterns are almost the same. The only contrast is the difference of each term, so from the difference in each term we can determine the number pattern.

Thematization is a construction that links separate actions, processes, and objects to a particular object then it produces a scheme (Ed Dubinsky, 2001; DeVries, 2001). Thematize a number sequence involves a special relationship between a number sequence with the concept of functions. A student has been able to mathematize the sequence of numbers as a scheme if she/he can show the relationship of a sequence by relating it to the concept of function.

Conclusion

The conclusions of the results of this study about number pattern generalization process by Provincial Mathematics Olympiad winner students based on APOS framework is at the stage of action, if given a sequence using number sequence patterns, the subject can determine the other term. At the process stage, the subject interiorizing the action by calculating the next term repeatedly and explaining the process of determining the next term. The subject explains the difference of a sequence by spotting to the number pattern of several terms and reflects the transformation steps manually/procedurally. At the process stage, the subject coordinates by arranging the patterns of each number sequence, compiling information to make strategies, arranging patterns to determine the next term. At the process stage, the subject reverses by tracing previous knowledge related to the problem. At the object stage, the subject encapsulates the process by showing that the number pattern has certain characteristics, then explains the relation of this problem to the problem previously understood and determines the next term conceptually. The subject describes the characteristics and patterns of each number sequence, then outlines the next term of the pattern found. The subject de-encapsulates by assessing the observed pattern and checking the number pattern found. At the schema stage, the subject explains the generalization characteristics of number sequence patterns by linking the actions, processes, objects of a concept with other concepts. At the stage of the scheme, the subject performs thematization by linking the concept of the existing pattern with the general sequence.

Recommendations

The results of this study provide an overview of the generalization of number patterns in Olympiad winner students that can be used as a basis for designing mathematics learning in the number pattern material in junior high school students. The findings of this study also provide opportunities for other researchers to uncover the generalization of picture patterns and contextual problem patterns.

Limitations of the Study

The scope of this study is limited to mathematics learning of generalization of numbers based on APOS theory. Therefore, researchers in the field of education should carry out further research with the material, and different levels of education (classes) to add to the repertoire of knowledge, especially APOS theory both in learning and in solving problems.

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