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NILPOTENT AND LINEAR COMBINATION OF IDEMPOTENT MATRICES

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Abstract. A ring R is Zhou nil-clean if every element in R is the sum of a nilpotent and two tripotents. Let R be a Zhou nil-clean ring. If R is of bounded index or 2-primal, we prove that every square matrix over R is the sum of a nilpotent and a linear combination of two idempotents. This provides a large class of rings over which every square matrix has such decompositions by nilpotent and linear combination of idempotent matrices.

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1. Introduction

Throughout, all rings are associative with an identity. Very recently, Zhou investigated a class of rings in which elements are the sum of a nilpotent and two tripotents that commute (see [7]). We call such ring a Zhou nil-clean ring. Many elementary properties of such rings are investigated in [4].

Decomposition of a matrix into the sum of simple matrices is of interest. In this paper, we consider a linear combination of the form

$$P = N + c_1P_1 + c_2P_2,$$

where N is a nilpotent matrix and P_1, P_2 are idempotent matrices and c_1 and c_2 are scalars. Such decomposition of matrices over Zhou nil-clean rings is thereby determined in this way. A ring R is of bounded index if there exists $m \in \mathbb{N}$ such that $x^m = 0$ for all nilpotent $x \in R$. A ring R is 2-primal if its primal radical coincides with the set of nilpotents in R [3]. For instance, every commutative (reduced) ring is 2-primal. Let R be Zhou nil-clean. If R is of bounded index or 2-primal, we prove that every square matrix over R is the sum of a nilpotent and linear combination of two idempotent matrices. This provides a large class of rings over which every square matrix has such decompositions by nilpotent and linear combination of idempotent matrices.

We use $N(R)$ to denote the set of all nilpotent elements in R. $\mathbb{N}$ stands for the set of all natural numbers.

2. Zhou nil-clean rings

Definition 2.1. A ring R is *a Zhou ring* if every element in R is the sum of two tripotents that commute.

The structure of Zhou rings was studied in [6]. We now investigate matrices over Zhou rings. We begin with

Lemma 2.2. *Every square matrix over $\mathbb{Z}_3$ is the sum of two idempotents and a nilpotent.*

Proof. See [5, Lemma 2.1]. □

Lemma 2.3. *Every square matrix over $\mathbb{Z}_5$ is the sum of a nilpotent and a linear combination of two idempotent matrices.*

Proof. As every matrix over $\mathbb{Z}_5$ is similar to a companion matrix, we may assume

$$A=\begin{pmatrix} 0 & & & & & c_0\\ 1 & 0 & & & & c_1\\ & 1 & 0 & & & c_2\\ & & & \ddots & & \vdots\\ & & & \ddots & 0 & c_{n-2}\\ & & & & 1 & c_{n-1}\end{pmatrix}.$$

Case I. $c_{n-1}=0$. Choose

$$W=\begin{pmatrix} 0 & & & & & 0\\ 1 & 0 & & & & 0\\ & 1 & 0 & & & 0\\ & & & \ddots & & \vdots\\ & & & \ddots & 0 & 0\\ & & & & 1 & 0\end{pmatrix}, E_1=\begin{pmatrix} 0 & & & & & c_0\\ 0 & 0 & & & & c_1\\ & 0 & 0 & & & c_2\\ & & & \ddots & & \vdots\\ & & & \ddots & 0 & c_{n-2}\\ & & & & 0 & 1\end{pmatrix},$$

$$E_2=\begin{pmatrix} 0 & & & & & 0\\ 0 & 0 & & & & 0\\ & 0 & 0 & & & 0\\ & & & \ddots & & \vdots\\ & & & \ddots & 0 & 0\\ & & & & 0 & 1\end{pmatrix}.$$

Then $E_1^2=E_1$ and $E_2^2=E_2$, and so $A=E_1+(-1)E_2+W$.

Case II. $c_{n-1}=1$. Choose

$$W=\begin{pmatrix} 0 & & & & & 0\\ 1 & 0 & & & & 0\\ & 1 & 0 & & & 0\\ & & & \ddots & & \vdots\\ & & & \ddots & 0 & 0\\ & & & & 1 & 0 \end{pmatrix}, E=\begin{pmatrix} 0 & & & & & c_0\\ 0 & 0 & & & & c_1\\ & 0 & 0 & & & c_2\\ & & & \ddots & & \vdots\\ & & & \ddots & 0 & c_{n-2}\\ & & & & 0 & 1 \end{pmatrix}.$$

Then $E^2=E$, and so $A=E+0+W$.

Case III. $c_{n-1}=-1$. Choose

$$W=\begin{pmatrix} 0 & & & & & 0\\ 1 & 0 & & & & 0\\ & 1 & 0 & & & 0\\ & & & \ddots & & \vdots\\ & & & \ddots & 0 & 0\\ & & & & 1 & 0 \end{pmatrix}, E=\begin{pmatrix} 0 & & & & & c_0\\ 0 & 0 & & & & c_1\\ & 0 & 0 & & & c_2\\ & & & \ddots & & \vdots\\ & & & \ddots & 0 & c_{n-2}\\ & & & & 0 & 1 \end{pmatrix}.$$

Then $E^2=E$, and so $A=(-1)E+0+W$.

Case IV. $c_{n-1}=2$. Choose

$$W=\begin{pmatrix} 0 & & & & & 0\\ 1 & 0 & & & & 0\\ & 1 & 0 & & & 0\\ & & & \ddots & & \vdots\\ & & & \ddots & 0 & 0\\ & & & & 1 & 0 \end{pmatrix}, E_1=\begin{pmatrix} 0 & & & & & c_0\\ 0 & 0 & & & & c_1\\ & 0 & 0 & & & c_2\\ & & & \ddots & & \vdots\\ & & & \ddots & 0 & c_{n-2}\\ & & & & 0 & 1 \end{pmatrix},$$

$$E_2=\begin{pmatrix} 0 & & & & & 0\\ 0 & 0 & & & & 0\\ & 0 & 0 & & & 0\\ & & & \ddots & & \vdots\\ & & & \ddots & 0 & 0\\ & & & & 0 & 1 \end{pmatrix}.$$

Then $E_1^2=E_1$ and $E_2^2=E_2$, and so $A=E_1+E_2+W$.

Case IV. $c_{n-1} = -2$. Choose

$$W = \begin{pmatrix} 0 & & & & & 0 \\ 1 & 0 & & & & 0 \\ & 1 & 0 & & & 0 \\ & & & \ddots & & \vdots \\ & & & \ddots & 0 & 0 \\ & & & & 1 & 0 \end{pmatrix}, E_1 = \begin{pmatrix} 0 & & & & & c_0 \\ 0 & 0 & & & & c_1 \\ & 0 & 0 & & & c_2 \\ & & & \ddots & & \vdots \\ & & & \ddots & 0 & c_{n-2} \\ & & & & 0 & 1 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0 & & & & & 0 \\ 0 & 0 & & & & 0 \\ & 0 & 0 & & & 0 \\ & & & \ddots & & \vdots \\ & & & \ddots & 0 & 0 \\ & & & & 0 & 1 \end{pmatrix}.$$

Then $E_1^2 = E_1$ and $E_2^2 = E_2$, and so $A = (-1)E_1 + (-1)E_2 + W$.

Therefore we complete the proof. □

Recall that a ring R is a Yaqub ring if it is the subdirect product of $\mathbb{Z}_3$'s. A ring R is a Bell ring if it is the subdirect product of $\mathbb{Z}_5$'s. We have

Lemma 2.4. *Every Zhou ring is isomorphic to a strongly nil-clean ring of bounded index, a Yaqub ring, a Bell ring or products of such rings.*

Proof. See [5, Lemma 2.3]. □

Lemma 2.5. (See [2, Lemma 6.6]) *Let R be of bounded index. If $J(R)$ is nil, then $J(M_n(R))$ is nil for all $n \in \mathbb{N}$.*

We are ready to prove the following.

Theorem 2.6. *Let R be a Zhou nil-clean ring of bounded index. Then every square matrix over R is the sum of a nilpotent and linear combination of two idempotent matrices.*

Proof. In view of Lemma 2.3, R is isomorphic to R_1, R_2, R_3 or the products of these rings, where R_1 is a strongly nil-clean ring of bounded index, R_2 is a Yaqub ring and R_3 is a Bell ring.

Step 1. Let $A \in M_n(R_1)$. In view of [2, Corollary 6.8], there exist an idempotent $E \in M_n(R_1)$ and $W \in N(M_n(R_1))$ such that $A = E + W$.

Step 2. Let $A \in M_n(R_2)$, and let S be the subring of R_2 generated by the entries of A. That is, S is formed by finite sums of monomials of the form: $a_1a_2 \cdots a_m$, where $a_1, \dots, a_m$ are entries of A. Since R_2 is a commutative ring in which $3 = 0$, S is a finite ring in which $x = x^3$ for all $x \in S$. Thus, S is isomorphic to finite direct product of $\mathbb{Z}_3$. As $A \in M_n(S)$, it follows by Lemma 2.1 that A is the sum of two idempotents and a nilpotent matrix over S.

Step 3. Let $A \in M_n(R_3)$, and let S be the subring of R_3 generated by the entries of A. Analogously, S is isomorphic to finite direct product of $\mathbb{Z}_5$. As $A \in M_n(S)$, it follows by Lemma 2.2 that A is the sum of a linear combination of two idempotents and a nilpotent matrix over S.

Let $A \in M_n(R)$. We may write $A = (A_1, A_2, A_3)$ in $M_n(R_1) \times M_n(R_2) \times M_n(R_3)$, where $A_1 \in M_n(R_1), A_2 \in M_n(R_2), A_3 \in M_n(R_3)$. According to the preceding discussion, we obtain the result. □

Example 2.7. Let $n \geq 2$ be an integer, if $n = 2^k 3^l 5^m$, then every square matrix over $R = \mathbb{Z}_n$ is a linear combination of two idempotents and a nilpotent.

Proof. It is obvious by [5, Example 3.5] that R is a Zhou nil-clean ring, also it is clear that R is of bounded index. Then the result follows from Theorem 2.5. □

3. 2-Primal rings

An element w in a ring R is called strongly nilpotent if any chain $x_1 = x, x_2, x_3, \dots$ with $x_{n+1} \in x_n R x_n$ forces $x_m = 0$ for some $m \in \mathbb{N}$. Let $P(R)$ be the primal radical of R, i.e., the intersection of all prime ideals of R. Then $P(R)$ is exactly the set of all strongly nilpotents in R [1, Remark 2.8]. We derive

Theorem 3.1. *Let R be a ring. Then the following are equivalent:*

(1) *R is 2-primal and Zhou nil-clean.*
(2) *$a - a^5 \in R$ is strongly nilpotent for all $a \in R$.*
(3) *$R/P(R)$ has the identity $x = x^5$.*
(4) *Every element in R is the sum of two tripotents and a strongly nilpotent that commute.*

Proof. (1) ⇒ (2) This is obvious, as every nilpotent in R is strongly nilpotent.

(2) ⇒ (3) Since every strongly nilpotent in R is contained in $P(R)$, we are through.

(3) ⇒ (4) Let $a \in R$. Then $\overline{a} = \overline{a}^5$; hence, $a - a^5 \in P(R)$ is nilpotent. Thus, R is Zhou nil-clean. In view of [7, Theorem 2.11], every element in R is the sum of

two tripotents u, v and a nilpotent w that commute. Write $w^n = 0 (n \in \mathbb{N})$. Then $\overline{w} = \overline{w}^{5n} \in R/P(R)$. Hence, $w \in P(R)$, i.e., w is strongly nilpotent, as desired.

(4) $\Rightarrow$ (1) As every strongly nilpotent in R is nilpotent, R is Zhou nil-clean, by [7, Theorem 2.11]. In view of [7, Theorem 2.11], $2 \times 3 \times 5 \in N(R)$. Write $2^n \times 3^n \times 5^n = 0 (n \in \mathbb{N})$. Since $(2, 3, 5) = 1$, by the Chinese Remainder Theorem, $R \cong R_1 \times R_2 \times R_3$, where $R_1 = R/2^nR, R_2 = R/3^nR$ and $R_3 = R/5^nR$. Step 1. Let $a \in N(R_1)$. Then $a = e + w$ with $e^3 = e, w \in P(R)$ and $ae = ea$. As $2 \in N(R_1)$, we see that $2 \in P(R_1)$, as it is central. Hence, $a^2 - a^4 \in P(R)$, and so $a(a - a^3) \in P(R)$. As $P(R)$ is an ideal, we see that $(a - a^3)^2 \in P(R)$. Hence, $(a^3 - a^5)^2 \in P(R)$. It follows that $(a - a^5)^2 \in P(R)$. This implies that $a^2 \in P(R)$. This implies that $e^2 \in P(R)$, and so $e = e^3 \in P(R)$. Therefore $a \in P(R)$. Thus, $N(R) \subseteq P(R)$; hence, R_1 is 2-primal.

Step 2. Let $a \in N(R_2)$. Then $a = e + w$ with $e^3 = e, w \in P(R)$ and $ae = ea$. As $3 \in N(R_1)$, we see that $3 \in P(R_1)$, as it is central. Hence, $a - a^3 \in P(R)$. Hence, $a^3 - a^5 = a^2(a - a^3) \in P(R)$. It follows that $a - a^5 = (a - a^3) + (a^3 - a^5) \in P(R)$. This implies that $a \in P(R)$, and so $N(R) \subseteq P(R)$; hence, R_2 is 2-primal.

Step 3. Let $a \in N(R_3)$. Then there exist two tripotent $e, f \in R$ and a strongly nilpotent $w \in R$ that commute such that $a = e + f + w$. As $5 \in N(R_3)$, we easily see that $5 \in P(R_3)$, as it is central. Hence, $a^5 \equiv e^5 + f^5 (mod\ P(R))$. Hence, $a^5 \equiv e + f = a$, and so $a \in P(R)$. This shows that R_3 is 2-primal.

Therefore R is 2-primal, as asserted. □

Corollary 3.2. *Let R be a ring. Then the following are equivalent:*

(1) *R is 2-primal and Zhou nil-clean.*
(2) *Every element in R is the sum of four idempotents and a strongly nilpotent that commute.*

Proof. (1) $\Rightarrow$ (2) This is obvious, by [4, Theorem 2.5]. (2) $\Rightarrow$ (1) Let $a \in R$. Then there exist idempotents $e, f, g, h \in R$ and a strongly nilpotent $w \in R$ that commute such that $2 - a = e + f + g + h + w$. Hence, $a = (1 - e) - f + (1 - g) - h - w$. Obviously, $(1 - e) - f, (1 - g) - h \in R$ are both tripotents. Therefore a is the sum of two tripotents and a strongly nilpotent that commute. According to Theorem 3.1, R is 2-primal and Zhou nil-clean. □

Theorem 3.3. *Every subring of 2-primal Zhou nil-clean rings is 2-primal Zhou nil-clean.*

Proof. Let S be a subring of a 2-primal Zhou nil-clean R. For any $a \in S$, we have $a \in R$. By virtue of Theorem 3.1, $a - a^5 \in P(R)$.

Given any chain $x_1 = a-a^5, x_2, x_3, \dots$ in S with $x_{n+1} \in x_nSx_n$, we see that this chain is a chain in R with $x_{n+1} \in x_nRx_n$. Thus, we can find some $m \in \mathbb{N}$ such that $x_m = 0$. This implies that $a - a^5 \in S$ is strongly nilpotent. Hence, $a - a^5 \in P(S)$. By using Theorem 3.1 again, S is a 2-primal Zhou nil-clean ring. □

Consequently the center of a 2-primal Zhou nil-clean ring is 2-primal Zhou nil-clean. Every corner of 2-primal Zhou nil-clean rings is 2-primal Zhou nil-clean.

Corollary 3.4. *Every finite subdirect product of 2-primal Zhou nil-clean rings is 2-primal Zhou nil-clean ring.*

Proof. Let R be the subdirect product of 2-primal Zhou nil-clean rings $R_1, \dots, R_n$. Then R is isomorphic to the subring of $R_1 \times \cdots \times R_n$. In view of Theorem 5.3, R is a 2-primal Zhou nil-clean ring. □

Example 3.5. Let R be a ring. Set $S = \{(x, y) \in R \times R \mid x - y \in J(R)\}$, which is a subring of $R \times R$. Then R is 2-primal Zhou nil-clean if and only if S is 2-primal Zhou nil-clean.

Proof. $\Rightarrow$ Clearly, S is a subring of $R \times R$. Thus, S is 2-primal Zhou nil-clean.

$\Leftarrow$ Since R is a homomorphic image of S, we easily obtain the result. □

Example 3.6. Let V be a countably-infinite-dimensional vector space over $\mathbb{Z}_5$, with $\{v_1, v_2, \dots\}$ a basis, let

$$A = \{f \in End(V) \ \mid \ rank(f) < \infty, f(v_i) \in \sum_{k=1}^{i} v_k\mathbb{Z}_5 \text{ for all } i \in \mathbb{N}\};$$

and let R be the $\mathbb{Z}_5$-algebra of $End(V)$ generated by A and the identity endomorphism. Then R is 2-primal Zhou nil-clean.

Proof. In view of [3, Example 4.2.20],

$$P(R) = \{f \in A \ \mid \ f(v_i) \in \sum_{k=1}^{i-1} v_k\mathbb{Z}_5 \text{ for all } i \in \mathbb{N}\},$$

and then $R/P(R)$ is isomorphic to the ring of all eventually-constant sequences in the direct product of $\mathbb{Z}_5' s$; hence, $R/P(R)$ has the identity $x = x^5$. Therefore $a - a^5 \in P(R)$ for all $a \in R$. By using Theorem 3.1, R is a 2-primal Zhou nil-clean ring, as asserted. □

Proposition 3.7. *Let R be a ring. Then the following are equivalent:*

(1) *R is 2-primal Zhou nil-clean.*

(2) $T_n(R)$ *is 2-primal Zhou nil-clean for some* $n \in \mathbb{N}$.

(3) $T_n(R)$ *is 2-primal Zhou nil-clean for all* $n \in \mathbb{N}$.

Proof. (1) $\Rightarrow$ (3) Let $I = \{\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in T_n(R) \mid \text{each } a_{ii} = 0\}$. Then I is a nilpotent ideal of $T_n(R)$. Since $T_n(R)/I \cong \bigoplus_{i=1}^{n} R_i$ with each $R_i = R$, the finite direct product $\bigoplus_{i=1}^{n} R_i$ is Zhou nil-clean. It is obvious that $T_n(R)$ is Zhou nil-clean. Let $x \in N(T_n(R))$. Then $\overline{x} \in N(T_n(R)/I)$. Given any chain $x_1 = x, x_2, x_3, \dots$ in $T_n(R)$ with $x_{m+1} \in x_m T_m(R) x_m$, we get a chain $\overline{x_1} = \overline{x}, \overline{x_2}, \overline{x_3}, \dots$ in $T_m(R)/I$ with $\overline{x_{m+1}} \in \overline{x_m}(T_m(R)/I)\overline{x_m}$. As $\overline{x} \in T_n(R)/I$ is strongly nilpotent, we see that $\overline{x_k} = \overline{0}$ for some $k \in \mathbb{N}$, i.e., $x_k \in I$. Since $I^n = 0$, we see that $x_{k+n} \in I^n = 0$, and so $x \in T_n(R)$ is strongly nilpotent. Hence, $T_n(R)$ is 2-primal, as asserted.

(3) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) Clearly, R is isomorphic to a subring of $T_n(R)$, thus we obtain the result by Theorem 3.3. □

Theorem 3.8. *Let R be a 2-primal Zhou nil-clean ring. Then every square matrix over R is the sum of a nilpotent and linear combination of two idempotent matrices.*

Proof. Since R is a Zhou nil-clean ring, it follows by [7, Theorem 2.11] that $J(R)$ is nil and $R/J(R)$ has the identity $x = x^5$. Hence, $R/J(R)$ is Zhou nil-clean of bounded index 5. By virtue of Theorem 2.5, every matrix in $M_n(R/J(R))$ is the sum of a nilpotent and linear combination of two idempotent matrices. Clearly, $J(R) \subseteq N(R) = P(R) \subseteq J(R)$, we have $J(R) = P(R)$. Therefore $M_n(J(R)) = M_n(P(R)) = P(M_n(R))$ is nil. It follows from $M_n\big(R/J(R)\big) \cong M_n(R)/M_n(J(R))$ that every matrix in $M_n(R)$ is the sum of a nilpotent and linear combination of two idempotent matrices. □

Corollary 3.9. *Let R be a commutative Zhou nil-clean ring. Then every square matrix over R is the sum of a nilpotent and linear combination of two idempotent matrices.*

Proof. Since every commutative ring is 2-primal, we obtain the result by Theorem 3.8. □

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