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ON STAR COLORING OF MODULAR PRODUCT OF GRAPHS

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Abstract. A star coloring of a graph G is a proper vertex coloring in which every path on four vertices in G is not bicolored. The star chromatic number $\chi_s(G)$ of G is the least number of colors needed to star color G. In this paper, we find the exact values of the star chromatic number of modular product of complete graph with complete graph $K_m \diamond K_n$, path with complete graph $P_m \diamond K_n$ and star graph with complete graph $K_{1,m} \diamond K_n$.

1. Introduction

All graphs in this paper are finite, simple, connected and undirected graph and we follow [2,3,7] for terminology and notation that are not defined here. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. Branko Grünbaum introduced the concept of star chromatic number in 1973. A star coloring [1, 5, 6] of a graph G is a proper vertex coloring in which every path on four vertices uses at least three distinct colors. The star chromatic number $\chi_s(G)$ of G is the least number of colors needed to star color G.

During the years star coloring of graphs has been studied extensively by several authors, for instance see [1, 4, 5].

Definition 1. *A trail is called a path if all its vertices are distinct. A closed trail whose origin and internal vertices are distinct is called a cycle.*

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Definition 2. *A graph G is complete if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n.*

Definition 3. *A star graph is a complete bipartite graph in which $m-1$ vertices have degree 1 and a single vertex have degree $(m-1)$. It is denoted by $K_{1,m}$.*

Definition 4. *The modular product [8] $G \diamond H$ of two graphs G and H is the graph with vertex set $V(G)\times V(H)$, in which a vertex (v, w) is adjacent to a vertex (v', w') if and only if either*

- *$v = v'$ and w is adjacent to w', or*
- *$w = w'$ and v is adjacent to v', or*
- *v is adjacent to v' and w is adjacent to w', or*
- *v is not adjacent to v' and w is not adjacent to w'.*

2. Main Results

In this section, we find the exact values of the star chromatic number of modular product of complete graph with complete graph $K_m \diamond K_n$, path with complete graph $P_m \diamond K_n$ and star graph with complete graph $K_{1,m} \diamond K_n$.

2.1. Star chromatic number of $K_m \diamond K_n$.

Theorem 1. *For any positive integers $m, n \geq 2$,*

$$\chi_s(K_m \diamond K_n) = \begin{cases} m, & \text{when } n = 2 \\ n(m-1), & \text{Otherwise.} \end{cases}$$

Proof. Let K_m be the complete graph on m vertices and K_n be the complete graph on n vertices. Let

$$V(K_m) = \{u_i : 1 \leq i \leq m\}$$

and

$$V(K_n) = \{v_j : 1 \leq j \leq n\}.$$

By the definition of the modular product, the vertices of $K_m \diamond K_n$ is denoted as follows:

$$V(K_m \diamond K_n) = \bigcup_{i=1}^{m} \{(u_i, v_j) : 1 \leq j \leq n\}.$$

Case(i): When $m \geq 2$ and $n = 2$

Let $\{c_1, c_2, \ldots, c_m\}$ be the set of m distinct colors. The vertices (u_i, v_j) where $1 \leq i \leq m$ and $1 \leq j \leq 2$ can be colored with color c_i. Thus $\chi_s(K_m \diamond K_n) = m$. Suppose $\chi_s(K_m \diamond K_n) < m$, say $m-1$. Then the vertices (u_i, v_j) where $2 \leq i \leq m, 1 \leq j \leq 2$ has to be colored with one of the existing colors $\{1, 2, \ldots, m-1\}$ which results in improper coloring and also gives bicolored paths on four vertices (since the vertices (u_i, v_1), $1 \leq i \leq m$ and the vertices (u_i, v_2), $1 \leq i \leq m$ forms bipartite graphs) and so contradicts the star coloring. Hence $\chi_s(K_m \diamond K_n) = m$.

Case(ii): When $m \geq 2$ and $n > 2$
Let $\{c_1, c_2, \ldots, c_{n(m-1)}\}$ be the set of $n(m-1)$ distinct colors. For $1 \leq i \leq 2$ and $1 \leq j \leq n$, the vertices (u_i, v_j) can be colored with color c_j, and for $3 \leq i \leq m$ and $1 \leq j \leq n$, the vertices (u_i, v_j) can be colored with color $c_{(i-2)n+j}$. Thus $\chi_s(K_m \diamond K_n) = n(m-1)$ when $m \geq 2$, $n \geq 3$.
Suppose $\chi_s(K_m \diamond K_n) < n(m-1)$, say $n(m-1)-1$. Then the vertex (u_m, v_n) has to be colored with one of the existing colors $\{1, 2, \ldots, n(m-1)-1\}$ which results in improper coloring and also gives bicolored paths on four vertices (since (u_m, v_n) is adjacent to every vertices (u_i, v_j), $1 \leq i \leq m-1, 1 \leq j \leq n-1$) and this contradicts the star coloring. Hence $\chi_s(K_m \diamond K_n) = n(m-1)$. □

2.2. Star chromatic number of $P_m \diamond K_n$.

Theorem 2. *For any positive integers $m, n > 1$,*

$$\chi_s(P_m \diamond K_n) = \begin{cases} 3, & \textit{when } m > 4,\ n = 2 \\ n, & \textit{when } m = 2, 3 \textit{ and } n > 2 \\ n+1, & \textit{when } m = 4,\ n \geq 2 \\ 2n, & \textit{Otherwise.} \end{cases}$$

Proof. Let P_m be the path graph on m vertices and K_n be the complete graph on n vertices. Let
$$V(P_m) = \{u_i : 1 \leq i \leq m\}$$
and
$$V(K_n) = \{v_j : 1 \leq j \leq n\}.$$
By the definition of the modular product, the vertices of $P_m \diamond K_n$ is denoted as follows:
$$V(P_m \diamond K_n) = \bigcup_{i=1}^{m} \{(u_i, v_j) : 1 \leq j \leq n\}.$$

Case(i): When $m > 4$ and $n = 2$
Let $\{c_1, c_2, c_3\}$ be the set of 3 distinct colors. Then the vertices (u_i, v_j) where $1 \leq i \leq \lceil \frac{m}{2} \rceil$ and $1 \leq j \leq 2$ are colored with color c_1. For $i \equiv 2 \pmod 4, 1 \leq i \leq m$ and $1 \leq j \leq 2$, the vertices (u_i, v_j) can be colored with color c_2. Similarly, the vertices (u_i, v_j) where $i \equiv 0 \pmod 4, 1 \leq i \leq m$ and $1 \leq j \leq 2$ can be colored with color c_3. It is obvious that $\chi_s(P_m \diamond K_n) = 3$ when $m > 4$ and $n = 2$.

Case(ii): When $m = 2, 3$ and $n > 2$
Let $\{c_1, c_2, \ldots, c_n\}$ be the set of n distinct colors. The vertices (u_i, v_j) where $1 \leq j \leq n$ and $i = 1, 2, 3$ can be colored with color c_j. Thus $\chi_s(P_m \diamond K_n) = n$ when $m = 2, 3$ and $n > 2$.
Suppose $\chi_s(P_m \diamond K_n) < n$, say $n-1$. Then the vertices (u_i, v_n), $1 \leq i \leq m$ has to be colored with one of the existing colors $\{1, 2, \ldots, n-1\}$ which results in improper coloring since the vertices (u_i, v_n), $1 \leq i \leq m$ is adjacent to the vertices colored with colors $1, 2, \ldots, n-1$ and so contradicts the star coloring. Hence $\chi_s(P_m \diamond K_n) = n$.

Case(iii): When $m = 4$ and $n \geq 2$
Let $\{c_1, c_2, \ldots, c_{n+1}\}$ be the set of $n+1$ distinct colors. For $1 \leq i \leq 3$ and $1 \leq j \leq n$, the vertices (u_i, v_j) can be colored with color c_j. And the vertices (u_4, v_j), $1 \leq j \leq n$, can be given the color c_{j+1}. Thus $\chi_s(P_m \diamond K_n) = n+1$ when $m = 4$ and $n \geq 2$.
Suppose $\chi_s(P_m \diamond K_n) < n+1$, say n. Then the vertices (u_4, v_j), $1 \leq j \leq n$ has to be colored with the j^{th} color which results in bicolored paths on four vertices and so contradicts the star coloring. Hence $\chi_s(P_m \diamond K_n) = n+1$.

Case(iv): When $m > 4$ and $n \geq 3$
Let $\{c_1, c_2, \ldots, c_{2n}\}$ be the set of $2n$ distinct colors. The vertices (u_i, v_j) where $i \equiv 1, 2, 3 \pmod 4, 1 \leq i \leq m$ and $1 \leq j \leq n$ can be colored with color c_j, and the vertices (u_i, v_j) where $i \equiv 0 \pmod 4, 1 \leq i \leq m$ and $1 \leq j \leq n$ can be given the color c_{n+j}. Thus $\chi_s(P_m \diamond K_n) = 2n$ when $m > 4$ and $n \geq 3$.
Suppose $\chi_s(P_m \diamond K_n) < 2n$, say $2n-1$. Then the vertices (u_i, v_n) where $i \equiv 0 \pmod 4, 1 \leq i \leq m$ has to be colored with one of the colors $\{1, 2, \ldots, 2n-1\}$ which results in bicolored paths on four vertices and so contradicts the star coloring. Hence $\chi_s(P_m \diamond K_n) = 2n$. □

2.3. Star chromatic number of $K_{1,m} \diamond K_n$.

Theorem 3. *For any positive integers $m \geq 2$ and $n \geq 3$,*

$$\chi_s(K_{1,m} \diamond K_n) = n.$$

Proof. Let $K_{1,m}$ be the star graph on $m+1$ vertices and K_n be the complete graph on n vertices. Let

$$V(K_{1,m}) = \{u_1\} \cup \{u_i : 2 \leq i \leq m+1\}$$

and

$$V(K_n) = \{v_j : 1 \leq j \leq n\}.$$

By the definition of the modular product, the vertices of $K_{1,m} \diamond K_n$ is denoted as follows:

$$V(K_{1,m} \diamond K_n) = \bigcup_{i=1}^{m+1} \{(u_i, v_j) : 1 \leq j \leq n\}.$$

Let $\{c_1, c_2, \ldots, c_n\}$ be the set of n distinct colors. The vertices (u_i, v_j) where $1 \leq i \leq m+1$ and $1 \leq j \leq n$ can be colored with the color c_j. Thus $\chi_s(K_{1,m} \diamond K_n) = n$.
Suppose $\chi_s(K_{1,m} \diamond K_n) < n$, say $n-1$. Then the vertices (u_i, v_n), $1 \leq i \leq m+1$ has to be colored with one of the existing colors $\{1, 2, \ldots, n-1\}$ which results in improper coloring (since the vertices (u_i, v_n), $2 \leq i \leq m+1$ is adjacent to every vertices (u_1, v_j), $1 \leq j \leq n-1$ which are colored $1, 2, \ldots, n-1$ and also since the vertex (u_1, v_n) is adjacent to every vertices (u_i, v_j), $2 \leq i \leq m+1, 1 \leq j \leq n-1$ which are colored $1, 2, \ldots, n-1$ and this contradicts the star coloring. Hence $\chi_s(K_{1,m} \diamond K_n) = n$, when $m \geq 1, n \geq 3$. □

References

[1] Albertson, M.O., Chappell, G.G., Kierstead, H.A., Kündgen, A., Ramamurthi, R., Coloring with no 2-colored P_4's. *The Electronic Journal of Combinatorics* 11 (2004), R26, `doi:10.37236/1779`.

[2] Bondy, J.A., Murty, U.S.R. Graph theory with applications, MacMillan, London 1976.

[3] Clark, J., Holton, D. A., A first look at graph theory, World Scientific, 1991, `doi:10.1142/1280`.

[4] Coleman, T.F., Moré, J., Estimation of sparse Hessian matrices and graph coloring problems, *Mathematical Programming*, 28(3) (1984), 243–270, `doi:10.1007/BF02612334`.

[5] Fertin, G., Raspaud, A., Reed, B., On Star coloring of graphs, *Journal of Graph theory*, 47(3) (2004), 163–182, `doi:10.1002/jgt.20029`

[6] Grünbaum, B., Acyclic colorings of planar graphs, *Israel Journal of Mathematics*, 14 (1973), 390–408, `doi:10.1007/BF02764716`

[7] Harary, F., Graph theory, Narosa Publishing Home, New Delhi, 1969.

[8] Imrich, W., Klavžar, S., Product Graphs: Structure and Recognition, Wiley, New York 2000.