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INTEGER MULTIPLIERS OF REAL POLYNOMIALS WITHOUT NONNEGATIVE ROOTS

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Abstract. For a given real polynomial f without nonnegative roots we study monic integer polynomials g such that the product gf has positive (nonnegative, respectively) coefficients. We show that monic integer polynomials g with these properties can effectively be computed, and we give lower and upper bounds for their degrees.

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1. Introduction

For a polynomial f with real coefficients and positive leading coefficient the quantities

$$\delta(f) = \inf \{\deg(g) \ : \ g \in \mathbb{R}[X], \ gf \text{ has only positive coefficients}\}$$

and

$$\delta_0(f) = \inf \{\deg(g) \ : \ g \in \mathbb{R}[X] \setminus \{0\}\,, \ gf \text{ has only nonnegative coefficients}\}$$

have been introduced by J.-P. Borel [2]. It was shown by E. Meissner [16] and A. Durand (see [2, Théorème 2]) that $\delta(f)$ is finite if and only if f does not have a real nonnegative root; furthermore, $\delta_0(f)$ is finite if and only if f does not have a positive root. Moreover, if f does not have a real nonnegative root, a monic real polynomial t such that tf has only nonnegative coefficients and $\deg(g) = \delta_0(f)$ can effectively be computed; such a polynomial t is sometimes called a δ_0-multiplier of f. The analogous statement holds for f without a real positive root (see [6, Theorem 12]). In [2, Théorème 6] an upper bound for $\delta(f)$ is given provided that f admits only simple non-real roots of modulus one (see also [7, Theorem 20]).

We refer the reader to [2] for the historical roots of these quantities in uniform distribution theory. De Angelis [10] characterized those polynomials whose sufficiently large powers have all positive coefficients. A generalization of these results to multivariate homogeneous polynomials and further references to applications is provided by C. Tan – W.-K. To [20].

In this short note we consider monic real polynomials f without real nonnegative roots and extend our results in [4]. We show that for any natural number r a monic integer polynomial s such that sf has only positive coefficients (nonnegative coefficients, respectively) and $s(0) \le r^{\deg(s)}$ can effectively be computed. Further, we give lower and upper bounds for their degrees and an application to positively algebraic numbers. Finally, we illustrate our results by several examples.

2. Integer multipliers of elements of $\mathcal{F}$

We need some notation and auxiliary results to deal with the set of polynomials

$$\mathcal{F} = \{f \in \mathbb{R}[X] \;:\; f \text{ monic and } f \text{ does not have a root in } [0,\infty)\}\,.$$

For brevity, we denote by $\mathbb{R}_{>0}$ ($\mathbb{R}_{\ge 0}$, respectively) the set of positive (nonnegative, respectively) real numbers.

Our main effort is focused on quadratic polynomials. More precisely, we consider

$$q := X^2 - bX + c \in \mathbb{R}[X]$$

with $b > 0$ and $b^2 < 4c$. We know from [6, Lemma 1] that for every positive r the constant

$$\nu_q(r) := \min\{n \in \mathbb{N} \;:\; (X+r)^n q \in \mathbb{R}_{>0}[X]\}$$

can be calculated in finitely many steps. In the sequel the product

$$(X+r)^n \cdot q = r^n c + r^{n-1}(cn - br)X + \sum_{k=2}^{n} p_k X^k + (nr - b)X^{n+1} + X^{n+2} \tag{2.1}$$

with

$$p_k := \binom{n}{k-2} r^{n-k+2} - b\binom{n}{k-1} r^{n-k+1} + c\binom{n}{k} r^{n-k} = \binom{n}{k-1} r^{n-k} f(k) \tag{2.2}$$

and

$$f(k) := \frac{n+1-k}{k}c - br + \frac{k-1}{n+2-k}r^2 \qquad (2 \le k \le n) \tag{2.3}$$

is frequently used. Let us mention in passing that for small b we can easily find a suitable r such that $\nu_q(r)$ is also small.

Lemma 2.1. *Let $b, c \in \mathbb{R}_{>0}$ and $q := X^2 - bX + c$.*

(i) *$\nu_q(r) = 1$ for some $r \in \mathbb{R}_{>0}$ if and only if $b^2 < c$. In this case, we have $r \in (b,\ c/b)$.*

(ii) $\nu_q(r) = 2$ *for some* $r \in \mathbb{R}_{>0}$ *if and only if* $c \le b^2 < 4c/3$. *In this case, we have*

$$r \in \bigl(c/(2b),\ 2c/b\bigr)\,.$$

Proof. (i) Clear by (2.1).

(ii) Let $\nu_q(r) = 2$ with some $r \in \mathbb{R}_{>0}$, hence $b^2 \ge c$ by (i), and (2.1) yields

$$r \in (b/2,\ 2c/b) \qquad \text{and} \qquad |r - b| > \sqrt{b^2 - c}\,,$$

and we are left to show $b^2 < 4c/3$. If $r < b$ then we have

$$\sqrt{b^2 - c} < b - r < \frac{b}{2}$$

which immediately yields our assertion. If $r \ge b$ we deduce

$$r > b + \sqrt{b^2 - c}\,,$$

which easily shows that the relation $b^2 \ge 4c/3$ is impossible.

Now suppose $c \le b^2 < 4c/3$ and pick $r \in \bigl(c/(2b),\ 2c/b\bigr)$. We immediately check

$$r^2 - 2br + c > \frac{c^2}{4b^2} > 0\,,$$

and our claim drops out by verifying (2.1). □

Our central lemma specifies a seemingly useful upper bound for $\nu_q(r)$ for a quadratic $q \in \mathcal{F} \setminus \mathbb{R}_{\ge 0}[X]$ and arbitrary positive r, and its proof describes how $\nu_q(r)$ can effectively be computed.

Lemma 2.2. *Let* $b, c, r \in \mathbb{R}_{>0}$ *with* $b^2 < 4c$. *For* $q := X^2 - bX + c$ *we can effectively compute* $\nu_q(r)$ *using the inequalities*

$$\max\left\{\frac{b}{r},\ \frac{br}{c},\ \delta(q) - 1\right\} < \nu_q(r) \le \min\left\{m \in \mathbb{N}\ :\ m > \max\left\{\frac{b}{r},\ \frac{br}{c},\ w\right\}\right\},$$

where we set

$$w := \frac{\beta + \sqrt{\beta^2 + A\gamma}}{Ar} > \frac{r}{2\sqrt{c} - b} + 2 \cdot \frac{b^2 + b\sqrt{c} - 2c}{4c - b^2}$$

with

$$A := 4c - b^2, \quad \beta := br^2 + 2(b^2 - 2c)r + bc \quad \text{and} \quad \gamma := r^4 + 4br^3 + 2(2b^2 - c)r^2 + 4bcr + c^2$$

and

$$\delta(q) = \left\lfloor \frac{\pi}{\arcsin\sqrt{1 - \frac{b^2}{4c}}} \right\rfloor - 1\,.$$

Further, we can effectively compute

$$\nu_{q,0}(r) := \min\{n \in \mathbb{N}\ :\ (X + r)^n q \in \mathbb{R}_{\ge 0}[X]\}$$

using the inequalities

$$\max\left\{\frac{b}{r},\ \frac{br}{c},\ \delta_0(q)\right\} \le \nu_{q,0}(r) \le \nu_q(r)$$

with

$$\delta_0(q) = \left\lceil \frac{\pi}{\arcsin\sqrt{1-\frac{b^2}{4c}}} \right\rceil - 2\,.$$

Proof. The values of $\delta(q)$ and $\delta_0(q)$ are well-known [9,17,6]. Thus in view of (2.1) the left hand side inequalities are clear. The remainder of the proof is essentially given in [4, Lemma 1], however, for the convenience of the reader we repeat the details here.

We immediately check

$$\begin{aligned}
\beta^2 + A\gamma &= b^2r^4 + 2\big(2(b^2-2c)r+bc\big)br^2 + \big(2(b^2-2c)r+bc\big)^2 \\
&\quad + A\big(r^4+4br^3+2(2b^2-c)r^2+4bcr+c^2\big) \\
&= 4cr^4 + 4(b^2-2c+A)br^3 + 2\big(b^2c+(4c-b^2)(2b^2-c)\big)r^2 \\
&\quad + 4(b^2-2c)^2r^2 + 4(b^2-2c)bcr + b^2c^2 + A\big(4bcr+c^2\big) \\
&= 4cr^4 + 8bcr^3 + 2\big(10b^2c-2b^4-4c^2+2(b^2-2c)^2\big)r^2 \\
&\quad + 4(b^2-2c+A)bcr + b^2c^2 + (4c-b^2)c^2 \\
&= 4cr^4 + 8bcr^3 + 2\big(10b^2c-2b^4-4c^2+2(b^4-4b^2c+4c^2)\big)r^2 + 8bc^2r + 4c^3 \\
&= 4cr^4 + 8bcr^3 + 4(b^2+2c)cr^2 + 8bc^2r + 4c^3 > 0\,.
\end{aligned}$$

Therefore, the largest root of

$$h(x) := Ar^2x^2 - 2\beta rx - \gamma$$

equals w, and we have

$$\begin{aligned}
w &> \frac{1}{A}\big(br+2(b^2-2c)+2\sqrt{c}\sqrt{(r+b)^2+2c}\big) > \frac{1}{A}\big(br+2(b^2-2c)+2\sqrt{c}(r+b)\big) \\
&= \frac{1}{A}\big((b+2\sqrt{c})r+2b\sqrt{c}+2(b^2-2c)\big) = \frac{r}{2\sqrt{c}-b} + 2\cdot\frac{b^2+b\sqrt{c}-2c}{4c-b^2}\,.
\end{aligned}$$

Thus for

$$n := \min\left\{m\in\mathbb{N}\ :\ m > \max\left\{\frac{b}{r},\ \frac{br}{c},\ w\right\}\right\}$$

we have

$$h(n) > 0\,. \tag{2.4}$$

Using (2.3) we observe

$$k(n-k+2)f(k) = g(k)$$

for $2 \le k \le n$, where we set

$$g(x) := \delta x^2 - ((2c + br)n + \sigma)x + c(n^2 + 3n + 2)$$

with

$$\delta := r^2 + br + c \qquad \text{and} \qquad \sigma := r^2 + 2br + 3c\,.$$

In view of (2.4) we immediately check that the discriminant of g is negative, where we use the equalities

$$4\delta c - (2c + br)^2 = Ar^2$$

and

$$((2c + br)n + \sigma)^2 - 4\delta c(n^2 + 3n + 2) = -h(n)\,.$$

Thus $g(x) > 0$ for $x \in \mathbb{R}$, hence for $k \in \{2, \dots, n\}$ we have $f(k) > 0$ by (2.3) and then $p_k > 0$ by (2.2). In view of (2.1) we conclude $p \in \mathbb{R}_{>0}[X]$. If

$$(X + r)^{n-1} q \notin \mathbb{R}_{>0}[X]$$

we have found

$$\nu_q(r) = n\,.$$

Otherwise, successively decreasing n we determine

$$m := \max\left\{k \in \{1, \dots, n - \delta(q)\} \;:\; (X + r)^{n-k} q \in \mathbb{R}_{>0}[X]\right\},$$

and thus we find

$$\nu_q(r) = n - m\,.$$

Finally, similarly as above we determine

$$\nu_{q,0}(r) = \nu_q(r) - \max\left\{k \in \{0, \dots, \nu_q(r) - \delta_0(q)\} \;:\; (X + r)^{\nu_q(r)-k} q \in \mathbb{R}_{\ge 0}[X]\right\}.$$

□

Let us note an immediate consequence of the result above.

Corollary 2.3. *Let $b, c \in \mathbb{R}$ with $b^2 < 4c$ and $b \ge c$. For $r \in \mathbb{N}$ we have*

$$\nu_{X^2 - bX + c}(r) \ge r + 1\,.$$

Proof. In view of $(br)/c \ge r$ clear by Lemma 2.2. □

Now we turn to quadratic polynomials without a real root. For brevity we denote by $\mathbb{N}_0$ the set of nonnegative rational integers.

Corollary 2.4. *Let $r \in \mathbb{N}$, $\omega \in \mathbb{C} \setminus \mathbb{R}$ and set*

$$q := (X - \omega)(X - \bar{\omega}).$$

Further, set $d_0(\omega) := d(\omega) := 0$ if $\Re(\omega) \leq 0$, otherwise $d_0(\omega) := \delta_0(q)$ and $d(\omega) := \delta(q)$.

(i) *We can effectively compute a constant $K_0(r, \omega) \in \mathbb{N}_0$ and a monic integer polynomial t with*

$$d_0(\omega) \leq \deg(t) \leq K_0(r, \omega), \quad t \cdot q \in \mathbb{R}_{\geq 0}[X] \quad \text{and} \quad t(0) = r^{\deg(t)}. \tag{2.5}$$

(ii) *We can effectively compute a constant $K(r, \omega) \in \mathbb{N}_0$ and a monic integer polynomial s with*

$$d(\omega) \leq \deg(s) \leq K(r, \omega), \quad s \cdot q \in \mathbb{R}_{>0}[X] \quad \text{and} \quad s(0) = r^{\deg(s)}.$$

Proof. For $\Re(\omega) \leq 0$ we set $K(r, \omega) = K_0(r, \omega) := 0$ and $s = t = 1$, and for $\Re(\omega) > 0$ we apply Lemma 2.2 with $K(r, \omega) := \nu_q(r)$ and $K_0(r, \omega) := \nu_{q,0}(r)$. □

We have shown in [4, Lemma 3] that for $f \in \mathcal{F}$ and any non-constant $t \in \mathbb{R}_{>0}[X]$ there exists some $m \in \mathbb{N}$ bounded by an effectively computable constant such that $t^m f$ has only positive coefficients. Our main result here gives an explicit bound for the degree and the constant coefficient of a monic integer polynomial t such that tf has only nonnegative coefficients. The bound for the degree of t is given explicitly in terms of the nonreal roots of f. To this end we need the (possibly empty) multiset $\mathrm{Z}_{f,+}$ of roots α of f with positive real and imaginary parts.

Theorem 2.5. *Let f be a monic real polynomial without real nonnegative roots and r a natural number.*

(i) *There exists an effectively computable monic integer polynomial t with $1 \leq t(0) \leq r^{\deg(t)}$ and*

$$\sum_{\alpha \in \mathrm{Z}_{f,+}} d_0(\alpha) \leq \deg(t) \leq \sum_{\alpha \in \mathrm{Z}_{f,+}} K_0(r, \alpha)$$

such that the product tf has only nonnegative coefficients.

(ii) *There exists an effectively computable monic integer polynomial s with $s(0) \leq r^{\deg(s)}$ and*

$$\sum_{\alpha \in \mathrm{Z}_{f,+}} d(\alpha) \leq \deg(s) \leq \sum_{\alpha \in \mathrm{Z}_{f,+}} K(r, \alpha)$$

such that the product sf has only positive coefficients.

Proof. We only show the first part and leave the proof of the second part to the reader. Let us proceed by induction on the cardinality c_f of $\mathrm{Z}_{f,+}$. If $c_f = 0$ then every monic irreducible factor of f has the form

$$X + r \qquad (r > 0) \qquad \text{or} \qquad X^2 + bX + c \qquad (b \geq 0,\, c > 0),$$

and we may choose $t = 1$; note that in this case the right hand side equals 0.

Now let $c_f > 0$, pick $\alpha \in \mathrm{Z}_{f,+}$ and set

$$q := (X - \alpha)(X - \bar{\alpha})$$

and $g := f/q$. Clearly, $g \in \mathcal{F}$ and $c_g < c_f$. By induction hypothesis there exists a monic polynomial $s \in \mathbb{Z}[X]$ with

$$\sum_{\beta \in \mathrm{Z}_{g,+}} d_0(\beta) \leq \deg(s) \leq \sum_{\beta \in \mathrm{Z}_{g,+}} K_0(r, \beta)$$

and

$$s \cdot g \in \mathbb{R}_{\geq 0}[X] \quad \text{and} \quad s(0) = r^{\deg(s)}\,.$$

Corollary 2.4 yields a monic polynomial $t \in \mathbb{Z}[X]$ with (2.5), and we observe

$$(st)f = (tq)(sg) \in \mathbb{R}_{\geq 0}[X] \qquad \text{and} \qquad (st)(0) = r^{\deg(s)+\deg(t)} = r^{\deg(st)}$$

and

$$\begin{aligned} \sum_{\gamma \in \mathrm{Z}_{f,+}} d_0(\gamma) &= d_0(\alpha) + \sum_{\beta \in \mathrm{Z}_{g,+}} d_0(\beta) \leq \deg(t) + \deg(s) \\ &= \deg(st) \leq K_0(r, \alpha) + \sum_{\beta \in \mathrm{Z}_{g,+}} K_0(r, \beta) = \sum_{\gamma \in \mathrm{Z}_{f,+}} K_0(r, \gamma)\,. \end{aligned}$$

□

Recall that the algebraic number α is positively algebraic if neither α nor any of its conjugates over $\mathbb{Q}$ is a nonnegative real number [14, Section 2]. These numbers were characterized by A. Dubickas [12], and T. Zaïmi [21] gave an upper bound for the degree (in terms of the degree, the discriminant and the Mahler measure of α) of a polynomial with positive rational coefficients which nullifies α (see also [3,15]). Our result immediately yields the following.

Corollary 2.6. *Let f be the minimal polynomial of the algebraic number $\alpha \neq 0$. Then α is positively algebraic if and only if there exists a monic integer polynomial t with $t(0) = 1$ and*

$$\deg(t) \leq \sum_{\omega \in \mathrm{Z}_{f,+}} K_0(1, \omega)$$

such that the product tf has only nonnegative coefficients.

Some simple numerical examples illustrate our statements.

Example 2.7. (i) Let $b > 0$ and $c \geq b^2$. Then there exists some $r \in \mathbb{N}$ with $p := (X + r) \cdot (X^2 - bX + c) \in \mathbb{R}_{>0}[X]$ if and only if $\left(b,\ \frac{c}{b}\right) \cap \mathbb{N} \neq \emptyset$.

Similarly,

$$p \in \mathbb{R}_{\geq 0}[X] \iff \left[b,\ \frac{c}{b}\right] \cap \mathbb{N} \neq \emptyset .$$

For instance, for

$$q := X^2 - 2X + 6 \in \mathbb{Z}[X]$$

Lemma 2.1 tells us $\nu_q(r) > 1$ for every natural r, but

$$p := (X + \frac{5}{2}) \cdot q = X^3 + \frac{1}{2}X^2 + X + 15 \in \mathbb{R}_{>0}[X].$$

(ii) Certainly, powers of linear integer polynomials cannot be expected to yield multipliers of smallest degree. For instance, for

$$q := X^2 - 4X + 12 \in \mathbb{Z}[X]$$

we have

$$\delta(q) = \delta_0(q) = \nu_{q,0}(2) = 2,\ \nu_q(2) = 4,$$

and

$$(X^3 + 5X^2 + 9X + 2) \cdot q = X^5 + X^4 + X^3 + 26X^2 + 100X + 24 \in \mathbb{N}[X].$$

Another interesting example is given in [7, Lemma 18].

(iii) It should be pointed out that a multiplier of f can have negative coefficients if the degree of f is larger than two (e.g., see by [1, Example pp. 247-249]).

(iv) By [14, Section 2] the number $(4 + 3i)/5 \in \mathbb{C}$ is positively algebraic, and its minimal polynomial is $q = X^2 - (8/5)X + 1$ with $\delta(q) = \delta_0(q) = 3$. Following the proof of Lemma 2.2, for a few natural integers r we compute lower and upper bounds $L(r)$ and $U(r)$ for $\nu_q(r)$ and then the constants $\nu_q(r)$ and $\nu_{q,0}(r)$. The results are listed in Table 1.

r	$L(r)$	$U(r)$	$\nu_q(r)$	$\nu_{q,0}(r)$
1	4	9	9	7
2	4	10	9	9
3	5	12	12	11
4	7	14	14	14
5	9	17	17	16

TABLE 1

Further, we exhibit the polynomials

$$p_r := (X+r)^{\nu_{q,0}(r)} \cdot q \ :$$

$$p_1 = X^9 + \frac{27}{5}X^8 + \frac{54}{5}X^7 + \frac{42}{5}X^6 + \frac{42}{5}X^3 + \frac{54}{5}X^2 + \frac{27}{5}X + 1\,,$$

$$p_2 = X^{11} + \frac{82}{5}X^{10} + \frac{581}{5}X^9 + \frac{2298}{5}X^8 + \frac{5424}{5}X^7 + \frac{7392}{5}X^6 + \frac{4704}{5}X^5 + \frac{192}{5}X^4 + \frac{1536}{5}X^3 + \frac{7168}{5}X^2 + \frac{7424}{5}X + 512$$

$$p_3 = X^{13} + \frac{157}{5}X^{12} + \frac{2216}{5}X^{11} + 3696X^{10} + 20097X^9 + 73953X^8 + \frac{919512}{5}X^7 + \frac{1475496}{5}X^6 + 264627X^5 + 72171X^4 + \frac{1102248}{5}X^2 + \frac{1830519}{5}X + 177147$$

$$p_4 = X^{16} + \frac{272}{5}X^{15} + \frac{6837}{5}X^{14} + \frac{105112}{5}X^{13} + \frac{1102192}{5}X^{12} + \frac{8316672}{5}X^{11} + \frac{46382336}{5}X^{10} + \frac{192997376}{5}X^9 + \frac{595685376}{5}X^8 + \frac{1330774016}{5}X^7 + \frac{2033647616}{5}X^6 + \frac{1860698112}{5}X^5 + \frac{667942912}{5}X^4 + \frac{117440512}{5}X^3 + \frac{1459617792}{5}X^2 + \frac{2550136832}{5}X + 268435456$$

$$p_5 = X^{18} + \frac{392}{5}X^{17} + 2873X^{16} + 65280X^{15} + 1028500X^{14} + 11900000X^{13} + 104422500X^{12} + 707200000X^{11} + 3722468750X^{10} + 15193750000X^9 + 47480468750X^8 + 110500000000X^7 + 181289062500X^6 + 185937500000X^5 + 83007812500X^4 + 103759765625X^2 + 244140625000X + 152587890625$$

(v) Certainly, strict inequality on the left hand side in Lemma 2.2 is possible. E. g., for

$$q := X^2 - \frac{11}{5}X + \frac{7}{2}$$

we have $\delta(q) = 2$ and

$$(X^2 + 4X + 6) \cdot q = X^4 + \frac{9}{5}X^3 + \frac{7}{10}X^2 + \frac{4}{5}X + 21 \in \mathbb{R}_{>0}[X]\,.$$

Thus we have a monic integer multiplier of degree less than $\nu_q(r)$ for every $r \in \mathbb{R}_{>0}$ (see Lemma 2.1).

3. Concluding remarks

The proof of Lemma 2.2 allows to formulate an algorithm for the computation of the bounds and the multipliers given in Theorem 2.5. The examples in Table 1 seem to suggest that for quadratic polynomials the ν-values always lie close to the upper bounds given by Lemma 2.2. It might be an interesting problem to quantify this observation.

Let p be a CNS polynomial[1]. S. Akiyama [8] predicted that the leading coefficient of the canonical representative of -1 with respect to p equals 1. Since p belongs to $\mathcal{F}$ Theorem 2.5 shows that there is a monic polynomial $t \in \mathbb{Z}[X]$ with the properties

$$t \cdot p \in \mathbb{N}_0[X] \qquad \text{and} \qquad t(0) = 1\,.$$

To prove Akiyama's Conjecture it suffices to show that apart from the leading and the constant terms all coefficients of tp belong $\{0, \ldots, p(0) - 1\}$ (see [8, Lemma 4.6]). Unfortunately, our approach here does not provide a further progress in this direction.

Using convolutions H. G. Diamond and M. Essen [11, Theorem 5.1] determined analogues of the quantities defined above under the additional assumption that all coefficients of a multiplier of f are nonnegative.

The study of the class of polynomials without nonnegative roots naturally leads to polynomials with positive coefficients which admit irreducible factors with negative coefficients (e.g., the irreducible quadratic polynomial $X^2 - X + 4$ admits the linear polynomial $X + 2$ as a multiplier, and this yields the factorization $X^3 + X^2 + 2X + 8 = (X^2 - X + 4) \cdot (X + 2)$). Thus there might be a connection of our problem here with the factorization of real polynomials with nonnegative coefficients (e.g., see [18] and the literature cited therein).

Finally, we should remark that Theorem 2.5 corrects the proof of [5, Theorem 21]: In its proof it was not shown that an integer multiplier can be found with leading coefficient 1.

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