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ON GRADED UJ-RINGS

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Abstract. In this paper, graded rings are S-graded rings inducing S, that is, rings whose additive groups can be written as a direct sum of a family of their additive subgroups indexed by a nonempty set S, and such that the product of two homogeneous elements is again a homogeneous element. As a generalization of the recently introduced notion of a UJ-ring, we define a graded UJ-ring. Graded nil clean rings which are graded UJ are described. We also investigate the graded UJ-property under some graded ring constructions.

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1. Introduction

Recently, and independently, in [4,19], a *UJ-ring* is introduced as an associative ring R with identity 1 such that $1+J(R)=U(R)$, where $J(R)$ and $U(R)$ denote the Jacobson radical of R and the group of units of R, respectively. As it is noticed in [19], a ring R is a UJ-ring if and only if the set $Q(R)$ of all quasi-regular elements of R coincides with $J(R)$. Motivated by this characterization, in this paper, we define a *graded UJ-ring.*

We observe an *S-graded ring inducing S* [15,16,17], that is, a ring R whose additive group can be written as a direct sum of a family of additive subgroups of R indexed by a partial groupoid S, and such that $R_sR_t\subseteq R_{st}$ whenever st is defined, and $R_sR_t\neq 0$ implies that st is defined. The set $H_R=\bigcup_{s\in S}R_s$ is called the *homogeneous part of* R and elements of H_R are called *homogeneous elements of* R. The unique $s\in S$ for which $0\neq x\in R_s$ is called the *degree of* x, and is denoted by $\deg(x)$. We put $0_S=\deg(0)$, and we may without loss of generality assume that $0_S\in S$ since the zero element of R can be viewed as a component of R. Throughout the article, we make S a groupoid by putting $st=0_S$ for those pairs $(s,t)\in S\times S$ for which the product st was not originally defined (in which case $R_sR_t=0$). We

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denote $S^* = S \setminus \{0_S\}$. Clearly, $R = \bigoplus_{s\in S} R_s = \bigoplus_{s\in S^*} R_s$.

In [2,6,7,21] one can find a notion of a graded ring equivalent to the notion of an S-graded ring inducing S. Graded rings there are studied from the so-called homogeneous point of view by observing the homogeneous part of a graded ring with induced partial addition and everywhere defined multiplication. This approach goes back to [20].

Throughout the paper, if R is an S-graded ring inducing S, we assume that S is cancellative. If S is cancellative, then, by [6], a homogeneous element x from R_s is *graded* (*right*) *quasi-regular* if and only if either s is not a nonzero idempotent element of S, or if s is a nonzero idempotent element of S, then x is a classical (right) quasi-regular element of the ring R_s. The *graded Jacobson radical* $J^g(R)$ [6] is the largest graded quasi-regular right ideal of R, that is, the largest homogeneous right ideal all of whose homogeneous elements are graded right quasi-regular elements (in that case, they are all graded quasi-regular). We define an S-graded ring inducing S, not necessarily with identity, to be *graded* UJ if the homogeneous part of $J^g(R)$ coincides with the set of all graded quasi-regular elements of R.

We give basic properties of graded UJ-rings and discuss their connection to rings introduced in [13], namely to *graded nil clean rings*. More precisely, we characterize graded nil clean graded UJ-rings. We also give some answers to the question of when the UJ-property of components which correspond to nonzero idempotent elements of S imply the graded UJ-property of the whole S-graded ring inducing S. Moreover, we examine the graded UJ-property of corner rings and polynomial rings.

All rings in this paper are assumed to be associative. Also, unless otherwise specified, rings are not necessarily with identity.

A subring I of an S-graded ring R inducing S is said to be *homogeneous* if $I = \bigoplus_{s\in S} R_s \cap I$.

If R is an S-graded ring inducing S and R' is an S'-graded ring inducing S', then a ring homomorphism $f : R \to R'$ is called *homogeneous* [2,7,21] if $f(H_R) \subseteq H_{R'}$, and if $f(x)$ being a nonzero homogeneous element of R' implies that x is a homogeneous element of R. If f is moreover bijective, R is said to be *graded isomorphic* to R'.

If I is a homogeneous ideal (two-sided) of R, and $I_s = R_s \cap I$, then $R/I = \bigoplus_{s\in S} R_s/I_s$ is an S-graded ring inducing S [2,7,16,21] (see also [14]).

2. Graded UJ-rings

Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S. By $Q^g(R)$ we denote the set of all graded quasi-regular elements of R. Also, let us denote the set of all idempotent elements of S by $E(S)$. As usual, $E(S)^*$ stands for $E(S) \setminus \{0_S\}$. We say that R is trivially graded if $R = R_e$, where $e \in E(S)^*$, and $R_s = 0$ for every $e \neq s \in S$.

Definition 2.1. The ring R is said to be a *graded UJ-ring* if $Q^g(R) = J^g(R) \cap H_R$.

As it is stated in [19], if R is a ring not necessarily with identity, we may define R to be a *UJ-ring* if $Q(R) = J(R)$. In this paper, by a UJ-ring we mean exactly this, and it is in accordance with the notion of a UJ-ring in case of rings with identity by Lemma 2.1 in [19]. Therefore, we see that every UJ-ring can be viewed as a graded UJ-ring with the trivial grading.

The following theorem will be frequently used throughout the article.

Theorem 2.2 ([6,7])**.** *Let R be an S-graded ring inducing S. If $e \in E(S)$, then $J(R_e) = J^g(R) \cap R_e$.*

Example 2.3. Let A be a Jacobson radical ring, and let $R = M_2(A)$ be the ring of 2×2 matrices with coefficients in A. Then R is an S-graded ring inducing S with components $R_{(1,1)} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, $R_{(1,2)} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$, $R_{(2,1)} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$, $R_{(2,2)} = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$, where $S^* = \{(i,j) \mid i,j = 1,2\}$, and $S = S^* \cup \{0_S\}$ is cancellative (see [17]). It is known that $J(R) = M_2(J(A))$, and hence, $J(R) = R$. Since, according to [6], the largest homogeneous ideal contained in $J(R)$ is contained in $J^g(R)$, it follows that $J^g(R) = J(R) = R$. Hence, $H_R = J^g(R) \cap H_R$ is the set of graded quasi-regular elements of R. So R is a graded UJ-ring.

Proposition 2.4. *Let R be an S-graded ring inducing S. Then the following hold:*

i) Every graded Jacobson radical S-graded ring inducing S, that is, every S-graded ring inducing S which coincides with its graded Jacobson radical, is graded UJ.

ii) If R is a graded UJ-ring, then R_e is a UJ-ring for every $e \in E(S)^*$.

iii) R is a graded UJ-ring if and only if $R/J^g(R)$ has no nonzero graded quasi-regular elements.

iv) Let I be a homogeneous ideal of R contained in $J^g(R)$. Then R is a graded UJ-ring if and only if R/I is a graded UJ-ring.

Proof. *i*) If $J^g(R) = R$, then it follows that $J^g(R) \cap H_R = Q^g(R)$ like in the previous example.

ii) Let $e \in E(S)^*$. By Theorem 2.2 we have that $J(R_e) = J^g(R) \cap R_e$. However, R is graded UJ, that is, $J^g(R) \cap H_R = Q^g(R)$. We therefore have that $J(R_e) = J^g(R) \cap R_e = (J^g(R) \cap H_R) \cap R_e = Q^g(R) \cap R_e = Q(R_e)$. Hence R_e is a UJ-ring.

iii) Let R be a graded UJ-ring and let $x + J^g(R)$ be a graded quasi-regular element of $R/J^g(R)$. If the degree of $x + J^g(R)$ is a nonzero idempotent e, then there exists $y + J^g(R) \in (R/J^g(R))_e$ such that $x + y - xy \in J(R_e)$. Since according to *ii*), $J(R_e) = Q(R_e)$, we have that $x + y - xy \in Q(R_e)$. Hence there exists $z \in R_e$ such that $x+y-xy+z-xz-yz+xyz = 0$, that is, $x+(y+z-yz)-x(y+z-yz) = 0$. Therefore, x is a right quasi-regular element of R_e. Symmetrically we prove that x is also a left quasi-regular element of R_e. Therefore, by *ii*), we have that $x \in Q(R_e) = J(R_e)$, which implies that $x+J^g(R) = 0+J^g(R)$. Now, let $x+J^g(R) \in (R/J^g(R))_s$, where s is not a nonzero idempotent element of S. It follows that $x \in R_s$ is a graded quasi-regular element of R. Since R is graded UJ, it follows that $x \in J^g(R)$. Hence, again $x + J^g(R) = 0 + J^g(R)$.

Conversely, let us assume that $R/J^g(R)$ has no nonzero graded quasi-regular elements. Let $x \in Q(R_e)$, where $e \in E(S)^*$. Then $x + J^g(R) = x + J(R_e)$ is a quasi-regular element of $R_e/J(R_e)$. By assumption, $x \in J(R_e)$. Hence, for every $e \in E(S)^*$, we have that R_e is a UJ-ring. Now, let $x \in R_s$, where $s \notin E(S)^*$. Then x is a graded quasi-regular element of R. Therefore, $x + J^g(R)$ is a graded quasi-regular element of $R/J^g(R)$. By assumption, $x \in J^g(R)$. Hence, all of the graded quasi-regular elements of R are contained in $J^g(R)$, that is, R is a graded UJ-ring.

iv) As in the classical case, since $I \subseteq J^g(R)$, it can be easily verified that $(R/I)/J^g(R/I)$ and $R/J^g(R)$ are isomorphic as graded rings. The claim now follows by *iii*). □

Remark 2.5. Let us notice that Proposition 2.4 *iv*) in the case of a trivially graded ring gives a generalization of Proposition 2.3(5) in [19] for rings which are not necessarily with identity.

Next we want to characterize graded UJ-rings which are graded nil clean. Let us recall, if $R = \bigoplus_{s\in S} R_s$ is an S-graded ring inducing S, then it is said to be *graded nil clean* [13] if every homogeneous element of R can be written as a sum of a homogeneous idempotent and a homogeneous nilpotent element. Therefore, if R is graded nil clean, it follows that every R_e, where $e \in E(S)$, is a nil clean ring. According to Proposition 2.4 *i*), if S is cancellative, then every graded-nil

S-graded ring inducing S, that is, a graded ring all of whose homogeneous elements are nilpotent, can serve as an example of a graded nil clean graded UJ-ring.

We give some preparatory results which are interesting in their own right.

In order to generalize the notion of a *clean ring* (see [22]) to rings which are not necessarily with identity, the notion of a *clean general ring* is introduced (see [5,24]) as a ring in which every element can be written as a sum of an idempotent and a quasi-regular element. The following definition represents a graded analogue of that notion.

Definition 2.6. Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S. A homogeneous element of R is said to be *graded clean general* if it can be written as a sum of a homogeneous idempotent and a graded quasi-regular element. The ring R is said to be *graded clean general* if every of its homogeneous elements is graded clean general.

According to [3], a ring R is called *J-clean* if every element can be written as a sum of an idempotent and an element from $J(R)$.

Definition 2.7. Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S. A homogeneous element of R is said to be *graded J-clean* if it can be written as a sum of a homogeneous idempotent element from R and a homogeneous element from $J^g(R)$. The ring R is said to be *graded J-clean* if every of its homogeneous elements is graded J-clean.

Remark 2.8. Since we assume that S is cancellative, if R is a graded J-clean ring, then, for every $e \in E(S)$ we have that R_e is a J-clean ring. Namely, by Theorem 2.2, if $e \in E(S)$, then $J^g(R) \cap R_e = J(R_e)$. Similarly, if S is cancellative, and R a graded clean general ring, for every $e \in E(S)$ we have that R_e is a clean general ring. In fact, if S is cancellative, graded clean general rings are S-graded rings inducing S whose components, which correspond to $s \in E(S)$, are clean general rings.

In [19] it is proved that a ring with identity is UJ if and only if every clean element is J-clean. The following proposition generalizes this fact.

Proposition 2.9. *Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S. Then R is graded UJ if and only if every graded clean general element of R is graded J-clean.*

Proof. Let us assume that R is a graded UJ-ring and let r be a graded clean general element of R. Then r is a homogeneous element of R. Let $\deg(r) = s$. If $s \notin E(S)^*$, then, since r is graded clean general, we have that $r \in Q^g(R)$, according to definition. Since R is by assumption graded UJ, we have that $r \in J^g(R)$. Hence, r is graded J-clean. Now, let $s = e \in E(S)^*$. Then r is a clean general element in the ring R_e. Hence, $r = f + q$, for some idempotent $f \in R_e$ and $q \in Q(R_e) = J(R_e) = J^g(R) \cap R_e$.

Conversely, let us assume that every graded clean general element is graded J-clean. Let $r \in Q^g(R)$. Then $r \in R_s$ for some $s \in S$. Assume first that $s \notin E(S)^*$. Then $r \in J^g(R) \cap R_s$ by assumption. Hence, for every $s \in S$ which is not a nonzero idempotent element, we have that $Q^g(R) \cap R_s \subseteq J^g(R) \cap R_s$. Now, let $s = e \in E(S)^*$. Then, since S is cancellative, $r \in Q(R_e)$. If $J(R_e) = R_e$, there is nothing to prove. So, let $J(R_e) \neq R_e$. Since r is a graded clean general element, it is by assumption graded J-clean. Hence, $r = f + j$ for some $f^2 = f \in R_e$ and $j \in J^g(R) \cap R_e = J(R_e)$. Let $r' \in R_e$ be such that $r + r' - rr' = 0$. Then $f + j + r' - (f + j)r' = 0$. It follows that $f + r' - fr' \in J(R_e)$. This implies that $f + J(R_e)$ is a right quasi-regular element in $R_e/J(R_e)$. Similarly, it is also left quasi-regular, hence quasi-regular. On the other hand, $f + J(R_e)$ is an idempotent element in $R_e/J(R_e)$. Hence $f \in J(R_e)$. It follows that $r = f + j \in J(R_e)$. So, again, $r \in J^g(R)$. Therefore, $Q^g(R) = J^g(R) \cap H_R$, that is, R is a graded UJ-ring. □

We are now ready to give a generalization of Theorem 4.2 in [19]. Let us recall, if I is a one-sided ideal of a ring R, then it is said that idempotents *lift strongly* modulo I if $x^2 - x \in I$ implies $e - x \in I$ for some idempotent element $e \in xR$ (equivalently $e \in Rx$) (see [23]).

Theorem 2.10. *Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S. The following statements are equivalent:*

- *i)* *R is a graded clean general graded UJ-ring;*
- *ii)* *$R/J^g(R) = \bigoplus_{e\in E(S)} R_e/J(R_e)$. Moreover, for every $e \in E(S)$, the ring $R_e/J(R_e)$ is Boolean and homogeneous idempotents of $R/J^g(R)$ lift strongly modulo $J^g(R)$;*
- *iii)* *R is a graded J-clean graded UJ-ring;*
- *iv)* *R is a graded J-clean ring.*

Proof. *i)* $\Rightarrow$ *ii)* Since R is graded clean general, we have that $R/J^g(R)$ is graded clean general too. Also, since R is graded UJ, we have that $Q^g(R/J^g(R)) = 0$ according to Proposition 2.4 *iii)*. Now, let $r + J^g(R) \in (R/J^g(R))_s$. If $s \notin E(S)^*$, then $r \in Q^g(R)$, and therefore, $r + J^g(R) = 0 + J^g(R)$. If $s = e \in E(S)^*$, then $r \in R_e$, and R_e is a clean general UJ-ring by Remark 2.8. Since R_e is clean general, $r = f + q$ for some $f^2 = f \in R_e$ and $q \in Q(R_e)$. On the other hand, since

R_e is UJ, we have that $Q(R_e) = J(R_e)$. Therefore $r + J^g(R) = f + J^g(R)$, that is $R_e/J(R_e)$ is a Boolean ring. Moreover, we have proved that the homogeneous part of $R/J^g(R)$ equals $\bigcup_{e\in E(S)} R_e/J(R_e)$. Therefore, homogeneous idempotents of $R/J^g(R)$ lift strongly modulo $J^g(R)$ according to Corollary 4 in [24] applied to R_e, where $e \in E(S)$.

ii) $\Rightarrow$ *iii)* By the proof of Theorem 4.2 in [19], we have that R_e is a J-clean ring for every $e \in E(S)^*$. Since $R_e/J(R_e)$ is a Boolean ring for every $e \in E(S)^*$, it follows that $Q^g(R/J^g(R)) \cap R_e/J(R_e) = 0$ for every $e \in E(S)^*$. Let $r \in R_s$, where $s \notin E(S)^*$. Since the homogeneous part of $R/J^g(R)$ coincides with $\bigcup_{e\in E(S)} R_e/J(R_e)$, we have that $r \in J^g(R)$. Hence, R is graded J-clean. Moreover, $Q^g(R/J^g(R)) \cap (R/J^g(R))_s = 0$ for every $s \notin E(S)^*$. Hence R is graded UJ by Proposition 2.4 *iii)*.

iii) $\Rightarrow$ *iv)* This implication is clear.

iv) $\Rightarrow$ *i)* This follows by Proposition 2.9, since obviously, every graded J-clean element is graded clean general. □

Remark 2.11. According to Theorem 2.10, a graded ring from Example 2.3 is both graded clean general and graded J-clean.

Proposition 2.12. *Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S. If R is graded nil clean, then $J^g(R)$ is a graded-nil ideal of R.*

Proof. Let $x \in J^g(R) \cap R_s$, where $s \notin E(S)$. Then, as R is graded nil clean, we have that x is a nilpotent element. Now, let $x \in J^g(R) \cap R_e$, where $e \in E(S)$. According to Theorem 2.2, $x \in J(R_e)$. On the other hand, since R is graded nil clean, we have that R_e is nil clean. By Proposition 3.16 in [5], it follows that $J(R_e)$ is nil, hence x is nilpotent. □

Now, Theorem 2.10 and Proposition 2.12 together with Lemma 3.28 in [13] imply the next result which characterizes graded nil clean rings which are also graded UJ-rings. Let us just mention that Lemma 3.28 in [13] assumes the existence of identity in a ring in order to use the fact that idempotents lift modulo nil ideals of a ring with identity. However, according to [18], the same holds true for rings without an identity. Moreover, let us mention that Lemma 3.28 in [13] says that R is graded nil clean if and only if R/I is graded nil clean provided that I is a graded-nil ideal of R and that the grading set is cancellative.

Theorem 2.13. *Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S. Then the following statements are equivalent:*

i) *R is a graded clean general graded UJ-ring with the graded-nil graded Jacobson radical $J^g(R)$;*

ii) *$R/J^g(R) = \bigoplus_{e\in E(S)} R_e/J(R_e)$. Moreover, for every $e \in E(S)$, the ring $R_e/J(R_e)$ is Boolean and $J^g(R)$ is graded-nil;*

iii) *R is a graded nil clean graded UJ-ring.*

Remark 2.14. Let us notice that in case R is trivially graded, and with identity, the previous theorem gives the equivalence of statements (1)-(3) of Theorem 4.3 in [19].

Having in mind Proposition 2.4 *ii)*, it is natural to ask what can be said of the following implication:

$$(\forall e \in E(S)^*)\ R_e \text{ is a } UJ\text{-ring} \Rightarrow R = \bigoplus_{s\in S} R_s \text{ is a graded } UJ\text{-ring.} \tag{1}$$

Implication (1) does not hold in general, as the following example shows.

Example 2.15. Let $R = M_2(\mathbb{F}_2)$ be the ring of 2×2 matrices over the field $\mathbb{F}_2$ with two elements. Then R is an S-graded ring inducing S with the same grading as in Example 2.3. The generalized matrix ring is a graded ring inducing a partial operation on the indexing set, and, according to [17], its Jacobson radical is homogeneous. Hence, in particular, the Jacobson radical of R is homogeneous. Since R is Artinian, we have that $J^g(R) = J(R)$ according to [6]. Moreover, $J(R) = 0$, and so, $J^g(R) \cap H_R \neq Q^g(R)$ since $R_{(i,j)} \subseteq Q^g(R)$ for $i \neq j$, $i, j = 1, 2$. Hence, R is not a graded UJ-ring. However, since $\mathbb{F}_2$ is a UJ-ring (see [19]), we have that the rings $R_{(i,i)}$, $i = 1, 2$, are UJ.

We conclude this section by giving some sufficient conditions under which the implication (1) holds true.

Theorem 2.16. *Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S. Assume also that R has a unique maximal homogeneous right ideal. If there exists a nonzero idempotent $e \in S$ such that $st = e \Rightarrow s = e \vee t = e$ $(s, t \in S)$, and if R_e is a UJ-ring, then R is a graded UJ-ring.*

Proof. By Proposition 2.4 *iv)* applied to R_e, we have that $R_e/J(R_e)$ is a UJ-ring. By Theorem 2.2, the e-component of $R/J^g(R)$ is $R_e/J(R_e)$. According to the proof of Theorem 3.27 in [13] (see also Theorem 3.2 in [11]), we have that every nonzero homogeneous element from $R/J^g(R)$ can be identified with exactly one nonzero element from $R_e/J(R_e)$ and vice-versa. Hence, $R/J^g(R)$ is graded UJ. By Proposition 2.4 *iv)*, we have that R is a graded UJ-ring. $\square$

3. Graded UJ-property under graded ring constructions

In the final part of this paper, we deal with the graded UJ-property of corner rings and polynomial rings, thus obtaining graded versions of the corresponding results in [19].

Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S, and let $e_1, \dots, e_n$ be mutually orthogonal homogeneous idempotent elements of R, that is, $e_i^2 = e_i$ for all i and $e_ie_j = 0$ for all $i \neq j$. If $f = \sum_{i=1}^n e_i$, then, following [7], by $(1-f)R$ and $R(1-f)$ we denote the sets $\{x - fx \mid x \in R\}$ and $\{x - xf \mid x \in R\}$, respectively. It can be proved analogously to the case of classical rings, that fRf is a homogeneous subring of R, that the Peirce decomposition $R = fRf \oplus (1-f)Rf \oplus fR(1-f) \oplus (1-f)R(1-f)$ holds, and that $fJ^g(R)f = J^g(fRf)$ (see [7]).

Theorem 3.1. *Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S. Also, let $e_1, \dots, e_n$ be mutually orthogonal homogeneous idempotents of R and $f = \sum_{i=1}^n e_i$. Then the following conditions are equivalent:*

i) R is a graded UJ-ring;

ii) fRf and $(1-f)R(1-f)$ are graded UJ-rings and $fR(1-f)$, $(1-f)Rf \subseteq J^g(R)$.

Proof. Let R be a graded UJ-ring. By [7] we know that $fR(1-f) = fR \cap R(1-f)$ is a homogeneous subring of R. If $x \in fR(1-f)$ is a nonzero homogeneous element, then we must have that $\deg(x) \notin E(S)^*$. Hence, x is a graded quasi-regular element of R. Therefore, since R is graded UJ, it must belong to $J^g(R)$. Thus, $fR(1-f) \subseteq J^g(R)$. Similarly, $(1-f)Rf \subseteq J^g(R)$. Next we prove that fRf is a graded UJ-ring. Let $x \in Q^g(fRf)$. Suppose $\deg(x) = s$ and $s \notin E(S)^*$. Then $x \in Q^g(R)$. However, since R is a graded UJ-ring, $x \in J^g(R)$. Hence $x \in fRf \cap J^g(R)$. On the other hand, $fRf \cap J^g(R) = fJ^g(R)f = J^g(fRf)$. Hence $Q^g(fRf) \cap R_s \subseteq J^g(fRf) \cap R_s$ for every $s \notin E(S)^*$. If $s = e \in E(S)^*$, then $x \in Q(fR_ef)$ since for every homogeneous element $x \in R$, we have that fx, xf and fxf are all homogeneous and they all have the same degree as x. In particular, $fR_ef = e_iR_ee_i$, where $\deg(e_i) = e$. As a quasi-regular element of $fR_ef \subseteq R_e$, we have that $x \in J(R_e)$ according to Proposition 2.4 *ii*). Hence $x \in J(R_e) \cap e_iR_ee_i = e_iJ(R_e)e_i = J(e_iR_ee_i) = J(fR_ef)$. According to Theorem 2.2, we have that $J(fR_ef) = J^g(fRf) \cap fR_ef$. Therefore, again we have that $x \in J^g(fRf)$. Thus, $J^g(fRf) \cap H_{fRf} = Q^g(fRf)$. Similarly, $(1-f)R(1-f)$ is a graded UJ-ring.

Conversely, assume that fRf and $(1-f)R(1-f)$ are graded UJ-rings, and moreover assume that $fR(1-f)$, $(1-f)Rf \subseteq J^g(R)$. Then, using the Peirce decomposition $R = fRf \oplus (1-f)Rf \oplus fR(1-f) \oplus (1-f)R(1-f)$, we obtain that R is a graded UJ-ring. Namely, $R/J^g(R) = fRf/J^g(fRf) \oplus (1-f)R(1-f)/J^g((1-f)R(1-f))$. Since fRf and $(1-f)R(1-f)$ are graded UJ-rings, it follows that $fRf/J^g(fRf)$ and $(1-f)R(1-f)/J^g((1-f)R(1-f))$ are graded UJ-rings as well by Proposition 2.4 *iv*). Let $x + J^g(R) \in Q^g(R/J^g(R))$. Then $x + J^g(R) = (fxf + J^g(fRf)) + ((1-f)x(1-f) + J^g((1-f)R(1-f)))$. If $\deg(x) \notin E(S)^*$, then fxf and $(1-f)x(1-f)$ are graded quasi-regular elements of fRf and $(1-f)R(1-f)$, respectively. Since fRf and $(1-f)R(1-f)$ are graded UJ, according to Proposition 2.4 *iii*), we have that $fxf \in J^g(fRf)$ and $(1-f)x(1-f) \in J^g((1-f)R(1-f))$. Hence, $x + J^g(R) = 0 + J^g(R)$. If on the other hand $\deg(x) \in E(S)^*$, then we come to the same conclusion with the use of Proposition 2.4 *ii*) and also of Proposition 2.4 *iii*) applied to the case of a trivially graded ring, that is, to the case of a classical ring. Therefore, $R/J^g(R)$ has no nonzero graded quasi-regular elements. The claim now follows by Proposition 2.4 *iv*). □

According to Proposition 3.4 in [19], if R is a 2-primal UU-ring with identity, and X a set of commuting indeterminates, then the polynomial ring $R[X]$ is a UJ-ring. We finish this article by observing a similar problem for a polynomial ring over an S-graded ring inducing S. For the sake of simplicity, we will only observe the case of $|X| = 1$.

Let us recall, an associative ring R with identity 1 is called a *UU-ring* [1] if every unit of R is a unipotent, that is, a sum of 1 and a nilpotent element of R. It is known that a ring is a UU-ring if and only if every quasi-regular element is nilpotent (see [1]). We may use this characterization in order to define a UU-ring which does not necessarily have an identity, but also to define the notion of a *graded UU-ring*.

Definition 3.2. Let R be an S-graded ring inducing S. Then R is said to be *graded UU* if every graded quasi-regular element of R is nilpotent.

Proposition 3.3. *Let $R = \bigoplus_{s \in S} R_s$ be an S-graded ring inducing S. If R is graded UU, then R_e is a UU-ring for every $e \in E(S)$.*

Proof. Let $e \in E(S)^*$. It is enough to prove that every quasi-regular element of R_e is nilpotent. Since S is cancellative, $Q^g(R) \cap R_e = Q(R_e)$, hence the claim. □

Let us recall from [8] that an S-graded ring R inducing S is said to be *graded prime* if for every two homogeneous ideals A, B of R, the condition $AB = 0$ implies

$A = 0$ or $B = 0$. A homogeneous ideal A of R is said to be *graded prime* if R/A is graded prime. The *graded prime radical* of R, denoted by $P^g(R)$, is defined as the intersection of all graded prime ideals of R (see [8]). For the graded versions of other classical radicals of rings, see [9,10,11].

Definition 3.4. Let R be an S-graded ring inducing S. Then R is said to be *graded 2-primal* if the set of all homogeneous nilpotent elements of R coincides with the homogeneous part of the graded prime radical of R.

Now, let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring inducing S, x an indeterminate, $\overline{S}$ an overgroupoid of S, such that $0_S\bar{s} = \bar{s}0_S = 0_S$ for all $\bar{s} \in \overline{S}$, where $\theta : \{x\} \to Z_S(\overline{S})$, and $Z_S(\overline{S}) = \{\bar{s} \in \overline{S} \mid (\forall s \in S)\ \bar{s}s = s\bar{s}\}$. We also assume that $\theta(x)$ is not a zero divisor in $\overline{S}$. Then, by [2], the polynomial ring $R[x]$ can be made into an $\overline{S}$-graded ring inducing $\overline{S}$, which, since it depends on the choice of θ, is denoted by $R_\theta[x]$. This graded ring construction is too technical to present all of the details here. We refer the reader to [2,21] or for a more general case of semigroup rings, to [12]. Here we do not need to know the explicit construction of $R_\theta[x]$, only its properties.

Lemma 3.5. *Let R be an S-graded ring inducing S. Also, let us assume that R is with identity. If R is graded 2-primal graded UU-ring, then $R_e[x]$ is a UJ-ring for every $e \in E(S)^*$.*

Proof. It is known that for any graded ring with identity, graded by some cancellative groupoid, every ring component is with identity. Since R is with identity and S is cancellative, we in particular have that R_e is a ring with identity for every $e \in E(S)^*$ (see also [7]). Since R is graded UU and graded 2-primal, we have that $Q^g(R) = P^g(R) \cap H_R$. On the other hand, $P^g(R) \cap H_R \subseteq J^g(R) \cap H_R \subseteq Q^g(R)$. It follows that $P^g(R) = J^g(R)$. Let $e \in E(S)^*$. By Theorem 8.17 in [8], we have that $P^g(R) \cap R_e \subseteq P(R_e)$, where $P(R_e)$ denotes the classical prime radical of R_e. However, $J^g(R) \cap R_e = J(R_e)$ by Theorem 2.2. Hence $J(R_e) = P(R_e) = P^g(R) \cap R_e$. Therefore, R_e is a 2-primal ring, and by Proposition 3.3, a UU-ring. Hence, if $e \in E(S)^*$, we have that $R_e[x]$ is a UJ-ring according to Proposition 3.4 in [19]. □

Theorem 3.6. *Let R be a graded 2-primal graded UU-ring with identity. If both S and $\overline{S}$ are cancellative, then $R_\theta[x]$ is a graded UJ-ring.*

Proof. According to Theorem 3.12 in [12], we have $P^g(R_\theta[x]) = (P^g(R))_\theta[x]$. Also, let $r \in R_s$, where $s \notin E(S)$. Then, since S is cancellative, we have that r is nilpotent. This, and the fact that R is graded 2-primal, together imply that $R/P^g(R) = \bigoplus_{e\in E(S)}(R/P^g(R))_e$. More precisely, by the proof of Lemma 3.5, we have that $R/P^g(R) = \bigoplus_{e\in E(S)} R_e/P(R_e)$. Also, according to Lemma 3.5, we have that $R_e[x]$ is a UJ-ring for every $e \in E(S)^*$. However, $R_\theta[x]/P^g(R_\theta[x])$ is graded isomorphic to $(R/P^g(R))_\theta[x]$ (see [12]). Hence, $R_\theta[x]/P^g(R_\theta[x])$ is a graded UJ-ring. Therefore, since $\overline{S}$ is cancellative, $R_\theta[x]$ is a graded UJ-ring according to Proposition 2.4 *iv*). □

If R is a ring with identity, then, according to Proposition 3.5 in [19], the nilness of the Jacobson radical $J(R)$ is a necessary condition for lifting the UJ-property from R to $R[x]$. By Theorem 3.6 in [19], the problem of lifting the UJ-property from a ring with nil Jacobson radical to its polynomial ring is equivalent to Köthe's Conjecture for algebras over $\mathbb{F}_2$. In order to encourage further research, we point out that it would be interesting to observe a graded version of Proposition 3.5 in [19], and therefore of Theorem 3.6 in [19]. One of the first missing ingredients is to check whether the graded Jacobson radical has the graded Amitsur property (see [12]).

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