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INVARIANT SUBMANIFOLDS IN GOLDEN RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we study invariant submanifolds of a golden Riemannian manifold with the aid of induced structures on them by the golden structure of the ambient manifold. We demonstrate that any invariant submanifold in a locally decomposable golden Riemannian manifold leaves invariant the locally decomposability of the ambient manifold. We give a necessary and sufficient condition for any submanifold in a golden Riemannian manifold to be invariant. We obtain some necessary conditions for the totally geodesicity of invariant submanifolds. Moreover, we find some facts on invariant submanifolds. Finally, we present an example of an invariant submanifold.

1. Introduction

The differential geometry of submanifolds has occupied an important place in natural and engineering sciences since some particular types of submanifolds have been used as a geometric tool to solve many problems concerning these disciplines. In particular, invariant submanifolds have a key role in applied mathematics and theoretical physics as a method, such as for determining non-linear normal modes in non-linear systems [1] and constructing the reduced description for dissipative systems of reaction kinetics [2]. When considered from this point of view, invariant submanifolds have a special meaning in differential geometry. Invariant submanifolds are one of typical classes among all submanifolds of an ambient manifold. It is well known that in general, an invariant submanifold inherits almost all properties of the ambient manifold. Therefore, invariant submanifolds are an active and fruitful research field playing a significant role in the development of modern differential

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geometry. Also, the papers related to invariant submanifolds have appeared in various ambient manifolds, such as almost contact Riemannian manifolds [3, 4], normal contact metric manifolds [5], Sasakian manifolds [6], almost product Riemannian manifolds [7], CR-manifolds [8] etc.

Recently, C^∞-differentiable manifolds endowed with golden structures, i.e., golden manifolds have become a popular topic in differential geometry. In [9], M. C. Crâşmăreanu and C. E. Hreţcanu have shown that there exists a close relationship between golden and almost product structures. In this sense, F. Etayo, R. Santamaría and A. Upadhyay have analyzed almost golden Riemannian manifolds by use of the corresponding almost product structures in [10], where the concept of a golden manifold was defined as a C^∞-differentiable manifold admitting an integrable golden structure. In [11], M. Gök, S. Keleş and E. Kılıç have examined the Schouten and Vrănceanu connections on golden manifolds. The different kind of classes of submanifolds in a golden Riemannian manifold have been defined according to the behaviour of their tangent bundles with respect to the action of the golden structure of the ambient manifold and studied by several geometers in [12, 13, 14, 15, 16]. Invariant submanifolds, which are one of important and known classes of submanifolds, have been investigated in a golden Riemannian manifold for the first time by C. E. Hreţcanu and M. C. Crâşmăreanu with the help of induced structures on them by the golden structure of the ambient manifold in [17] we can find their some fundamental properties. The authors have obtained a characterization for any submanifold in a golden Riemannian manifold to be invariant and proved that the Nijenhuis tensor of the induced structure vanishes identically on invariant submanifolds in the case that the ambient manifold is a locally decomposable golden Riemannian manifold. Also, an example of an invariant submanifold regarding a product of two spheres in an Euclidean space has been given in [18].

The main purpose of this paper is to examine invariant submanifolds of a golden Riemannian manifold by means of induced structures on them by the golden structure of the ambient manifold.

The paper has three sections and is organized as follows: Section 2 is devoted to preliminaries containing basic definitions, concepts, formulas, notations and results for golden Riemannian manifolds and their submanifolds. Section 3 deals with an investigation of invariant submanifolds of a golden Riemannian manifold. We prove that any invariant submanifold of a locally decomposable golden Riemannian manifold is also locally decomposable. We obtain a characterization for any submanifold in a golden Riemannian manifold to be invariant. We find some necessary conditions for any invariant submanifold to be totally geodesic. Also, we get other results on invariant submanifolds. Lastly, we construct an induced structure on a product of hyperspheres in an Euclidean space as an example of a golden Riemannian structure.

2. Preliminaries

In this section, we recall some basic facts on golden Riemannian manifolds and their submanifolds.

A non-trivial C^∞-tensor field f of type $(1,1)$ on a C^∞-differentiable manifold $\overline{M}$ is called a polynomial structure of degree n if it satisfies the algebraic equation

$$Q\left(x\right)=x^n+a_nx^{n-1}+\cdots+a_2x+a_1I=0, \tag{1}$$

where I is the identity $(1,1)$-tensor field on $\overline{M}$ and $f^{n-1}\left(p\right), f^{n-2}\left(p\right), \ldots, f\left(p\right), I$ are linearly independent for every point $p\in\overline{M}$. Also, the monic polynomial $Q\left(x\right)$ is named the structure polynomial [19].

A polynomial structure $\overline{\Phi}$ of degree 2 with the structure polynomial $Q\left(x\right)=x^2-x-1$ on a C^∞-differentiable real manifold $\overline{M}$ is called a golden structure. That is, the golden structure $\overline{\Phi}$ is a tensor field of type $(1,1)$ satisfying the algebraic equation

$$\overline{\Phi}^2=\overline{\Phi}+I. \tag{2}$$

In this case, we say that $\overline{M}$ is a golden manifold. We denote by $\Gamma\left(T\overline{M}\right)$ the Lie algebra of differentiable vector fields on $\overline{M}$. If there exists a Riemannian metric $\overline{g}$ on $\overline{M}$ endowed with a golden structure $\overline{\Phi}$ such that $\overline{g}$ and $\overline{\Phi}$ verify the relation

$$\overline{g}\left(\overline{\Phi}X,Y\right)=\overline{g}\left(X,\overline{\Phi}Y\right) \tag{3}$$

for any vector fields $X,Y\in\Gamma\left(T\overline{M}\right)$, then the pair $\left(\overline{g},\overline{\Phi}\right)$ is said to be a golden Riemannian structure and the triple $\left(\overline{M},\overline{g},\overline{\Phi}\right)$ is called a golden Riemannian manifold. The eigenvalues of the golden structure $\overline{\Phi}$ are $\phi=\frac{1+\sqrt{5}}{2}$ and $1-\phi=\frac{1-\sqrt{5}}{2}$ being the roots of the algebraic equation $x^2-x-1=0$, where the former is the golden ratio [9, 17, 18].

Let M be an n-dimensional submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M},\overline{g},\overline{\Phi}\right)$. We denote by T_pM and $T_pM^\perp$ its tangent and normal spaces at a point $p\in M$, respectively. Then the tangent space $T_p\overline{M}$ admits the decomposition

$$T_p\overline{M}=T_pM\oplus T_pM^\perp$$

for each point $p\in M$. The induced Riemannian metric on M is given by

$$g\left(X,Y\right)=\overline{g}\left(i_*X,i_*Y\right) \tag{4}$$

for any vector fields $X,Y\in\Gamma(TM)$, where i_* is the differential of the immersion $i:M\longrightarrow\overline{M}$. We consider a local orthonormal frame $\{N_1,\ldots,N_r\}$ of the normal bundle $TM^\perp$. For every tangent vector field $X\in\Gamma(TM)$, the vector fields $\overline{\Phi}\left(i_*X\right)$ and $\overline{\Phi}\left(N_\alpha\right)$ on the ambient manifold $\overline{M}$ can be decomposed into tangential and normal components as follows:

$$\overline{\Phi}\left(i_*X\right)=i_*\left(\Phi\left(X\right)\right)+\sum_{\alpha=1}^{r}u_\alpha\left(X\right)N_\alpha \tag{5}$$

and

$$\overline{\Phi}\left(N_{\alpha}\right)=\varepsilon i_{*}\left(\xi_{\alpha}\right)+\sum_{\beta=1}^{r} a_{\alpha \beta} N_{\beta}, \varepsilon=\pm 1, \tag{6}$$

respectively, where Φ is a tensor field of type $(1,1)$ on M, ξ_{α}'s are tangent vector fields on M, u_{α}'s are differential 1-forms on M and $(a_{\alpha\beta})$ is a matrix of type $r\times r$ of real functions on M for any $\alpha,\beta\in\{1,\ldots,r\}$. Thus, we obtain a structure $\left(\Phi, g, u_{\alpha}, \varepsilon\xi_{\alpha}, (a_{\alpha\beta})_{r\times r}\right)$ induced on M by the golden Riemannian structure $(\overline{g},\overline{\Phi})$. We denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections on $\overline{M}$ and M, respectively. Then the Gauss and Weingarten formulas of M in $\overline{M}$ are given, respectively, by

$$\overline{\nabla}_{i_{*} X} i_{*} Y=i_{*} \nabla_{X} Y+\sum_{\alpha=1}^{r} h_{\alpha}(X, Y) N_{\alpha} \tag{7}$$

and

$$\overline{\nabla}_{i_{*} X} N_{\alpha}=-i_{*} A_{\alpha} X+\sum_{\beta=1}^{r} l_{\alpha \beta}(X) N_{\beta} \tag{8}$$

for any vector fields $X,Y\in\Gamma(TM)$, where h_{α}'s are the second fundamental tensors corresponding to N_{α}'s, A_{α}'s are the shape operators in the direction of N_{α}'s and $l_{\alpha\beta}$'s are the 1-forms on M corresponding to the normal connection $\nabla^{\perp}$ for any $\alpha,\beta\in\{1,\ldots,r\}$. Also, the following relations hold:

$$h(X, Y)=\sum_{\alpha=1}^{r} h_{\alpha}(X, Y) N_{\alpha}, \tag{9}$$

$$h_{\alpha}(X, Y)=h_{\alpha}(Y, X), \tag{10}$$

$$h_{\alpha}(X, Y)=g\left(A_{\alpha} X, Y\right), \tag{11}$$

$$\nabla_{X}^{\perp} N_{\alpha}=\sum_{\beta=1}^{r} l_{\alpha \beta}(X) N_{\beta} \tag{12}$$

and

$$l_{\alpha \beta}=-l_{\beta \alpha} \tag{13}$$

for any vector fields $X,Y\in\Gamma(TM)$ [17].

As it is well known, the submanifold M is called totally geodesic if $h=0$. Besides, the mean curvature vector H of M is defined by

$$H=\sum_{i=1}^{n} h\left(e_{i}, e_{i}\right),$$

where $\{e_1,\ldots,e_n\}$ is orthonormal basis of the tangent space T_pM at a point $p\in M$. If the mean curvature vector H vanishes identically, then M is said to be a minimal submanifold. If $h(X,Y)=g(X,Y)H$ for any vector fields $X,Y\in\Gamma(TM)$, then M is named a totally umbilical submanifold [20].

The triple $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$ is called a locally decomposable golden Riemannian manifold if the golden structure $\overline{\Phi}$ is parallel with respect to the Levi-Civita connection $\overline{\nabla}$, i.e., the covariant derivative $\overline{\nabla}\,\overline{\Phi}$ is identically zero. Also, under the assumption that the induced structure is a golden structure, the same definition can be applied to the submanifold (M, g, Φ) in terms of the Levi-Civita connection ∇ of M.

3. Invariant Submanifolds of Golden Riemannian Manifolds

This section is mainly concerned with invariant submanifolds in golden Riemannian manifolds. We show that any invariant submanifold in a locally decomposable golden Riemannian manifold preserves the locally decomposability of the ambient manifold. We get an equivalent expression to the invariance of any submanifold in a golden Riemannian manifold. We give some necessary conditions for the totally geodesicity of invariant submanifolds. Besides, we obtain some results on invariant submanifolds.

As a beginning, we remember that the notion of an invariant submanifold in golden Riemannian manifolds. Any invariant submanifold M of a golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$ is submanifold such that the golden structure $\overline{\Phi}$ of the ambient manifold $\overline{M}$ carries each tangent vector of the submanifold M into its corresponding tangent space in the ambient manifold $\overline{M}$, in other words,

$$\overline{\Phi}\left(T_pM\right) \subseteq T_pM$$

for any point $p \in M$.

Let M be an n-dimensional invariant submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. Then we have $\xi_\alpha = 0$ and $u_\alpha = 0$ for any $\alpha \in \{1, \ldots, r\}$. Hence, (5) and (6) can be expressed in the following forms:

$$\overline{\Phi}\left(i_*X\right) = i_*\left(\Phi\left(X\right)\right) \tag{14}$$

and

$$\overline{\Phi}\left(N_\alpha\right) = \sum_{\beta=1}^{r} a_{\alpha\beta} N_\beta, \tag{15}$$

respectively.

Theorem 1. [18, Remark 3.1] *Let M be an n-dimensional invariant submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. Then the induced structure $\left(\Phi, g, u_\alpha = 0, \varepsilon\xi_\alpha = 0, \left(a_{\alpha\beta}\right)_{r\times r}\right)$ on M by the golden Riemannian structure $\left(\overline{g}, \overline{\Phi}\right)$ satisfies the following relations:*

$$\Phi^2\left(X\right) = \Phi\left(X\right) + X, \tag{16}$$

$$a_{\alpha\beta} = a_{\beta\alpha}, \tag{17}$$

$$\sum_{\gamma=1}^{r} a_{\alpha\gamma} a_{\beta\gamma} = \delta_{\alpha\beta} + a_{\alpha\beta}, \tag{18}$$

$$g\left(\Phi\left(X\right),Y\right)=g\left(X,\Phi\left(Y\right)\right) \tag{19}$$

and

$$g\left(\Phi\left(X\right),\Phi\left(Y\right)\right)=g\left(\Phi\left(X\right),Y\right)+g\left(X,Y\right) \tag{20}$$

for any vector fields $X,Y\in\Gamma(TM)$.

Theorem 2. [18, Theorem 3.2] *Let* M *be an* n*-dimensional submanifold of codimension* r*, isometrically immersed in an* m*-dimensional golden Riemannian manifold* $\left(\overline{M},\overline{g},\overline{\Phi}\right)$*. Then* M *is an invariant submanifold if and only if the induced structure* (Φ,g) *on* M *is a golden Riemannian structure whenever* Φ *is non-trivial.*

Theorem 3. [17, Theorem 2.1] *Let* M *be an* n*-dimensional invariant submanifold of codimension* r*, isometrically immersed in an* m*-dimensional locally decomposable golden Riemannian manifold* $\left(\overline{M},\overline{g},\overline{\Phi}\right)$*. Then the induced structure* $\left(\Phi,g,u_\alpha=0,\varepsilon\xi_\alpha=0,\left(a_{\alpha\beta}\right)_{r\times r}\right)$ *on* M *by the golden Riemannian structure* $\left(\overline{g},\overline{\Phi}\right)$ *verifies the following relations:*

$$\left(\nabla_X\Phi\right)Y=0, \tag{21}$$

$$h_\alpha\left(X,\Phi Y\right)=\sum_{\beta=1}^{r}h_\beta\left(X,Y\right)a_{\beta\alpha}, \tag{22}$$

$$\Phi\left(A_\alpha X\right)=\sum_{\beta=1}^{r}a_{\alpha\beta}A_\beta X \tag{23}$$

and

$$X\left(a_{\alpha\beta}\right)=\sum_{\gamma=1}^{r}l_{\alpha\gamma}\left(X\right)a_{\gamma\beta}+\sum_{\gamma=1}^{r}l_{\beta\gamma}\left(X\right)a_{\alpha\gamma} \tag{24}$$

for any vector fields $X,Y\in\Gamma(TM)$.

Theorem 4. *Let* M *be an* n*-dimensional invariant submanifold of codimension* r*, isometrically immersed in an* m*-dimensional locally decomposable golden Riemannian manifold* $\left(\overline{M},\overline{g},\overline{\Phi}\right)$*. Then* M *is a locally decomposable golden Riemannian manifold whenever* Φ *is non-trivial.*

Proof. Taking into consideration Theorem 2, the proof is obvious from (21). □

Theorem 5. *Let* M *be an* n*-dimensional submanifold of codimension* r*, isometrically immersed in an* m*-dimensional golden Riemannian manifold* $\left(\overline{M},\overline{g},\overline{\Phi}\right)$*. Then* M *is an invariant submanifold if and only if there exists a local orthonormal frame of the normal bundle* $TM^\perp$ *such that it consists of eigenvectors of the golden structure* $\overline{\Phi}$*.*

Proof. At first, we recall that it is possible to transform the local orthonormal frame $\{N_1, \ldots, N_r\}$ of the normal bundle $TM^{\perp}$ into another local orthonormal frame $\{N'_1, \ldots, N'_r\}$ such that $\xi'_\alpha = \sum_{\gamma=1}^{r} k_\alpha^\gamma \xi_\gamma$ and $a'_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}$, where (k_α^γ) is an orthogonal matrix of type $r \times r$ and λ_a's are the eigenvalues of the matrix $(a_{\alpha\beta})_{r\times r}$ for any $\alpha, \beta \in \{1, \ldots, r\}$. If M is an invariant submanifold, then the tangent vector fields ξ'_α's are zero. Hence, we obtain from (15) that

$$\overline{\Phi}\left(N'_\alpha\right) = \lambda_\alpha N'_\alpha,\ \alpha = 1, \ldots, r,$$

which shows that the normal vector fields N'_α's are eigenvectors of the golden structure $\overline{\Phi}$. Conversely, we assume that $\overline{\Phi}\left(N'_\alpha\right) = \lambda_\alpha N'_\alpha$ for any $\alpha \in \{1, \ldots, r\}$. Then it follows from (15) that

$$\xi'_\alpha = 0,\ \alpha = 1, \ldots, r,$$

from which we conclude that the submanifold M is invariant. □

Theorem 6. *Let M be an n-dimensional submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. Then M is a totally geodesic invariant submanifold if the following relations are satisfied:*

$$\overline{\Phi} i_* = \phi i_* \tag{25}$$

and

$$\overline{\Phi} N_\alpha = (1-\phi) N_\alpha,\ \alpha = 1, \ldots, r. \tag{26}$$

Proof. Using (25) and (26) in (5) and (6), respectively, we get

$$\Phi = \phi I \tag{27}$$

and

$$a_{\alpha\beta} = (1-\phi)\,\delta_{\alpha\beta} \tag{28}$$

for any $\alpha, \beta \in \{1, \ldots, r\}$. On the other hand, (25) and (26) mean that M is an invariant submanifold. Thus, in virtue of (27) and (28), it results by a simple computation from (23) that

$$\sqrt{5} A_\alpha = 0,\ \alpha = 1, \ldots, r,$$

which proves that the submanifold M is totally geodesic. As a result, the proof has been completed. □

Theorem 7. *Let M be an n-dimensional submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. Then M is a totally geodesic invariant submanifold if the following relations are verified:*

$$\overline{\Phi} i_* = (1-\phi)\, i_* \tag{29}$$

and

$$\overline{\Phi} N_\alpha = \phi N_\alpha,\ \alpha = 1, \ldots, r. \tag{30}$$

Proof. Applying (29) and (30) to (5) and (6), respectively, we deduce

$$\Phi = (1-\phi)\, I \tag{31}$$

and

$$a_{\alpha\beta} = \phi\delta_{\alpha\beta} \tag{32}$$

for any $\alpha, \beta \in \{1, \ldots, r\}$. On the other hand, it is clear from (29) and (30) that M is an invariant submanifold. Hence, taking into account (31) and (32), we obtain by a straightforward computation from (23) that

$$-\sqrt{5}A_\alpha = 0,\ \alpha = 1, \ldots, r,$$

which implies that the submanifold M is totally geodesic. Consequently, the proof has been shown. □

Theorem 8. *Let M be an n-dimensional submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. Then the second fundamental tensors h_θ's are zero for any $\theta \in \{1, \ldots, t < r\}$ if the following relations are satisfied:*

$$\overline{\Phi} i_* = \phi i_*, \tag{33}$$

$$\overline{\Phi} N_\theta = (1-\phi)\, N_\theta,\ \theta = 1, \ldots, t \tag{34}$$

and

$$\overline{\Phi} N_\mu = \phi N_\mu,\ \mu = t+1, \ldots, r. \tag{35}$$

Proof. Taking account of (33), (34) and (35), in view of (5) and (6), we obtain

$$\Phi = \phi I, \tag{36}$$

$$a_{\theta\vartheta} = (1-\phi)\, \delta_{\theta\vartheta},\ \theta, \vartheta = 1, \ldots, t \tag{37}$$

and

$$a_{\mu\nu} = \phi\delta_{\mu\nu},\ \mu, \nu = t+1, \ldots, r. \tag{38}$$

On the other hand, it follows from (33), (34) and (35) that the submanifold M is invariant. Hence, by means of (36), (37) and (38), (23) takes the form

$$\sqrt{5}A_\theta = 0,\ \theta = 1, \ldots, t < r,$$

from which we have

$$h_\theta = 0,\ \theta = 1, \ldots, t < r.$$

□

Theorem 9. *Let M be an n-dimensional submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. Then the second fundamental tensors h_θ's are zero for any $\theta \in \{1, \ldots, t < r\}$ if the following relations are verified:*

$$\overline{\Phi} i_* = (1-\phi)\, i_*, \tag{39}$$

$$\overline{\Phi} N_\theta = \phi N_\theta,\ \theta = 1, \ldots, t, \tag{40}$$

and

$$\overline{\Phi}N_{\mu} = (1-\phi)\,N_{\mu},\ \mu = t+1,\ldots,r. \tag{41}$$

Proof. By reason of (39), (40) and (41), we infer from (5) and (6) that

$$\Phi = (1-\phi)\,I, \tag{42}$$

$$a_{\theta\vartheta} = \phi\delta_{\theta\vartheta},\ \theta,\vartheta = 1,\ldots,t \tag{43}$$

and

$$a_{\mu\nu} = (1-\phi)\,\delta_{\mu\nu},\ \mu,\nu = t+1,\ldots,r. \tag{44}$$

On the other hand, it is obvious from (39), (40) and (41) that the submanifold M is invariant. Thus, using (42), (43) and (44), (23) is reduced to

$$-\sqrt{5}A_{\theta} = 0,\ \theta = 1,\ldots,t < r,$$

which implies

$$h_{\theta} = 0,\ \theta = 1,\ldots,t < r.$$

□

Theorem 10. *Let M be an n-dimensional totally umbilical invariant submanifold of codimension r, isometrically immersed in an m-dimensional golden Riemannian manifold $\left(\overline{M},\overline{g},\overline{\Phi}\right)$. If*

$$\{tr(\Phi)\}^2 \neq n\,\{n + tr(\Phi)\}\,,$$

or equivalently

$$\{tr(\Phi)\}^2 \neq \lambda^2 n^2,$$

then M is a totally geodesic submanifold, where λ is one of the eigenvalues of the golden structure $\overline{\Phi}$.

Proof. We denote by $\{e_1,\ldots,e_n\}$ an orthonormal basis of the tangent space T_pM at a point $p \in M$. Since the submanifold M is totally umbilical, there are constants σ_{α}'s such that $h_{\alpha} = \sigma_{\alpha}g$ for any $\alpha \in \{1,\ldots,r\}$. Then (22) is given by

$$\sigma_{\alpha}g\,(X,\Phi Y) = \sum_{\beta=1}^{r} a_{\beta\alpha}\sigma_{\beta}g\,(X,Y) \tag{45}$$

for any vector fields $X,Y \in \Gamma\,(TM)$. Putting $X_p = Y_p = e_i$ for any $i \in \{1,\ldots,n\}$ at the point $p \in M$ in (45), we have

$$\sigma_{\alpha}g\,(e_i,\Phi e_i) = g\,(e_i,e_i)\sum_{\beta=1}^{r} a_{\beta\alpha}\sigma_{\beta}. \tag{46}$$

Summing over i in (46), we get

$$\sum_{i=1}^{n}\sigma_{\alpha}g\,(e_i,\Phi e_i) = n\sum_{\beta=1}^{r} a_{\beta\alpha}\sigma_{\beta},$$

which implies

$$tr(\Phi)\sigma_\alpha = n\sum_{\beta=1}^{r} a_{\beta\alpha}\sigma_\beta. \tag{47}$$

Multiplying (47) by the matrix element $a_{\beta\alpha}$ and then summing over α, we obtain

$$tr(\Phi)\sum_{\alpha=1}^{r} a_{\beta\alpha}\sigma_\alpha = n\sum_{\gamma=1}^{r}\sum_{\alpha=1}^{r} a_{\beta\alpha}a_{\alpha\gamma}\sigma_\gamma. \tag{48}$$

Using (18), (48) takes the form

$$tr(\Phi)\sum_{\alpha=1}^{r} a_{\beta\alpha}\sigma_\alpha = n\sigma_\beta + n\sum_{\gamma=1}^{r} a_{\beta\gamma}\sigma_\gamma,$$

from which we have

$$\sigma_\beta = \frac{1}{n}\left(tr(\Phi) - n\right)\sum_{\alpha=1}^{r} a_{\beta\alpha}\sigma_\alpha. \tag{49}$$

Hence, substituting (49) into (47), we find

$$\left\{tr(\Phi)\left(tr(\Phi) - n\right) - n^2\right\}\sum_{\beta=1}^{r} a_{\alpha\beta}\sigma_\beta = 0. \tag{50}$$

On the other hand, on account of the fact that $\{tr(\Phi)\}^2 \neq n\{n + tr(\Phi)\}$, or equivalently $\{tr(\Phi)\}^2 \neq \lambda^2 n^2$ in the hypothesis, it follows from (50) that

$$\sum_{\beta=1}^{r} a_{\alpha\beta}\sigma_\beta = 0.$$

Therefore, we infer from (49) that

$$\sigma_\beta = 0,\ \beta = 1, \dots, r,$$

which demonstrates that the submanifold M is totally geodesic. □

Now, we give an example.

Example 11. *Let $\left(E^{2(p+q)}, \langle,\rangle\right)$ be the $2(p+q)$-dimensional Euclidean space, where p and q are two positive natural numbers. Hereafter we use the following abbreviations for a point and a tangent vector in the Euclidean space $E^{2(p+q)}$, respectively:*

$$\left(x^i, y^i, z^j, w^j\right) = \left(x^1, \dots, x^p, y^1, \dots, y^p, z^1, \dots, z^q, w^1, \dots, w^q\right)$$

and

$$\left(X^i, Y^i, Z^j, W^j\right) = \left(X^1, \dots, X^p, Y^1, \dots, Y^p, Z^1, \dots, Z^q, W^1, \dots, W^q\right).$$

We consider a tensor field $\overline{\Phi}$ of type $(1,1)$ defined by

$$\overline{\Phi}\left(X^i, Y^i, Z^j, W^j\right) = \left(\phi X^i, \phi Y^i, (1-\phi) Z^j, (1-\phi) W^j\right)$$

for any tangent vector $\left(X^i, Y^i, Z^j, W^j\right) \in T_{(x^i,y^i,z^j,w^j)}E^{2(p+q)}$*, where* ϕ *and* $1-\phi$ *are the roots of the algebraic equation* $x^2 - x - 1 = 0$*, i.e.,* $\phi = \frac{1+\sqrt{5}}{2}$ *and* $1-\phi = \frac{1-\sqrt{5}}{2}$*. In this case, it is easy to show that* $\left(\langle,\rangle, \overline{\Phi}\right)$ *is a golden Riemannian structure and* $\left(E^{2(p+q)}, \langle,\rangle, \overline{\Phi}\right)$ *is a golden Riemannian manifold.*

Because of the fact that $E^{2(p+q)} = E^p \times E^p \times E^q \times E^q$*, we have the following four hyperspheres:*

$$S^{p-1}(r_1) = \left\{ \left(x^1, \dots, x^p\right) : \sum_{i=1}^{p} \left(x^i\right)^2 = r_1^2 \right\},$$

$$S^{p-1}(r_2) = \left\{ \left(y^1, \dots, y^p\right) : \sum_{i=1}^{p} \left(y^i\right)^2 = r_2^2 \right\},$$

$$S^{q-1}(r_3) = \left\{ \left(z^1, \dots, z^q\right) : \sum_{j=1}^{q} \left(z^j\right)^2 = r_3^2 \right\}$$

and

$$S^{q-1}(r_4) = \left\{ \left(w^1, \dots, w^q\right) : \sum_{j=1}^{q} \left(w^j\right)^2 = r_4^2 \right\}.$$

We construct the product manifold $S^{p-1}(r_1) \times S^{p-1}(r_2) \times S^{q-1}(r_3) \times S^{q-1}(r_4)$ *in a similar way as in* [18]*. We denote it by* M *for simplicity. Its every point has the coordinates* $\left(x^i, y^i, z^j, w^j\right)$ *satisfying the equation*

$$\sum_{i=1}^{p} \left(x^i\right)^2 + \sum_{i=1}^{p} \left(y^i\right)^2 + \sum_{j=1}^{q} \left(z^j\right)^2 + \sum_{j=1}^{q} \left(w^j\right)^2 = R^2,$$

where $R^2 = r_1^2 + r_2^2 + r_3^2 + r_4^2$*. Then* M *is a submanifold of codimension* 4 *in the Euclidean space* $E^{2(p+q)}$ *and* M *is a submanifold of codimension* 3 *in the sphere* $S^{2(p+q)-1}(R)$*. Hence, there exist successive embeddings such that*

$$M \hookrightarrow S^{2(p+q)-1}(R) \hookrightarrow E^{2(p+q)}.$$

Also, its tangent space $T_{(x^i,y^i,z^j,w^j)}M$ *at a point* $\left(x^i, y^i, z^j, w^j\right)$ *is as follows:*

$$T_{(x^i,0^i,0^j,0^j)}S^{p-1}(r_1) \oplus T_{(0^i,y^i,0^j,0^j)}S^{p-1}(r_2) \oplus T_{(0^i,0^i,z^j,0^j)}S^{q-1}(r_3) \oplus T_{(0^i,0^i,0^j,w^j)}S^{q-1}(r_4).$$

As it is seen, any tangent vector $\left(X^i, Y^i, Z^j, W^j\right) \in T_{(x^i,y^i,z^j,w^j)}E^{2(p+q)}$ *belongs to* $T_{(x^i,y^i,z^j,w^j)}M$ *for every point* $\left(x^i, y^i, z^j, w^j\right) \in M$ *if and only if*

$$\sum_{i=1}^{p} x^i X^i = \sum_{i=1}^{p} y^i Y^i = \sum_{j=1}^{q} z^j Z^j = \sum_{j=1}^{q} w^j W^j = 0.$$

In addition, since $\left(X^i, Y^i, Z^j, W^j\right)$ *is a tangent vector on the sphere* $S^{2(p+q)-1}(R)$*, we have*

$$T_{(x^i,y^i,z^j,w^j)}M \subset T_{(x^i,y^i,z^j,w^j)}S^{2(p+q)-1}(R)$$

for every point $\left(x^i, y^i, z^j, w^j\right) \in M$.

Let us consider a local orthonormal basis $\{N_1, N_2, N_3, N_4\}$ *for the normal space* $T_{(x^i,y^i,z^j,w^j)}M^{\perp}$ *at a point* $\left(x^i, y^i, z^j, w^j\right)$*. Then we can choose the normal vectors* N_1*,* N_2*,* N_3 *and* N_4 *such that*

$$N_1 = \frac{1}{R}\left(x^i, y^i, z^j, w^j\right),$$

$$N_2 = \frac{1}{R}\left(\frac{r_2}{r_1}x^i, -\frac{r_1}{r_2}y^i, \frac{r_4}{r_3}z^j, -\frac{r_3}{r_4}w^j\right),$$

$$N_3 = \frac{1}{R}\left(\frac{r_3}{r_1}x^i, -\frac{r_4}{r_2}y^i, -\frac{r_1}{r_3}z^j, \frac{r_2}{r_4}w^j\right)$$

and

$$N_4 = \frac{1}{R}\left(\frac{r_4}{r_1}x^i, \frac{r_3}{r_2}y^i, -\frac{r_2}{r_3}z^j, -\frac{r_1}{r_4}w^j\right).$$

We identify i_*X *with* X *for any tangent vector* $X \in T_{(x^i,y^i,z^j,w^j)}M$*. From (6), we have*

$$\overline{\Phi}N_\alpha = \xi_\alpha + \sum_{\beta=1}^{4} a_{\alpha\beta}N_\beta \tag{51}$$

for any $\alpha \in \{1,2,3,4\}$*. Also, we remark that*

$$a_{\alpha\beta} = \left\langle \overline{\Phi}N_\alpha, N_\beta \right\rangle$$

for any $\alpha, \beta \in \{1,2,3,4\}$*. Then by straightforward computations, we obtain the elements of the matrix* $\mathcal{A} = \left(a_{\alpha\beta}\right)_{4\times 4}$ *as follows:*

$$a_{11} = a_{22} = \frac{1}{2R^2}\left(R^2 + \sqrt{5}\left(r_1^2 + r_2^2 - r_3^2 - r_4^2\right)\right),$$

$$a_{12} = a_{21} = a_{34} = a_{43} = 0,$$

$$a_{13} = a_{31} = -a_{24} = -a_{42} = \frac{\sqrt{5}}{R^2}\left(r_1r_3 - r_2r_4\right),$$

$$a_{14} = a_{41} = a_{23} = a_{32} = \frac{\sqrt{5}}{R^2}\left(r_1r_4 + r_2r_3\right),$$

$$a_{33} = a_{44} = \frac{1}{2R^2}\left(R^2 - \sqrt{5}\left(r_1^2 + r_2^2 - r_3^2 - r_4^2\right)\right).$$

Hence, using the matrix elements $a_{\alpha\beta}$*'s given above, it follows from (51) that*

$$\xi_1 = \xi_2 = \xi_3 = \xi_4 = 0_{2(p+q)}. \tag{52}$$

In this case, we have

$$\overline{\Phi}\left(T_{(x^i,y^i,z^j,w^j)}M^{\perp}\right) \subseteq T_{(x^i,y^i,z^j,w^j)}M^{\perp}.$$

From (5), we can write the following relation:

$$\overline{\Phi}\left(X^i, Y^i, Z^j, W^j\right) = \Phi\left(X^i, Y^i, Z^j, W^j\right) + \sum_{\alpha=1}^{4} u_\alpha\left(X^i, Y^i, Z^j, W^j\right)N_\alpha. \tag{53}$$

We recall that

$$u_\alpha\left(X^i, Y^i, Z^j, W^j\right) = \varepsilon\left\langle\left(X^i, Y^i, Z^j, W^j\right), \xi_\alpha\right\rangle$$

for any $\alpha \in \{1,2,3,4\}$*, where* $\varepsilon = \pm 1$*. Then we get from (52) that*

$$u_1 = u_2 = u_3 = u_4 = 0. \tag{54}$$

Thus, we infer from (53) and (54) that

$$\overline{\Phi}\left(X^i, Y^i, Z^j, W^j\right) = \Phi\left(X^i, Y^i, Z^j, W^j\right).$$

In the circumstances, we have

$$\overline{\Phi}\left(T_{(x^i,y^i,z^j,w^j)}M\right) \subseteq T_{(x^i,y^i,z^j,w^j)}M$$

and

$$\Phi^2 = \Phi + I.$$

Consequently, we establish an induced structure $\left(\Phi, \langle,\rangle, \varepsilon\xi_\alpha = 0_{2(p+q)}, u_\alpha = 0, \mathcal{A}\right)$ *on the product of hyperspheres* M *by the golden Riemannian structure* $\left(\langle,\rangle, \overline{\Phi}\right)$ *on the Euclidean space* $E^{2(p+q)}$*. Moreover,* $(\Phi, \langle,\rangle)$ *is a golden Riemannian structure and* M *is an invariant submanifold in the Euclidean space* $E^{2(p+q)}$*.*

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