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Research Article

Rings whose total graphs have small vertex-arboricity and arboricity

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Abstract

Let R be a commutative ring with non-zero identity, and $Z(R)$ be its set of all zero-divisors. The total graph of R, denoted by $T(\Gamma(R))$, is an undirected graph with all elements of R as vertices, and two distinct vertices x and y are adjacent if and only if $x+y \in Z(R)$. In this article, we characterize, up to isomorphism, all of finite commutative rings whose total graphs have vertex-arboricity (arboricity) two or three. Also, we show that, for a positive integer v, the number of finite rings whose total graphs have vertex-arboricity (arboricity) v is finite.

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1. Introduction

In [1], D.F. Anderson and A. Badawi introduced the total graph of ring R, denoted by $T(\Gamma(R))$, as the graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x+y \in Z(R)$, where $Z(R)$ is the set of zero-divisors of R. They studied some graph theoretical parameters of $T(\Gamma(R))$ such as diameter and girth. In addition, they showed that the total graph of a commutative ring is connected if and only if $Z(R)$ is not an ideal of R. In [7], H.R. Maimani et al. gave the necessary and sufficient conditions for the total graphs of finite commutative rings to be planar or toroidal and in [5] T. Chelvam and T. Asir characterized all commutative rings such that their total graphs have genus two.

Suppose that G is a graph, and let $V(G)$ and $E(G)$ be the vertex set and edge set of G, respectively. The *vertex-arboricity* of a graph G, denoted by $va(G)$, is the minimum positive integer k such that $V(G)$ can be partitioned into k sets $V_1, V_2 \ldots, V_k$ such that $G[V_i]$ is a forest for each $i \in \{1, 2, \ldots, k\}$, where $G[V_i]$ is the induced subgraph of G whose vertex set is V_i and its edge set consists of all of the edges in $E(G)$ that have both endpoints in V_i. This partition is called *acyclic partition*. The vertex-arboricity can be viewed as a vertex coloring f with k colors, where each color class V_i induces a forest; namely, $G[f^{-1}(i)]$ is an acyclic graph for each $i \in \{1, 2, \ldots, k\}$. Vertex-arboricity, also known as point arboricity, was first introduced by G. Chartrand, H.V. Kronk, and C.E.

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Wall [4] in 1968. Note that a graph with no cycles is a forest, and it has vertex-arboricity one.

Likewise, the arboricity of a graph G, denoted by $\nu(G)$, is the least number of line-disjoint spanning forests into which G can be partitioned, that is, there is some collection of $\nu(G)$ subgraphs of G, where each subgraph is a forest and each edge in G is in exactly one such subgraph. Arboricity of a graph was first introduced by C. St. J. A. Nash-Williams [4] in 1964.

The main purpose of this paper is to characterize all finite commutative rings whose total graph has vertex-arboricity (arboricity) two or three. In addition, we show that, for a positive integer v, there are only finitely many finite rings whose total graph has vertex-arboricity (arboricity) v.

Now, we recall some definitions of graph theory which are necessary in this article. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We use n and e to denote the number of vertices and the number of edges of G, respectively. A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. We use K_n to denote the complete graph with n vertices. A *bipartite graph* G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that the edge set of such a graph consists of precisely those edges which join vertices in V_1 to vertices of V_2. In particular, if $E(G)$ consists of all possible such edges, then G is called the *complete bipartite graph* and denoted by the symbol $K_{r,s}$, where $|V_1| = r$ and $|V_2| = s$. For a vertex $x \in V(G)$, $\deg(x)$ is the *degree of vertex* x, $\delta(G) = \min\{deg(x) : x \in V(G)\}$, $\Delta(G) = \max\{\deg(x) : x \in V(G)\}$. For a nonnegative integer d, a graph is called d*-regular* if every vertex has degree d. Let $S \subset V(G)$ be any subset of vertices of G. Then the *induced subgraph* $G[S]$ is the graph whose vertex set is S and whose edge set consists of all of the edges in $E(G)$ that have both endpoints in S. A *spanning subgraph* for G is a subgraph of G which contains every vertex of G. A graph without any cycle is called *acyclic graph*. A *forest* is an acyclic graph. Let G_1 and G_2 be subgraphs of G, we say that G_1 and G_2 are *disjoint* if they have no vertex and no edge in common. The *union* of two disjoint graphs G_1 and G_2, which is denoted by $G_1 \cup G_2$ is a graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. For any graph G, the disjoint union of k copies of G is denoted by kG. Graphs G and H are said to be *isomorphic* to one another, written $G \cong H$, if there exists a one-to-one correspondence $f : V(G) \to V(H)$ such that for each pair x, y of vertices of G, $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$. Also, for a rational number p, $\lceil p \rceil$ is the first integer number greater than or equal to p, and $\lfloor p \rfloor$ is the first integer number less than or equal to p.

2. Basic properties

First of all, let us recall some of the basic facts about total graphs and vertex arboricity, which we shall use in the rest of the paper.

Lemma 2.1 ([7, Lemma 1.1])**.** *Let x be a vertex of $T(\Gamma(R))$. Then the following statements are true.*

(i) *If $2 \in Z(R)$, then* $\deg(x) = |Z(R)| - 1$.
(ii) *If $2 \notin Z(R)$, then* $\deg(x) = |Z(R)| - 1$ *for every $x \in Z(R)$ and* $\deg(x) = |Z(R)|$ *for every vertex $x \notin Z(R)$.*

Remark 2.2. It is clear that $va(G) = 1$ if and only if G is acyclic. For a few classes of graphs, the vertex-arboricity is easily determined. For example, $va(C_n) = 2$, where C_n is a cycle graph with n vertices. If n is even, $va(K_n) = \frac{n}{2}$; while if n is odd, $va(K_n) = \frac{n+1}{2}$. So, in general, $va(K_n) = \lceil \frac{n}{2} \rceil$. Also, $va(K_{r,s}) = 1$ if $r = 1$ or $s = 1$, and $va(K_{r,s}) = 2$ otherwise.

Lemma 2.3 ([3, Lemma 1]). *Let G be the disjoint union of graphs $G_1, G_2, \ldots, G_k$. Then, for all i with $1 \leq i \leq k$,*

$$va(G) = \max\ va(G_i).$$

Now, we are ready to show that for a positive integer v, there are only finitely many finite rings whose total graph has vertex-arboricity v.

Theorem 2.4. *For any positive integer v, the number of finite rings whose total graphs have vertex-arboricity v is finite.*

Proof. Let R be a finite ring. We want to obtain a complete subgraph (with vertex set T) of $T(\Gamma(R))$. To achieve this, we consider the following two cases:

(a) R is local. In this case $Z(R)$ is the maximal ideal of R and $|R| \leq |Z(R)|^2$ [8]. In this situation, we put $T = Z(R)$.

(b) R is not local. Then there is a natural number $n \geq 2$ and there are local rings $R_1, R_2, \ldots, R_n$ such that $R = R_1 \times R_2 \times \cdots \times R_n$. We may assume that $|R_1| \leq |R_2| \leq \cdots \leq |R_n|$. Now put $R_1^* = 0 \times R_2 \times \cdots \times R_n$. Since $|R| = |R_1||R_1^*|$, we have $|R| \leq |R_1^*|^2$. In this situation, we put $T = R_1^*$.

Now, it is easy to see that, for every elements x and y of T, x is adjacent to y in $T(\Gamma(R))$. Thus there is an induced subgraph $K_{|T|}$ in $T(\Gamma(R))$. Hence Remark 2.2 implies that $va(K_{|T|}) \leq v$, and so $\lceil \frac{|T|}{2} \rceil \leq v$. Thus $|R| \leq 4v^2$, and so the proof is complete. □

Let $Reg(\Gamma(R))$ be the induced subgraph of $T(\Gamma(R))$ with vertices $Reg(R) = R - Z(R)$, and $Z(\Gamma(R))$ be the induced subgraph of $T(\Gamma(R))$ with vertices $Z(R)$. Next, we record some facts concerning total graphs. If $Z(R)$ is an ideal of R, then $Z(\Gamma(R))$ is a complete subgraph of $T(\Gamma(R))$ and is disjoint from $Reg(\Gamma(R))$. Thus, the following theorem of D.F. Anderson and A. Badawi gives a complete description of $T(\Gamma(R))$.

Theorem 2.5 ([1, Theorem 2.2]). *Let R be a commutative ring such that $Z(R)$ is an ideal of R, and let $|Z(R)| = n$ and $|\frac{R}{Z(R)}| = m$. Then the following statements hold.*

(i) *If $2 \in Z(R)$, then $Reg(\Gamma(R))$ is the union of $m-1$ disjoint K_n's.*
(ii) *If $2 \notin Z(R)$, then $Reg(\Gamma(R))$ is the union of $\frac{m-1}{2}$ disjoint $K_{n,n}$'s.*

Theorem 2.6. *Let R be a finite commutative ring with identity and I be a nontrivial ideal contained in $Z(R)$. Set $|I| = n$ and $|\frac{R}{I}| = m$. Then the following statements hold.*

(i) *If $2 \in I$, then $va(T(\Gamma(R))) \geq \lceil \frac{n}{2} \rceil$.*
(ii) *If $2 \notin I$, then $va(T(\Gamma(R))) \geq \max\{\lceil \frac{n}{2} \rceil, 2\}$.*

Proof. Let G be the spanning subgraph of $T(\Gamma(R))$ such that, for every two vertices $x, y \in R$, x is adjacent to y in G if $x + y \in I$. Now, since I is an ideal of R contained in $Z(R)$, by making obvious modification to the proof of Theorem 2.5, one can show that

$$G = \begin{cases} mK_n & \text{if } 2 \in I \\ K_n \bigcup (\frac{m-1}{2})K_{n,n} & \text{if } 2 \notin I. \end{cases}$$

Now, by Remark 2.2 in conjunction with Lemma 2.3, we have the following equalities

$$va(G) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } 2 \in I \\ \max\{\lceil \frac{n}{2} \rceil, 2\} & \text{if } 2 \notin I. \end{cases}$$

Now, since G is a subgraph of $T(\Gamma(R))$, we have that $va(G) \leq va(T(\Gamma(R)))$, and so the proof is complete. □

The following corollary is immediate from Theorem 2.5.

Corollary 2.7. *Let R be a finite commutative ring with identity, $Z(R)$ be nontrivial ideal of R and set $|Z(R)| = n$ and $|\frac{R}{Z(R)}| = m$. Then the following statements hold.*

(i) *If $2 \in Z(R)$, then $va(T(\Gamma(R))) = \lceil \frac{n}{2} \rceil$.*
(ii) *If $2 \notin Z(R)$, then $va(T(\Gamma(R))) = \max\{\lceil \frac{n}{2} \rceil, 2\}$.*

3. The vertex-arboricity of the total graph

For any graph G, the girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G ($gr(G) = \infty$ if G contains no cycles). The following Theorem of Anderson and Badawi implies that $T(\Gamma(R))$ has vertex-arboricity one if and only if either R is an integral domain or R is isomorphic to $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{(x^2)}$.

Theorem 3.1 ([2, Theorem 4.7]). *Let R be a commutative ring. Then $gr(T(\Gamma(R))) \in \{3, 4, \infty\}$. Moreover,*

(i) $gr(T(\Gamma(R))) = \infty$ *if and only if either R is an integral domain or R is isomorphic to $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{(x^2)}$,*

(ii) $gr(T(\Gamma(R))) = 4$ *if and only if R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and*

(iii) $gr(T(\Gamma(R))) = 3$ *otherwise.*

Now, we will classify, up to isomorphism, all finite commutative rings whose total graphs have vertex-arboricity two or three. We begin with a following result which is essentially due to Raghavendran.

Theorem 3.2 ([10, Theorem 2]). *Let R be a finite commutative local ring with nonzero identity and $U(R)$ be the set of all unit elements of R. Then $|R| = p^{nr}$, $|Z(R)| = p^{(n-1)r}$ and $|U(R)| = p^{(n-1)r}(p^r - 1)$ for some prime p and some positive integers n and r.*

In sequel, we state two remarks which we will use throughout this paper.

Remark 3.3. Let R_1 and R_2 be two finite commutative rings with $|R_1| = m$, $|R_2| = n$ and $m \leq n$. It is easy to see that the subgraph of the total graph of $R_1 \times R_2$ induced by the set $\{0\} \times R_2$ is a copy of K_n.

Remark 3.4. Let R_1, R_2, S_1 and S_2 be finite commutative rings such that $T(\Gamma(R_1)) \cong T(\Gamma(R_2)$ and $T(\Gamma(S_1)) \cong T(\Gamma(S_2)$. Then $T(\Gamma(R_1 \times S_1)) \cong T(\Gamma(R_2 \times S_2)$. However, this property does not hold in general for other widely studied graphs associated to rings (for example, the zero-divisor graphs).

Lemma 3.5. $va(T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))) = va(T(\Gamma(\mathbb{F}_4 \times \mathbb{F}_4))) = 3$.

Proof. First of all, note that, in view of Remark 3.3, $va(T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))) > 1$. Now, we show that $va(T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))) > 2$. To this, we consider a set of vertices of the graph $T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))$ of the form

$$A = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

Let the set $\{V_1, V_2\}$ be an acyclic partition of $V(T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)))$. Since $G[A]$ is a complete graph isomorphic to K_4 and $G[V_i](1 \leq i \leq 2)$ have no triangle, so $|A \cap V_1| = |A \cap V_2| = 2$. Without the loss of generality, we may assume that $(0, 0, 0), (1, 0, 0) \in V_1$ and $(0, 1, 0), (0, 0, 1) \in V_2$. Now, consider the vertex $(0, 1, 1)$ of $T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))$. It is clear that $(0, 1, 1) \in V_1$. Therefore, each of the remaining vertex of the graph $T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))$ forms a triangle with two vertices of V_1. Hence, all of these vertices must be in V_2, which is a contradiction.

Now, consider the partition of $V(T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)))$ with sets $V_1 = \{(0, 0, 0), (0, 1, 0), (1, 1, 1)\}$, $V_2 = \{(1, 0, 0), (0, 0, 1), (0, 1, 1)\}$ and $V_3 = \{(1, 0, 1), (1, 1, 0)\}$. It is clear that the subgraphs of $T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))$ induced by sets V_1, V_2 and V_3 are acyclic. Hence $va(T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))) = 3$.

By Remark 3.3, we have $va(T(\Gamma(\mathbb{F}_4 \times \mathbb{F}_4))) > 1$. Assume that $B_y = \{(a, y) : a \in \mathbb{F}_4\}$ and $C_x = \{(x, b) : b \in \mathbb{F}_4\}$ for all $x, y \in \mathbb{F}_4$. Obviously, $\{B_y : y \in \mathbb{F}_4\}$ and $\{C_x : x \in \mathbb{F}_4\}$ both form partitions for $V(T(\Gamma(\mathbb{F}_4 \times \mathbb{F}_4)))$. Let $\{V_1, V_2\}$ be an acyclic partition of $V(T(\Gamma(\mathbb{F}_4 \times \mathbb{F}_4)))$. Since the subgraphs of $T(\Gamma(\mathbb{F}_4 \times \mathbb{F}_4))$ induced by sets V_1 and V_2 have no triangles, each of these sets has exactly two vertices of the sets B_y and C_x for all

$x, y \in \mathbb{F}_4$. Hence, each of the sets V_1 and V_2 has exactly two vertices such that their first components are the same and have exactly two vertices such that the second components are the same. So, each vertex in V_1 and V_2 has degree 2, which is a contradiction, since the subgraphs of $T(\Gamma(\mathbb{F}_4 \times \mathbb{F}_4))$ induced by the sets V_1 and V_2 are union of cycles. Thus we have $va(T(\Gamma(\mathbb{F}_4 \times \mathbb{F}_4))) > 2$.

Now, according to the Figure 1, we have $va(T(\Gamma(\mathbb{F}_4 \times \mathbb{F}_4))) = 3$.

□

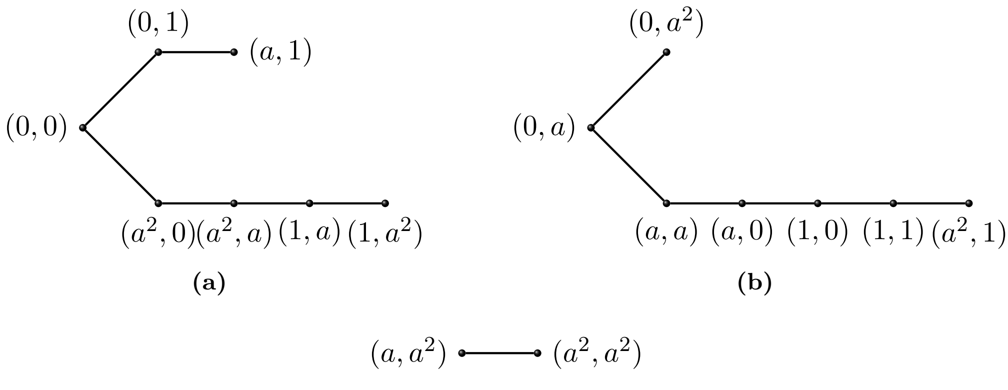

Figure 1

Theorem 3.6. *Let R be a finite commutative ring such that $va(T(\Gamma(R))) = 2$. Then the following statements hold.*

(i) *If R is local, then R is isomorphic to one of the following rings:* $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{(x^2)}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{(x^3)}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$, $\frac{\mathbb{Z}_2[x,y]}{(x,y)^2}$, $\frac{\mathbb{Z}_4[x]}{(2,x)^2}$, $\frac{\mathbb{F}_4[x]}{(x^2)}$, $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$.

(ii) *If R is not local, then R is isomorphic to one of the following rings:* $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_6$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{F}_4$.

Proof. (i) Assume that R is a local ring, and let $|Z(R)| = n$ and $|\frac{R}{Z(R)}| = m$. Then by Theorem 2.5, $T(\Gamma(R))$ has an induced subgraph isomorphic to K_n and so by Remark 2.2, $|Z(R)| \leq 4$. Now, we consider the following two cases:

(a) If $2 \in Z(R)$, then by Theorem 3.2, $|R| = 2^k$ and $k \leq 4$. Since $va(T(\Gamma(R))) = 2$, Theorem 3.1 implies that $|R| = 16, 8$. According to Corbas and Williams [6] there are two non-isomorphic rings of order 16 with maximal ideals of order 4, namely $\frac{\mathbb{F}_4[x]}{(x^2)}$ and $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$ (see also Redmond [11]), so for these rings have $T(\Gamma(R)) \cong 4K_4$. Therefore, by Remark 2.2, these rings have vertex-arboricity 2. In [6] it is also shown that there are 5 local rings of order 8 (except $\mathbb{F}_8$) as follows:

$$\mathbb{Z}_8, \frac{\mathbb{Z}_2[x]}{(x^3)}, \frac{\mathbb{Z}_4[x]}{(2x, x^2-2)}, \frac{\mathbb{Z}_2[x,y]}{(x,y)^2}, \frac{\mathbb{Z}_4[x]}{(2,x)^2}.$$

In all of these rings we have $|Z(R)| = 4$ and hence $T(\Gamma(R)) \cong 2K_4$. Then, by Remark 2.2, these rings have vertex-arboricity 2.

(b) If $2 \notin Z(R)$, then $|Z(R)| = 3$. According to [6], there are two rings of order 9 namely, $\mathbb{Z}_9$ and $\frac{\mathbb{Z}_3[x]}{(x^2)}$. For these rings, we have $T(\Gamma(R)) \cong K_3 \bigcup K_{3,3}$. Hence, by Corollary 2.7, these rings have vertex-arboricity 2.

(ii) Suppose that R is not local. Since R is finite, there are finite local rings $R_1, \ldots, R_t$ (with $t \geq 2$) such that $R = R_1 \times R_2 \times \cdots \times R_t$. Now, according to Remarks 2.2 and 3.3,

we have the following candidates:
$\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_6$, $\mathbb{Z}_2\times\mathbb{Z}_4$, $\mathbb{Z}_2\times\frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_2\times\mathbb{F}_4$, $\mathbb{Z}_3\times\mathbb{Z}_3$, $\mathbb{Z}_3\times\mathbb{Z}_4$, $\mathbb{Z}_3\times\frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_3\times\mathbb{F}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_4\times\mathbb{Z}_4$, $\mathbb{Z}_4\times\frac{\mathbb{Z}_2[x]}{(x^2)}$, $\frac{\mathbb{Z}_2[x]}{(x^2)}\times\frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_4\times\mathbb{F}_4$, $\frac{\mathbb{Z}_2[x]}{(x^2)}\times\mathbb{F}_4$, $\mathbb{F}_4\times\mathbb{F}_4$.
Now we examine each of the above rings.

The total graph of the ring $\mathbb{Z}_2\times\mathbb{Z}_2$ is isomorphic to the cycle of size 4. We consider the acyclic partition $V_1=\{(0,0),(1,0)\}$ and $V_2=\{(0,1),(1,1)\}$ of $V(T(\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2)))$. Hence, the subgraphs of $T(\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2))$ induced by sets V_1 and V_2 are acyclic. Thus $va(T(\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2)))=2$.

For $\mathbb{Z}_6$, by considering the acyclic partition $V_1=\{0,1,3\}$ and $V_2=\{2,4,6\}$ of $V(T(\Gamma(\mathbb{Z}_6)))$, we have $va(T(\Gamma(\mathbb{Z}_6)))=2$.

For $\mathbb{Z}_2\times\mathbb{Z}_4$, we put $V_1=\{(0,0),(0,2),(1,1),(1,3)\}$ and $V_2=\{(0,1),(0,3),(1,0),(1,2)\}$. Now, it is easy to see that $va(T(\Gamma(\mathbb{Z}_2\times\mathbb{Z}_4)))=2$. Since $T(\Gamma(\mathbb{Z}_4))\cong T(\Gamma(\frac{\mathbb{Z}_2[x]}{(x^2)}))$, by Remark 3.4, we have $T(\Gamma(\mathbb{Z}_2\times\mathbb{Z}_4))\cong T(\Gamma(\mathbb{Z}_2\times\frac{\mathbb{Z}_2[x]}{(x^2)}))$. Thus $va(T(\Gamma(\mathbb{Z}_2\times\frac{\mathbb{Z}_2[x]}{(x^2)})))=2$.

For $\mathbb{Z}_2\times\mathbb{F}_4$, by using the acyclic partition

$$V_1=\{(0,0),(0,1),(1,0),(1,a)\}\text{ and }V_2=\{(0,a),(0,a^2),(1,1),(1,a^2)\}$$

of $V(T(\Gamma(\mathbb{Z}_2\times\mathbb{F}_4)))$, we have $va(T(\Gamma(\mathbb{Z}_2\times\mathbb{F}_4)))=2$.

For $\mathbb{Z}_3\times\mathbb{Z}_3$, we consider the acyclic partition $V_1=\{(0,0),(0,1),(1,0),(1,1),(2,1)\}$ and $V_2=\{(0,2),(2,0),(1,2),(2,2)\}$ of $V(T(\Gamma(\mathbb{Z}_3\times\mathbb{Z}_3)))$. Hence $va(T(\Gamma(\mathbb{Z}_3\times\mathbb{Z}_3)))=2$.

For $\mathbb{Z}_3\times\mathbb{Z}_4$, the graph $T(\Gamma(\mathbb{Z}_3\times\mathbb{Z}_4))$ has a complete graph K_6 as a subgraph with vertex set $\{(0,0),(1,0),(2,0),(0,2),(1,2),(2,2)\}$, and so, by Remark 2.2, we have $va(T(\Gamma(\mathbb{Z}_3\times\mathbb{Z}_4)))>2$. Also by Remark 3.4, we have $T(\Gamma(\mathbb{Z}_3\times\mathbb{Z}_4))\cong T(\Gamma(\mathbb{Z}_3\times\frac{\mathbb{Z}_2[x]}{(x^2)}))$. Thus $va(T(\Gamma(\mathbb{Z}_3\times\frac{\mathbb{Z}_2[x]}{(x^2)})))>2$.

For $\mathbb{Z}_3\times\mathbb{F}_4$, according to the Figure 2 we have $va(T(\Gamma(\mathbb{Z}_3\times\mathbb{F}_4)))=2$.

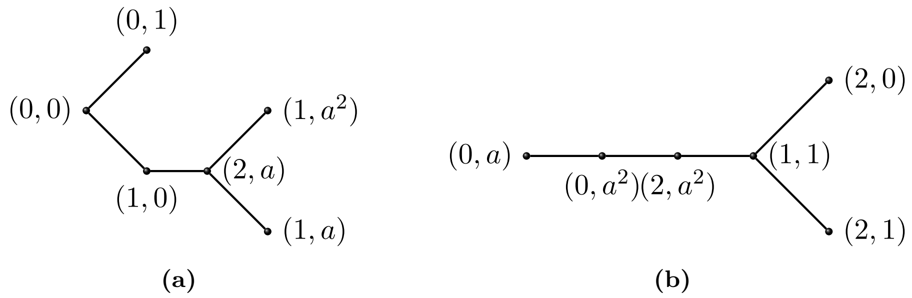

Figure 2

For $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$, by Lemma 3.5, we have $va(T(\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2)))>2$.

For $\mathbb{Z}_4\times\mathbb{Z}_4$, the graph $T(\Gamma(\mathbb{Z}_4\times\mathbb{Z}_4))$ has a K_8 as a subgraph with vertex set

$$\{(0,0),(1,0),(2,0),(3,0),(0,2),(1,2),(2,2),(3,2)\},$$

and so, by Remark 2.2, we have $va(T(\Gamma(\mathbb{Z}_4\times\mathbb{Z}_4)))>3$.

According to Remark 3.4, $T(\Gamma(\mathbb{Z}_4\times\mathbb{Z}_4))\cong T(\Gamma(\mathbb{Z}_4\times\frac{\mathbb{Z}_2[x]}{(x^2)}))\cong T(\Gamma(\frac{\mathbb{Z}_2[x]}{(x^2)}\times\frac{\mathbb{Z}_2[x]}{(x^2)}))$. So the vertex-arboricity of graphs $T(\Gamma(\mathbb{Z}_4\times\frac{\mathbb{Z}_2[x]}{(x^2)}))$ and $T(\Gamma(\frac{\mathbb{Z}_2[x]}{(x^2)}\times\frac{\mathbb{Z}_2[x]}{(x^2)}))$ is greater than three.

For $\mathbb{Z}_4\times\mathbb{F}_4$, the graph $T(\Gamma(\mathbb{Z}_4\times\mathbb{F}_4))$ has a K_8 as a subgraph with vertex set

$$\{(0,0),(0,1),(0,a),(0,a^2),(2,0),(2,1),(2,a),(2,a^2)\},$$

and so, by Remark 2.2, we have $va(T(\Gamma(\mathbb{Z}_4 \times \mathbb{F}_4))) > 3$. Also by Remark 3.4, $T(\Gamma(\mathbb{Z}_4 \times \mathbb{F}_4)) \cong T(\Gamma(\frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{F}_4))$. Therefore $va(T(\Gamma(\frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{F}_4))) > 3$.

For $\mathbb{F}_4 \times \mathbb{F}_4$, by Lemma 3.5, we have $va(T(\Gamma(\mathbb{F}_4 \times \mathbb{F}_4))) > 2$. □

Lemma 3.7. *For the ring $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, $va(T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3))) = 4$.*

Proof. First, by Remark 3.3, we have $va(T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3))) > 2$.

Now, let $T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)) = G$ and $A = A_0 \cup A_1$, where $A_0 = \{(0,0,z) : z \in \mathbb{Z}_3\}$ and $A_1 = \{(0,1,z) : z \in \mathbb{Z}_3\}$. Also put $B = B_0 \cup B_1$, where $B_0 = \{(1,0,z) : z \in \mathbb{Z}_3\}$ and $B_1 = \{(1,1,z) : z \in \mathbb{Z}_3\}$. It is clear that the two sets A and B are partition for $V(G)$. Let $\{V_1, V_2, V_3\}$ be an acyclic partition for $V(G)$. If $|V_j| \geq 5$ for some $j \in \{1,2,3\}$, then $|A \cap V_j| \geq 3$ or $|B \cap V_j| \geq 3$, which is impossible, since $G[A]$ and $G[B]$ are complete graphs isomorphic to K_6 and $G[V_i]$ $(1 \leq i \leq 3)$ are acyclic induced subgraphs of G. Therefore $|V_i| = 4$ for some $i \in \{1,2,3\}$.

We know that every vertex of $G[A_0]$ ($G[A_1]$) are adjacent to every vertex of $G[B_0]$ ($G[B_1]$) and $G[V_i]$ $(1 \leq i \leq 3)$ are acyclic induced subgraphs of G. Hence without the loss of generality we can assume that $|A_0 \cap V_1| = |B_1 \cap V_1| = 2$ and $|A_1 \cap V_2| = |B_0 \cap V_2| = 2$. Then $V_3 = \{a_0, a_1, b_0, b_1 : a_s \in A_s, b_t \in B_t, 0 \leq s, t \leq 1\}$. It follows that $G[V_3]$ is a cycle of length 4, which is a contradiction and so $va(G) > 3$.

Now, by using the following partition of $V(G)$, we have that $va(G) = 4$.

$$V_1 = \{(0,0,0), (1,0,0), (1,1,2)\}, \qquad V_2 = \{(0,1,0), (1,1,1), (1,0,1)\},$$
$$V_3 = \{(0,1,2), (0,0,2), (1,0,2)\}, \qquad V_4 = \{(0,0,1), (0,1,1), (1,1,0)\}.$$

□

Theorem 3.8. *Let R be a finite commutative ring such that $va(T(\Gamma(R))) = 3$. Then the following statements hold.*

(i) *If R is local, then R is isomorphic to $\mathbb{Z}_{25}$ or $\frac{\mathbb{Z}_5[x]}{(x^2)}$.*

(ii) *If R is not local, then R is isomorphic to one of the following rings:*

$\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{F}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_5 \times \mathbb{Z}_5$.

Proof. (i) Assume that R is a local ring. We consider the following two cases:

(a) If $2 \in Z(R)$, then, by Theorem 2.5, we have $T(\Gamma(R)) \cong mK_n$. Hence, by Remark 2.2, $5 \leq |Z(R)| \leq 6$. But, in this situation $2 \in Z(R)$, and so, there are no such local rings.

(b) If $2 \notin Z(R)$, then, by Theorem 2.5, we have $T(\Gamma(R)) \cong K_n \bigcup (\frac{m-1}{2}) K_{n,n}$. Hence, by Remark 2.2, $5 \leq |Z(R)| \leq 6$. Therefore $|Z(R)| = 5$ and so there exist two local rings, $\mathbb{Z}_{25}$ and $\frac{\mathbb{Z}_5[x]}{(x^2)}$ of order 25. For these rings we have $T(\Gamma(R)) \cong K_5 \bigcup 2K_{5,5}$. Hence, by Corollary 2.7, we have $va(T(\Gamma(R))) = 3$.

(ii) Suppose that R is not a local ring. Arguments similar to those used in proof of Theorem 3.6 (ii), in conjunction with Remarks 2.2 and 3.3 show that we have the following candidates:

$\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_6$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_3 \times \mathbb{F}_4$,
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\frac{\mathbb{Z}_2[x]}{(x^2)} \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_4 \times \mathbb{F}_4$, $\frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{F}_4$,
$\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_4 \times \mathbb{Z}_5$, $\frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{Z}_5$, $\mathbb{F}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_5 \times \mathbb{Z}_5$.

According to the proof of Theorem 3.6 (ii), we examine the following cases:

For $\mathbb{Z}_3 \times \mathbb{Z}_4$, we consider the partition

$$V_1 = \{(0,0), (1,1), (1,2), (1,3)\},$$
$$V_2 = \{(0,2), (2,0), (2,1), (2,3)\}$$

and

$$V_3 = \{(0,1), (0,3), (1,0), (2,2)\}$$

of $V(T(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_4)))$. The subgraphs of $T(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_4))$ induced by the sets V_1, V_2 and V_3 are acyclic graphs. Hence, we have $va(T(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_4))) = 3$. The Remark 3.4 implies that $T(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_4)) \cong T(\Gamma(\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}))$ and so $va(T(\Gamma(\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}))) = 3$.

For rings $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{F}_4 \times \mathbb{F}_4$, by Lemma 3.5, we have $va(T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))) = va(T(\Gamma(\mathbb{F}_4 \times \mathbb{F}_4))) = 3$.

For $\mathbb{Z}_2 \times \mathbb{Z}_5$, consider the acyclic partition $V_1 = \{(0,0),(0,1),(1,1),(1,2)\}$, $V_2 = \{(0,2),(0,3),(1,0),(1,4)\}$ and $V_3 = \{(0,4),(1,3)\}$ of $V(T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5)))$. Now, it is easy to see that $va(T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5))) = 3$.

For $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, by Lemma 3.7, we have $va(T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3))) > 3$.

For $\mathbb{Z}_3 \times \mathbb{Z}_5$, by using the acyclic partition

$$V_1 = \{(0,4),(1,0),(1,3),(2,3)\},$$

$$V_2 = \{(0,0),(0,1),(1,2),(1,4),(2,1)\}$$

and

$$V_3 = \{(0,2),(0,3),(1,1),(2,0),(2,2),(2,4)\}$$

of $V(T(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_5)))$, we have $va(T(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_5))) = 3$.

For $\mathbb{Z}_4 \times \mathbb{Z}_5$, the graph $T(\Gamma(\mathbb{Z}_4 \times \mathbb{Z}_5))$ has a complete graph K_{10} as a subgraph with vertex set $\{(0,0),(0,1),(0,2),(0,3),(0,4),(2,0),(2,1),(2,2),(2,3),(2,4)\}$, and so, we have $va(T(\Gamma(\mathbb{Z}_4 \times \mathbb{Z}_5))) \geq 5$. Also, Remark 3.4, $T(\Gamma(\mathbb{Z}_4 \times \mathbb{Z}_5)) \cong T(\Gamma(\frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{Z}_5))$ and so $va(T(\Gamma(\frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{Z}_5))) \geq 5$.

For $\mathbb{F}_4 \times \mathbb{Z}_5$, according to Figure 3, we have $va(T(\Gamma(\mathbb{F}_4 \times \mathbb{Z}_5))) = 3$.

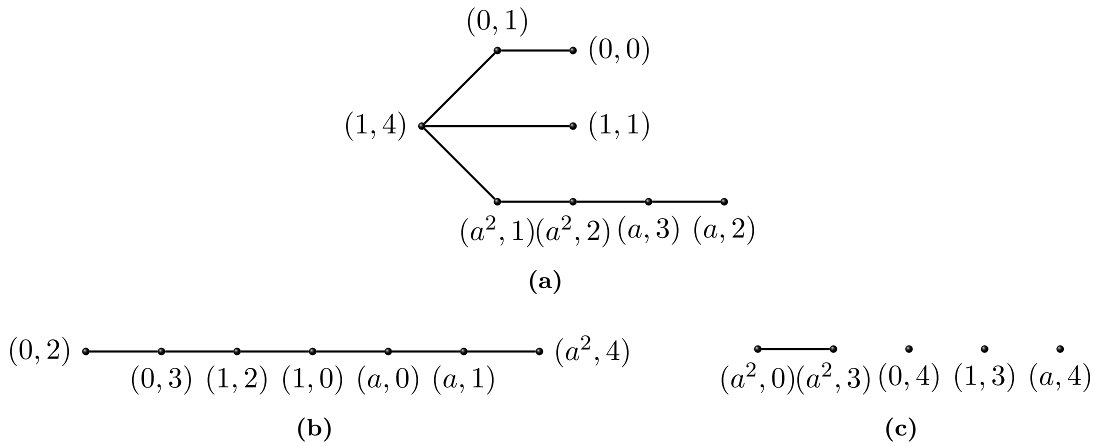

Figure 3

For $\mathbb{Z}_5 \times \mathbb{Z}_5$, by Figure 4, we conclude that $va(T(\Gamma(\mathbb{Z}_5 \times \mathbb{Z}_5))) = 3$.

Thus the proof is complete. □

4. The arboricity of the total graph

In this section, we characterize all finite commutative rings whose total graph has arboricity two or three. In addition, we show that, for a positive integer v, there are only finitely many finite rings whose total graph has arboricity v. We begin the section with the following result of C. St. J. A. Nash-Williams.

Theorem 4.1 ([9]). *For a graph G, $\nu(G) = \max\lceil \frac{e_H}{n_H - 1} \rceil$, where $n_H = |V(H)|$, $e_H = |E(H)|$ and H ranges over all non-trivial induced subgraphs of G.*

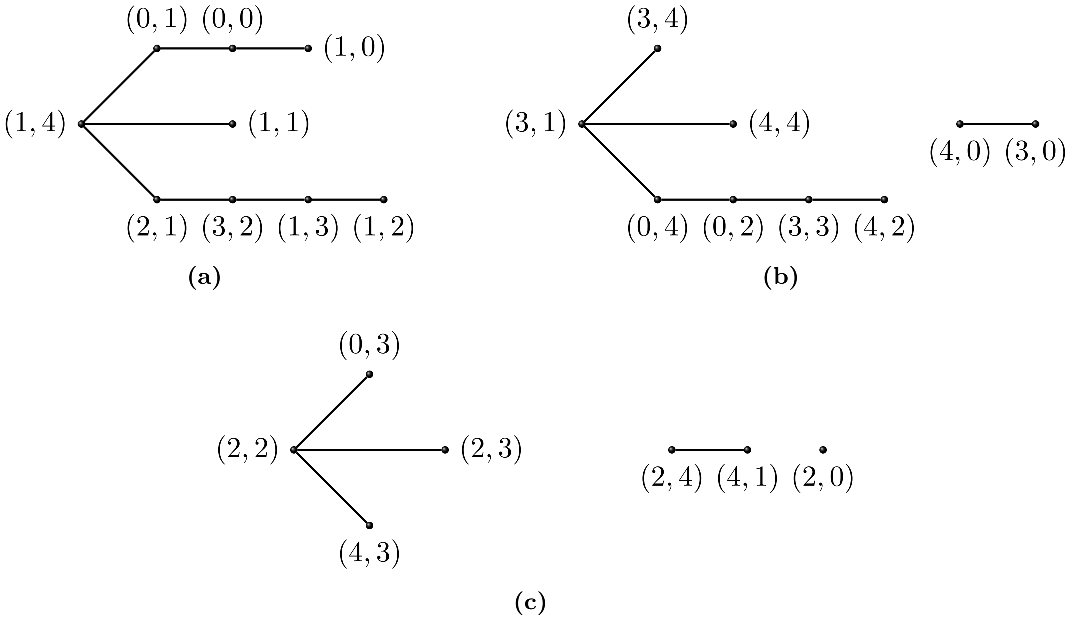

Figure 4

Theorem 4.2. *For a graph G, $\lceil \frac{\delta(G)+1}{2} \rceil \leq \nu(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$. In particular, if G is d-regular, then $\nu(G) = \lceil \frac{d+1}{2} \rceil = \lceil \frac{e}{n-1} \rceil$, where $n = |V(G)|$ and $e = |E(G)|$.*

Proof. First, it is clear that, if G has some isolated vertices, say $X = \{x_1, x_2, \ldots, x_k\}$, then $\nu(G) = \nu(G[V(G) \setminus X])$. So, we can assume that G has no isolated vertices. Let H be a subgraph of G with $|V(H)| = n'$ and $|E(H)| = e'$. Then we have

$$\frac{e'}{n'-1} \leq \frac{\Delta(H)n'}{2(n'-1)} = \frac{1}{2}(\Delta(H) + \frac{\Delta(H)}{n'-1}).$$

Since $\Delta(H) \leq \min\{\Delta(G), n'-1\}$, we have $\frac{e'}{n'-1} \leq \frac{\Delta(G)+1}{2}$, and hence, by Theorem 4.1, $\nu(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$. On the other hand $\frac{e}{n-1} \geq \frac{\delta(G)n}{2(n-1)} > \frac{\delta(G)}{2}$. Since $\nu(G)$ is an integer, $\nu(G) \geq \lceil \frac{\delta(G)+1}{2} \rceil$, as required. □

Clearly, in view of the above theorem, $\nu(K_n) = \lceil \frac{n}{2} \rceil$. So, by arguing as in the proof of Theorem 2.4, we have the following theorem.

Theorem 4.3. *For any positive integer v, the number of finite rings R whose total graph has arboricity v is finite.*

Theorem 3.1 implies that $T(\Gamma(R))$ has arboricity one if and only if either R is an integral domain or R is isomorphic to $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{(x^2)}$. Now, we will classify, up to isomorphism, all the finite commutative rings whose total graph has arboricity two or three.

Theorem 4.4. *Let R be a finite ring such that $\nu(T(\Gamma(R))) = 2$. Then the following statements hold.*

(i) *If R is local, then R is isomorphic to one of the following rings:*
$\mathbb{Z}_9, \frac{\mathbb{Z}_3[x]}{(x^2)}, \mathbb{Z}_8, \frac{\mathbb{Z}_2[x]}{(x^3)}, \frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}, \frac{\mathbb{Z}_2[x,y]}{(x,y)^2}, \frac{\mathbb{Z}_4[x]}{(2,x)^2}, \frac{\mathbb{F}_4[x]}{(x^2)}, \frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$.

(ii) *If R is not local, then R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_6$.*

Proof. (i) Assume that R is a local ring. If $2 \in Z(R)$, then, by Lemma 2.1 and Theorem 4.2, we have $|Z(R)| = 4$. Then by Theorem 3.2, $|R| = 16, 8$. Now, by same argument of

Theorem 3.6, R is isomorphic to one of the following rings:

$$\mathbb{Z}_8, \frac{\mathbb{Z}_2[x]}{(x^3)}, \frac{\mathbb{Z}_4[x]}{(2x, x^2-2)}, \frac{\mathbb{Z}_2[x,y]}{(x,y)^2}, \frac{\mathbb{Z}_4[x]}{(2,x)^2}, \frac{\mathbb{F}_4[x]}{(x^2)}, \frac{\mathbb{Z}_4[x]}{(x^2+x+1)}.$$

If $2 \notin Z(R)$, then $|Z(R)| = 3$. So, R is isomorphic to $\mathbb{Z}_9$ or $\frac{\mathbb{Z}_3[x]}{(x^2)}$.

(ii) If R is not a local ring, then, by Theorem 4.2, we have $3 \leq |Z(R)| \leq 4$. When $|Z(R)| = 3$, it is clear that R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Moreover, if $|Z(R)| = 4$, then R is isomorphic to $\mathbb{Z}_6$, and so the proof is complete. □

By slight modifications in the proof of Theorem 4.4, one can prove the following theorem.

Theorem 4.5. *Let R be a finite ring such that $\nu(T(\Gamma(R))) = 3$. Then the following statements hold.*

(i) *If R is local, then R is isomorphic to $\mathbb{Z}_{25}$ or $\frac{\mathbb{Z}_5[x]}{(x^2)}$.*

(ii) *If R is not local, then R is isomorphic to one of the following rings:*

$$\mathbb{Z}_2 \times \mathbb{F}_4, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}, \mathbb{Z}_2 \times \mathbb{Z}_5, \mathbb{Z}_3 \times \mathbb{F}_4.$$

In general, we can determine the arboricity of the total graph as in the following theorem.

Theorem 4.6. *Let R be a finite ring.*

(i) *If $2 \in Z(R)$, then $\nu(T(\Gamma(R))) = \lceil \frac{|Z(R)|}{2} \rceil$.*

(ii) *If $2 \notin Z(R)$, then the following statements hold.*

 (1) *If $|Z(R)| = 2k+1$, then $\nu(T(\Gamma(R))) = k+1$.*

 (2) *If $|Z(R)| = 2k$, then $k \leq \nu(T(\Gamma(R))) \leq k+1$.*

Proof. It follows from Lemma 2.1 and Theorem 4.2. □

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