BG-Volterra Integral Equations and Relationship with *BG*-Differential Equations

BG-Volterra İntegral Denklemleri ve BG-Diferansiyel Denklemlerle İlişkisi

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Abstract
In this study, the Volterra integral equations are defined in the sense of bigeometric calculus by the aid of bigeometric integral. The main aim of the study is to research the relationship between bigeometric Volterra integral equations and bigeometric differential equations.

Keywords: Bigeometric Calculus, Bigeometric Differential Equations, Bigeometric Volterra Integral Equations

Öz
Bu çalışmada, bigeometrik integral yardımıyla bigeometrik Volterra integral denklemleri tanımlanmıştır. Çalışmanın asıl amacı bigeometrik manada Volterra integral denklemleri ile bigeometrik manada diferansiyel denklemler arasındaki ilişkiyi araştırmaktır.

***Anahtar kelimeler**: Bigeometrik Hesap, Bigeometrik Diferansiyel Denklemler, Bigeometrik Volterra İntegral Denklemleri*

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1. Introduction

Grossman and Katz have built a new calculus, called non-Newtonian calculus, between years 1967-1970 as an alternative to classic calculus. They have defined infinite family of calculus consisting of classic, geometric, harmonic and quadratic calculus, then they have created bigeometric, biharmonic and biquadratic calculus in this progress. Non-Newtonian calculus provides a wide application area in science, engineering and mathematics. Such as studied on can be expressed as theory of elasticity in the economy, the viscosity of the blood, computer science including image processing and artificial intelligence, biology, differential equations, functional analysis and probability theory. Non-Newtonian calculus is researched by various researchers such as Grossman (1979); Çakmak and Başar (2012, 2014a,b, 2015); Türkmen ve Başar (2012a,b); Tekin and Başar (2013); Kadak and Özlük (2014); Duyar and Oğur (2017); Duyar and Sağır (2017); Erdoğan and Duyar (2018); Sağır and Erdoğan (2019); Güngör (2020). One of the most popular non-Newtonian calculus, namely, bigeometric calculus which is investigated especially by Volterra and Hostinsky (1938); Grossmann (1983); Rybaczuk and Stopel (2000) investigated the fractal growth in material science by using bigeometric calculus. Aniszewska and Rybaczuk (2005) used bigeometric calculus on a multiplicative Lorenz System. Córdova-Lepe (2006) studied on measure of elasticity in economics by aid of bigeometric calculus. Boruah and Hazarika (2018a,b) named Bigeometric calculus as G-calculus and investigated basic properties of derivative and integral in the sense of bigeometric calculus and also applications in numerical analysis. Boruah et al. (2018) researched solvability of bigeometric differential equations by using numerical methods.

Integral equations have used for the solution of several problems in engineering, applied mathematics and mathematical physics since 18th century. The integral equations have begun to enter the problems of engineering and other fields because of the relationship with differential equations which have wide range of applications and so their importance has increased in recent years. The reader may refer for relevant terminology on the integral equations to Smithies (1958); Krasnov et al. (1971); Zarnan (2016); Brunner (2017); Maturi (2019).

In this paper, we define Volterra integral equations in bigeometric calculus by using the concept of bigeometric integral and called BG-Volterra integral equations. We prove Leibniz formula in the sense of bigeometric calculus and demonstrate the converting the BG-Volterra integral equations to bigeometric differential equations by aid of this formula. By defining the bigeometric linear differential equations with constant coefficients and variable coefficients, we demonstrate that they are converted to BG-Volterra integral equations.

A generator is one-to-one function α whose domain is $\mathbb{R}$ the set of real numbers and whose range is a subset of $\mathbb{R}$. It is indicated by $\mathbb{R}_\alpha=\{\alpha(x):x\in\mathbb{R}\}$ the range of generator α. α-arithmetic operations are descriped as indicated, below:

$\alpha-$addition $\quad x\dot{+}y=\alpha\left[\alpha^{-1}(x)+\alpha^{-1}(y)\right]$

$\alpha-$subtraction $\quad x\dot{-}y=\alpha\left[\alpha^{-1}(x)-\alpha^{-1}(y)\right]$

$\alpha-$multiplication $\quad x\dot{\times}y=\alpha\left[\alpha^{-1}(x)\times\alpha^{-1}(y)\right]$

$\alpha-$division $\quad x\dot{/}y=\alpha\left[\alpha^{-1}(x)/\alpha^{-1}(y)\right]$

$\alpha-$order $\quad x\dot{<}y\Leftrightarrow\alpha^{-1}(x)<\alpha^{-1}(y).$

For $x,y\in\mathbb{R}_\alpha$. $(\mathbb{R}_\alpha,\dot{+},\dot{\times})$ is complete field. In particular, the identity function I generates classical arithmetic and the exponential function generates geometric arithmetic. The numbers $x\dot{>}\dot{0}$ are α-positive numbers and the numbers $x\dot{<}\dot{0}$ are α-negative numbers in $\mathbb{R}_\alpha$. α-zero and α-one numbers are denoted by $\alpha(0)=\dot{0}$ and $\alpha(1)=\dot{1}$, respectively. α-integers are obtained by successive α-addition of $\dot{1}$ to $\dot{0}$ and succesive α-subtraction of $\dot{1}$ from $\dot{0}$. Hence the α-integers are as follows:

$\ldots,\alpha(-2),\alpha(-1),\alpha(0),\alpha(1),\alpha(2),\ldots$

For each integer n, we set $\dot{n}=\alpha(n)$. If $\dot{n}$ is an α-positive integer, then it is n times sum of $\dot{1}$ (Grossman and Katz, 1972).

Definition 1. The α-absolute value of $x \in \mathbb{R}_\alpha$ determined by

$$|x|_\alpha = \begin{cases} x & , x > \dot{0} \\ \dot{0} & , x = \dot{0} \\ \dot{0} - x & , x < \dot{0} \end{cases}$$

and this value is equivalent to $\alpha\left(\left|\alpha^{-1}(x)\right|\right)$. For $x \in \mathbb{R}_\alpha$, $x^{p_\alpha} = \alpha\left\{\left[\alpha^{-1}(x)\right]^p\right\}$ and $\sqrt[p]{x}^{\alpha} = \alpha\left(\sqrt[p]{\alpha^{-1}(x)}\right)$ (Grossman and Katz, 1972).

Definition 2. An open α-interval on $\mathbb{R}_\alpha$ expressed with

$$\dot{(}r,s\dot{)} = \left\{x \in \mathbb{R}_\alpha : r \dot{<} x \dot{<} s\right\} = \left\{x \in \mathbb{R}_\alpha : \alpha^{-1}(r) < \alpha^{-1}(x) < \alpha^{-1}(s)\right\} = \alpha\left(\left(\alpha^{-1}(r), \alpha^{-1}(s)\right)\right)$$

Similarly, a closed α-interval on $\mathbb{R}_\alpha$ can be expressed (Grossman and Katz, 1972).

Definition 3. A point a is said to be an interior point of the subset $A \subset \mathbb{R}_\alpha$ if there is an open α-interval, contained entirely in the set A which contains this point :

$$a \in \dot{(}r,s\dot{)} \subset A \Leftrightarrow \alpha^{-1}(a) \in \alpha^{-1}\left(\dot{(}r,s\dot{)}\right) = \left(\alpha^{-1}(r), \alpha^{-1}(s)\right) \subset \alpha^{-1}(A).$$

According to this definition, a is an interior point of the subset $A \subset \mathbb{R}_\alpha$ iff $\alpha^{-1}(a)$ is an interior point of the subset $\alpha^{-1}(A) \subset \mathbb{R}$. If a subset $A \subset \mathbb{R}_\alpha$ whose all points are interior points, it is called α-open (Duyar and Oğur, 2017).

Definition 4. Let $\left(\mathbb{R}_\alpha, |.|_\alpha\right)$ be non-Newtonian metric space and $a \in \mathbb{R}_\alpha$. If $\left(\dot{(}a \dot{-} \varepsilon, a \dot{+} \varepsilon\dot{)} - \{a\}\right) \cap S \neq \varnothing$ for every $\varepsilon \dot{>} \dot{0}$ where $S \subset \mathbb{R}_\alpha$, then the point a is called α-accumulation point of the set S. The set of all α-accumulation points of S is indicated by $S^{'\alpha}$.(Sağır and Erdoğan, 2019).

Definition 5. Let (x_n) be sequence and x be a point in the non-Newtonian metric space $\left(\mathbb{R}_\alpha, |.|_\alpha\right)$. If for every $\varepsilon \dot{>} \dot{0}$, there exits $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $|x_n \dot{-} x|_\alpha \dot{<} \varepsilon$ for all $n \geq n_0$, then it is said that the sequence (x_n) α-convergent and denoted by ${}_\alpha\lim_{n\to\infty} x_n = x$ (Grossman and Katz, 1972; Çakmak and Başar, 2012).

Grosmann and Katz described the *-calculus with the help of two arbitrary selected generators. Let α and β are arbitrarily choosed generators and * is the ordered pair of arithmetic (α-arithmetic, β-arithmetic). The following notions are used.

	α – arithmetic	β – arithmetic
Realm	$A(\subseteq \mathbb{R}_\alpha)$	$B(\subseteq \mathbb{R}_\beta)$
Summation	$\dot{+}$	$\ddot{+}$
Subtraction	$\dot{-}$	$\ddot{-}$
Multiplication	$\dot{\times}$	$\ddot{\times}$
Division	$\dot{/}$ (or $-_\alpha$)	$\ddot{/}$ (or $-_\beta$)
Order	$\dot{<}$	$\ddot{<}$

If the generators α and β are choosen as one of I and exp, the following special calculus are obtained.

Calculus	α	β
Classic	I	I
Geometric	I	exp
Anageometric	exp	I
Bigeometric	exp	exp.

The ι (iota) which is an isomorphism from α-arithmetic to β-arithmetic uniquely satisfying the following three properties:

(1) ι is one to one,

(2) ι is on A and onto B,

(3) For any numbers u and v in A,

$$\iota(u \dot{+} v) = \iota(u) \ddot{+} \iota(v),$$
$$\iota(u \dot{-} v) = \iota(u) \ddot{-} \iota(v),$$
$$\iota(u \dot{\times} v) = \iota(u) \ddot{\times} \iota(v),$$
$$\iota(u \dot{/} v) = \iota(u) \ddot{/} \iota(v),\ v \neq \dot{0},$$
$$u \dot{<} v \Leftrightarrow \iota(u) \ddot{<} \iota(v).$$

It turns out that $\iota(u) = \beta\{\alpha^{-1}(u)\}$ for every u in A.(Grossman and Katz, 1972).

2. Bigeometric Calculus

Throughout this study, we interest Bigeometric calculus that is the one of the family of non-Newtonian calculus. As mentioned above, the bigeometric calculus is the $*$-calculus for which $\alpha = \beta = \exp$. That is to say, one uses geometric arithmetic on function arguments and values in the bigeometric calculus. Therefore, we begin with presenting the geometric aritmetic and its necessary properties.

If the function $\exp$ from $\mathbb{R}$ to $\mathbb{R}^+$ which gives $\alpha^{-1}(x) = \ln x$ is selected as a generator, that is to say that α-arithmetic turns into geometric arithmetic. The range of generator $\exp$ is denoted by $\mathbb{R}_{\exp} = \{e^x : x \in \mathbb{R}\}$.

geometric addition $\quad x \oplus y = \alpha[\alpha^{-1}(x) + \alpha^{-1}(y)] = e^{(\ln x + \ln y)} = x.y$

geometric subtraction $\quad x \ominus y = \alpha[\alpha^{-1}(x) - \alpha^{-1}(y)] = e^{(\ln x - \ln y)} = x / y,\ y \neq 0$

geometric multiplication $\quad x \odot y = \alpha[\alpha^{-1}(x) \times \alpha^{-1}(y)] = e^{(\ln x \times \ln y)} = x^{\ln y}$

geometric division $\quad x \oslash y = \alpha[\alpha^{-1}(x) / \alpha^{-1}(y)] = e^{(\ln x \div \ln y)} = x^{\frac{1}{\ln y}},\ y \neq 1$

geometric order $\quad x <_{\exp} y \Leftrightarrow \alpha^{-1}(x) = \ln x < \alpha^{-1}(y) = \ln y$

$(\mathbb{R}_{\exp}, \oplus, \odot)$ is a field with geometric zero 1 and geometric identity e. The geometric positive real numbers and geometric negative real numbers are denoted by $\mathbb{R}^+_{\exp} = \{x \in \mathbb{R}_{\exp} : x > 1\}$ and $\mathbb{R}^-_{\exp} = \{x \in \mathbb{R}_{\exp} : x < 1\}$, respectively. Now, we will give some useful and necessary relations between geometric and classical arithmetic operations. The geometric absolute valued of $x \in \mathbb{R}_{\exp}$ defined by

$$|x|_{\exp} = \begin{cases} x & ,x > 1 \\ 1 & ,x = 1 \\ 1/x & ,x < 1 \end{cases}$$

Thus $|x|_{\exp} \geq 1$. For all $x, y \in \mathbb{R}_{\exp}$, the following relations hold:

$x \oplus y = x.y$	$x^{-1_{\exp}} = e^{\frac{1}{\ln x}}$	$\lvert e^x \rvert_{\exp} = e^{\lvert x \rvert}$
$x \ominus y = x / y$	$x^{p_{\exp}} = x^{\ln^{p-1} x}$	$\lvert x \oslash y \rvert_{\exp} = \lvert x \rvert_{\exp} \oslash \lvert y \rvert_{\exp}$
$x \odot y = x^{\ln y} = y^{\ln x}$	$\sqrt{x}^{\exp} = e^{(\ln x)^{1/2}}$	$\lvert x \oplus y \rvert_{\exp} \leq_{\exp} \lvert x \rvert_{\exp} \oplus \lvert y \rvert_{\exp}$
$x \oslash y$ or $\frac{x}{y}_{\exp} = x^{\frac{1}{\ln y}}$	$e^n \odot x = x^n$	$\lvert x \oslash y \rvert_{\exp} = \lvert x \rvert_{\exp} \oslash \lvert y \rvert_{\exp}$
$x^{2_{\exp}} = x \odot x = x^{\ln x}$	$\sqrt{x^{2_{\exp}}}^{\exp} = \lvert x \rvert_{\exp}$	$\lvert x \ominus y \rvert_{\exp} \geq_{\exp} \lvert x \rvert_{\exp} \ominus \lvert y \rvert_{\exp}$
$x \odot e = x,\ x \oplus 1 = x$	$1 \ominus e \odot (x \ominus y) = y \ominus x$	

(Grossman and Katz, 1972; Grossman 1983; Türkmen and Başar, 2012a,b; Boruah and Hazarika, 2018a,b). The geometric factorial notation $!_{\exp}$ denoted by $n!_{\exp} = e^n \odot e^{n-1} \odot \ldots \odot e^2 \odot e = e^{n!}$ (Boruah and Hazarika, 2018a). For example,

$0!_{\exp} = e^{0!} = 1$

$1!_{\exp} = e^{1!} = e$

$2!_{\exp} = e^{2!} = e^2$

....

Definition 6. Let $f : X \subset \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$ be a function and $a \in X^{'\exp}$, $b \in \mathbb{R}_{\exp}$. If for every $\varepsilon >_{\exp} 1$ there is a number $\delta = \delta(\varepsilon) >_{\exp} 1$ such that $\left|f(x) \ominus b\right|_{\exp} <_{\exp} \varepsilon$ for all $x \in X$ whenever $1 <_{\exp} \left|x \ominus a\right|_{\exp} <_{\exp} \delta$, then it is said that the BG-limit function f at the point a is b and it is indicate by ${}_{BG}\lim_{x\to a} f(x) = b$ or $f(x) \overset{BG}{\to} b$. Here $1 <_{\exp} \left|x \ominus a\right|_{\exp} <_{\exp} \delta \Rightarrow \frac{a}{\delta} < x < a\delta$ and $\left|f(x) \ominus b\right|_{\exp} <_{\exp} \varepsilon \Rightarrow \frac{b}{\varepsilon} < f(x) < b\varepsilon$ (Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018a).

Definition 7. If the sequence $(f(x_n))$ $\exp$-converges to the number b for all sequences which $\exp$-converges to point a, then it is said that BG-limit of the function f at the point a is b and is denoted by ${}_{BG}\lim_{x\to a} f(x) = b$ (Grossman and Katz, 1972; Grossman 1983).

Definition 8. Let $a \in X$ and $f : X \subset \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$ be a function. If for every $\varepsilon >_{\exp} 1$ there is a number $\delta = \delta(\varepsilon) >_{\exp} 1$ such that $\left|f(x) \ominus f(a)\right|_{\exp} <_{\exp} \varepsilon$ for all $x \in X$ whenever $1 <_{\exp} \left|x \ominus a\right|_{\exp} <_{\exp} \delta$, then it is said that f is BG-continuous at point $a \in X$. The function f is BG-continuous at the point $a \in X$ iff this point a is an element of domain of the function f and ${}_{BG}\lim_{x\to a} f(x) = f(a)$ (Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018a).

Remark 1. ${}_{BG}\lim_{x\to a} f(x)$ and $\lim_{t\to \ln a} \ln f(t)$ coexist, and if they do exist ${}_{BG}\lim_{x\to a} f(x) = \exp\left\{\lim_{t\to \ln a} \ln f(e^t)\right\}$. Furthermore, f is BG-continuous at a iff $\ln f$ is continuous at $\ln a$ (Grossman and Katz, 1972; Grossman 1983).

Definition 9. Let $f : (r,s) \subset \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$ be a function and $a \in (r,s)$. If the following limit

$${}_{BG}\lim_{x\to a} \frac{f(x) \ominus f(a)}{x \ominus a}{}_{\exp} = \lim_{x\to a} \left[\frac{f(x)}{f(a)}\right]^{\frac{1}{\ln x - \ln a}}$$

exists, it is indicated by $f^{BG}(a)$ and called the BG-derivative of f at a and say that f is BG-differentiable. If the function f is BG-differentiable at all points of the $\exp$-open interval (r,s), then f is BG-differentiable on (r,s) and BG-derivative of f identified as

$${}_{BG}\lim_{h\to 1} \frac{f(x \oplus h) \ominus f(x)}{h}{}_{\exp} = \lim_{h\to 1} \left[\frac{f(hx)}{f(x)}\right]^{\frac{1}{\ln h}}$$

for $h \in \mathbb{R}_{\exp}$ and denoted by f^{BG} or $\frac{d^{BG} f}{dx^{BG}}$ (Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018a,b).

Remark 2. The derivatives $f^{BG}(a)$ and $\left(\ln f(\ln a)\right)'$ coexist, and if they do exist $f^{BG}(a)=\exp\left[\left(\ln f\left(e^{\ln a}\right)\right)'\right]=e^{a\frac{f'(a)}{f(a)}}$.

Therefore the relation between the BG-derivatives and classical derivatives can be written as follows:

$$f^{BG}(x)=e^{\frac{xf'(x)}{f(x)}}=e^{x\left(\ln f(x)\right)'}$$

(Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018a,b).

The second BG-derivative of $f(x)$ is defined as

$$\frac{d^{2_{BG}}f}{dx^{2_{BG}}}=f^{(2_{BG})}(x)={}_{BG}\lim_{h\to 1}\frac{f^{BG}(x\oplus h)\ominus f^{BG}(x)}{h}\exp=\lim_{h\to 1}\left[\frac{f^{BG}(hx)}{f^{BG}(x)}\right]^{\frac{1}{\ln h}}=\lim_{h\to 1}\left[\frac{e^{hx\left(\ln f(hx)\right)'}}{e^{x\left(\ln f(x)\right)'}}\right]^{\frac{1}{\ln h}}$$

$$=\lim_{h\to 1}\left(e^{x\left[h\left(\ln f(hx)\right)'-\left(\ln f(x)\right)'\right]}\right)^{\frac{1}{\ln h}}=\lim_{h\to 1}\left(e^{x\frac{1}{\ln h}\left[\left(\ln\frac{f(hx)}{f(x)}\right)'\right]}\right)=e^{x\lim_{h\to 1}\left[\left(\ln\left(\frac{f(hx)}{f(x)}\right)^{\frac{1}{\ln h}}\right)'\right]}$$

$$=e^{x\left[\left(\ln f^{BG}(x)\right)'\right]}=e^{x\left(\ln e^{x\left(\ln f(x)\right)'}\right)'}=e^{x^2\left(\ln f(x)\right)''+x\left(\ln f(x)\right)'}.$$

Similarly, the n^{th} order derivative is

$$\frac{d^{n_{BG}}f}{dx^{n_{BG}}}=f^{(n_{BG})}(x)={}_{BG}\lim_{h\to 1}\frac{f^{(n-1)_{BG}}(x\oplus h)\ominus f^{(n-1)_{BG}}(x)}{h}\exp=e^{x\left[\left(\ln f^{(n-1)_{BG}}(x)\right)'\right]}=e^{x^n\left(\ln f(x)\right)^{(n)}+\ldots+x\left(\ln f(x)\right)'}.$$

Theorem 1. If $f,g:(r,s)\subset\mathbb{R}_{\exp}\to\mathbb{R}_{\exp}$ are BG-differentiable functions and c is an arbitrary constant, then

(1) $\left(f(x)\oplus g(x)\right)^{BG}=f(x)^{BG}\oplus g(x)^{BG}$

(2) $\left(f(x)\ominus g(x)\right)^{BG}=f(x)^{BG}\ominus g(x)^{BG}$

(3) $\left(f(x)^c\right)^{BG}=\left(f(x)^{BG}\right)^c$

(Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018a,b).

Now, we will give some standart BG-derivatives:

$\frac{d^{BG}}{dx^{BG}}(c)=1$	$\frac{d^{BG}}{dx^{BG}}(\sin x)=e^{x\cot x}$	$\frac{d^{BG}}{dx^{BG}}(\cot x)=e^{-x\sec x\csc x}$
$\frac{d^{BG}}{dx^{BG}}(cf(x))=f^{BG}(x)$	$\frac{d^{BG}}{dx^{BG}}(\cos x)=e^{-x\tan x}$	$\frac{d^{BG}}{dx^{BG}}(\sec x)=e^{x\tan x}$
$\frac{d^{BG}}{dx^{BG}}(x^n)=e^n$	$\frac{d^{BG}}{dx^{BG}}(\tan x)=e^{x\sec x\csc x}$	$\frac{d^{BG}}{dx^{BG}}(\csc x)=e^{-x\cot x}$

(Boruah and Hazarika, 2018a,b).

Theorem 2. *(Mean Value Theorem of BG-Calculus)* If f is BG-continuous function on $[r,s]\subset\mathbb{R}_{\exp}$ and BG-differentiable on (r,s), there is $r<_{\exp}c<_{\exp}s$ such that $f^{BG}(c)=\frac{f(s)\ominus f(r)}{s\ominus r}\exp$ (Grossman and Katz, 1972; Grossman 1983; Kadak and Özlük, 2014).

Definition 10. The BG-average of a BG-continuous positive function f on $[r,s]\subset\mathbb{R}_{\exp}$ is defined as the $\exp$-limit of the $\exp$-convergent sequence whose n^{th} term is geometric average of $f(a_1), f(a_2),\ldots, f(a_n)$

where $a_1, a_2, \ldots, a_n$ is the n-fold $\exp$-partition of $[r,s]$ and denoted by $\overset{BG}{M_r^s} f$. The BG-integral of a BG-continuous function f on $[r,s]$ is the positive number $\left[\overset{BG}{M_r^s} f\right]^{[\ln(s)-\ln(r)]}$ and is denoted by ${}_{BG}\int_r^s f(x)dx^{BG}$ (Grossman and Katz, 1972; Grossman 1983).

Remark 3. If f is BG-continuous on $[r,s] \subset \mathbb{R}_{\exp}$, then ${}_{BG}\int_r^s f(x)dx^{BG} = \exp\left(\int_{\ln(r)}^{\ln(s)} \ln f(e^t)dt\right)$, i.e., the BG-integral of the function f is a BG-continuous positive function on $[r,s] \subset \mathbb{R}_{\exp}$ is defined by

$$ {}_{BG}\int_r^s f(x)dx^{BG} = e^{\int_r^s \frac{\ln f(x)}{x}dx} $$

(Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018b).

Theorem 3. If f and g are BG-continuous positive functions on $[r,s] \subset \mathbb{R}_{\exp}$ and c is an arbitrary constant, then

(1) $${}_{BG}\int_r^s \left(f(x) \oplus g(x)\right)dx^{BG} = {}_{BG}\int_r^s f(x)dx^{BG} \oplus {}_{BG}\int_r^s f(x)dx^{BG}$$

(2) $${}_{BG}\int_r^s \left(f(x) \ominus g(x)\right)dx^{BG} = {}_{BG}\int_r^s f(x)dx^{BG} \ominus {}_{BG}\int_r^s f(x)dx^{BG}$$

(3) $${}_{BG}\int_r^s \left(f(x)\right)^c dx^{BG} = \left({}_{BG}\int_r^s f(x)dx^{BG}\right)^c$$

(4) $${}_{BG}\int_r^s f(x)dx^{BG} = {}_{BG}\int_r^t f(x)dx^{BG} \oplus {}_{BG}\int_t^s f(x)dx^{BG} \text{ where } r <_{\exp} t <_{\exp} s$$

(Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018b).

Now, we will give some standart BG-integrations:

$${}_{BG}\int 1dx^{BG} = c \qquad {}_{BG}\int e^{x\cot x}dx^{BG} = \sin x \qquad {}_{BG}\int e^{-x\sec x\csc x}dx^{BG} = \cot x$$

$${}_{BG}\int e^n dx^{BG} = x^n \qquad {}_{BG}\int e^{-x\tan x}dx^{BG} = \cos x \qquad {}_{BG}\int e^{x\tan x}dx^{BG} = \sec x$$

$${}_{BG}\int e^x dx^{BG} = e^x \qquad {}_{BG}\int e^{x\sec x\csc x}dx^{BG} = \tan x \qquad {}_{BG}\int e^{-x\cot x}dx^{BG} = \csc x$$

$${}_{BG}\int x^n dx^{BG} = e^{n\frac{\ln^2 x}{2}}$$

(Boruah and Hazarika, 2018b).

Theorem 4. (*First Fundamental Theorem of* BG*-calculus)* If f is BG-continuous function on $[r,s] \subset \mathbb{R}_{\exp}$ and $g(x) = {}_{BG}\int_r^x f(t)dt^{BG}$ for every $x \in [r,s]$, then $g^{BG} = f$ on $[r,s]$(Grossman and Katz, 1972; Grossman 1983).

Theorem 5. (*Second Fundamental Theorem of* BG*-calculus)* If f^{BG} is BG-continuous function on $[r,s] \subset \mathbb{R}_{\exp}$, then ${}_{BG}\int_r^s \left[f^{BG}\right](x)dx^{BG} = f(s) \ominus f(r)$ (Grossman and Katz, 1972; Grossman 1983).

Definition 11. An equation involving BG-differential coefficient is called a BG- differential equation, i.e, the n^{th} order BG-differential equation is defined as $G\left(x, y, y^{BG}, \ldots, y^{(n-1)_{BG}}, y^{n_{BG}}(x)\right)=1,\ (x, y) \in \mathbb{R}_{\exp} \times \mathbb{R}_{\exp}$ (Boruah et al., 2018).

3. *BG* -Volterra Integral Equations

The equation is called BG-integral equation where an unknown function appears under the BG-integral sign. The equation

$$u(x)=f(x) \oplus\left(\lambda \odot {}_{BG}\int_a^x K(x,t) \odot u(t)\, dt^{BG}\right)$$

where $f(x)$ and $K(x,t)$ are known functions, $u(x)$ is unknown function and $\lambda \in \mathbb{R}_{\exp}$, is said to be BG-Volterra linear integral equation of the second kind. The function $K(x,t)$ is the kernel of BG-Volterra equation. If $f(x)=1$ then the equation is reduced to the following form

$$u(x)=\lambda \odot {}_{BG}\int_a^x K(x,t) \odot u(t)\, dt^{BG}$$

and it is called homogeneous BG-Volterra linear integral equation of the second kind. The equation

$$\lambda \odot {}_{BG}\int_a^x K(x,t) \odot u(t)\, dt^{BG}=f(x)$$

where $u(x)$ is unknown function is called BG-Volterra linear integral equation of the first kind.

Example 1. Demonstrate that $u(x)=e^x$ is a solution of the BG-Volterra integral equation $u(x)=(x \oplus e) \oplus {}_{BG}\int_1^x (x \ominus t) \odot u(t)\, dt^{BG}$.

Solution. Substituting the function e^x in place of $u(x)$ into the right side of the equation, then

$$(x \oplus e) \oplus {}_{BG}\int_1^x (x \ominus t) \odot u(t)\, dt^{BG}=(x \oplus e) \oplus {}_{BG}\int_1^x (x \ominus t) \odot e^t dt^{BG}=xe \cdot e^{\int_1^x \frac{\ln\left(\frac{x}{t}\right) \ln e^t}{t} dt}=xe \cdot e^{-\ln x+x-1}=e^x=u(x).$$

3.1. The Relationship with BG -Differantial Equations

3.1.1. The Conversion of the BG -Volterra Integral Equations to BG -Diferential Equations

In this section, we demonstrate the method of converting BG-Volterra integral equations into BG-differential equations. For this, we need Leibniz formula in Bigeometric calculus. Firstly, we will give some necessary definition and theorems.

Definition 12. Let f be a bipositive function with two variables. Then, we define its BG-partial derivatives as

$$f_x^{BG}(x, y)=\frac{\partial^{BG}}{\partial x^{BG}} f(x, y)={}_{BG}\lim_{h \rightarrow 1} \frac{f(x \oplus h, y) \ominus f(x, y)}{h}_{\exp}$$

and

$$f_y^{BG}(x, y)=\frac{\partial^{BG}}{\partial y^{BG}} f(x, y)={}_{BG}\lim_{h \rightarrow 1} \frac{f(x, y \oplus h) \ominus f(x, y)}{h}_{\exp}.$$

From the definition of BG-partial derivative, we find its relation with classical partial derivative, as follows:

$$ {}_{BG}\lim_{h\to 1}\frac{f(x\oplus h,y)\ominus f(x,y)}{h}{}_{\exp}=\lim_{h\to 1}\left[\frac{f(hx,y)}{f(x,y)}\right]^{\frac{1}{\ln h}}=\lim_{h\to 1}e^{\ln\left[\frac{f(hx,y)}{f(x,y)}\right]^{\frac{1}{\ln h}}}=\lim_{h\to 1}e^{\frac{\ln f(hx,y)-\ln f(x,y)}{\ln h}} $$

$$ =e^{\lim_{h\to 1}\frac{\ln f(hx,y)-\ln f(x,y)}{\ln h}}=e^{\lim_{h\to 1}hx\frac{f'(hx,y)}{f(hx,y)}}=e^{x\frac{f'_x(x,y)}{f(x,y)}}=e^{x\frac{\partial}{\partial x}\ln f(x,y)}. $$

Therefore, we can also write the definition of BG-partial derivative as

$$ f_x^{BG}(x,y)=\frac{\partial^{BG}}{\partial x^{BG}}f(x,y)=e^{x\frac{\partial}{\partial x}\ln f(x,y)} \text{ and } f_y^{BG}(x,y)=\frac{\partial^{BG}}{\partial y^{BG}}f(x,y)=e^{y\frac{\partial}{\partial y}\ln f(x,y)}. $$

Theorem 6. (BG-*Chain Rule*) Suppose that f has BG-partial derivatives of y and z with BG-continuous where $y=y(x)$ and $z=z(x)$ are BG-differentiable functions of x. Then $f(y(x),z(x))$ is BG-differentiable function of x and

$$ \frac{d^{BG}f(y(x),z(x))}{dx^{BG}}=\left(f_y^{BG}(y(x),z(x))\odot y^{BG}\right)\oplus\left(f_z^{BG}(y(x),z(x))\odot z^{BG}\right). $$

Proof. By using the definition of BG-derivative, we get

$$ \frac{d^{BG}f(y(x),z(x))}{dx^{BG}}={}_{BG}\lim_{h\to 1}\frac{f(y(x\oplus h),z(x\oplus h))\ominus f(y(x),z(x))}{h}{}_{\exp} $$

$$ =\lim_{h\to 1}e^{\ln\left[\frac{f(y(xh),z(xh))}{f(y(x),z(x))}\right]^{\frac{1}{\ln h}}} $$

$$ =e^{\lim_{h\to 1}\frac{\ln\left[\frac{f(y(xh),z(xh))}{f(y(x),z(x))}\right]}{\ln h}} $$

$$ =e^{x\frac{f'_x(y(x),z(x))}{f(y(x),z(x))}} $$

$$ =\left(f_y^{BG}(y(x),z(x))\right)^{\ln e^{x\frac{y'(x)}{y(x)}}}\oplus\left(f_z^{BG}(y(x),z(x))\right)^{\ln e^{x.\frac{z'(x)}{z(x)}}} $$

$$ =\left(f_y^{BG}(y(x),z(x))\right)\odot y^{BG}\oplus f_z^{BG}(y(x),z(x))\odot z^{BG}. $$

Theorem 7. Let Ω be an $\exp$-open set in $\mathbb{R}_{\exp}\times\mathbb{R}_{\exp}$. Assume that $f:\Omega\to\mathbb{R}_{\exp}$ be a function such that the BG-partial derivative $f_{xy}^{2_{BG}}(x,y)$, $f_{yx}^{2_{BG}}(x,y)$ exists in Ω and are BG-continuous, then

$$ \frac{\partial^{BG}}{\partial x^{BG}}\left(\frac{\partial^{BG}}{\partial y^{BG}}f(x,y)\right)=\frac{\partial^{BG}}{\partial y^{BG}}\left(\frac{\partial^{BG}}{\partial x^{BG}}f(x,y)\right). $$

Proof. Fix x and y and we define $F(h,k)$ as

$$ F(h,k)=\frac{1}{h}{}_{\exp}\odot\frac{1}{k}{}_{\exp}\odot\left[f(x\oplus h,y\oplus k)\ominus f(x\oplus h,y)\ominus f(x,y\oplus k)\oplus f(x,y)\right]. $$

By using the mean value theorem in the sense of BG-calculus, we obtain that

$$\begin{aligned}F(h,k) &= \frac{1}{h}{}_{\exp}\odot\frac{1}{k}{}_{\exp}\odot\left[\left(f(x\oplus h,y\oplus k)\ominus f(x,y\oplus k)\right)\ominus\left(f(x\oplus h,y)\ominus f(x,y)\right)\right]\\ &= \frac{1}{h}{}_{\exp}\odot\frac{\partial^{BG}}{\partial y^{BG}}\left(f(x\oplus h,y\oplus\lambda_1\odot k)\ominus f(x,y\oplus\lambda_1\odot k)\right)\\ &= \frac{\partial^{BG}}{\partial y^{BG}}\left(\frac{f(x\oplus h,y\oplus\lambda_1\odot k)\ominus f(x,y\oplus\lambda_1\odot k)}{h}{}_{\exp}\right)\\ &= \frac{\partial^{BG}}{\partial y^{BG}}\left(\frac{f(x\oplus\lambda_2\odot h,y\oplus\lambda_1\odot k)\ominus f(x\oplus\lambda_2\odot h,y\oplus\lambda_1\odot k)}{h}{}_{\exp}\right)\\ &= \frac{\partial^{BG}}{\partial y^{BG}}\frac{\partial^{BG}}{\partial x^{BG}}f(x\oplus\lambda_2\odot h,y\oplus\lambda_1\odot k)\end{aligned}$$

and

$$\begin{aligned}F(h,k) &= \frac{1}{k}{}_{\exp}\odot\frac{1}{h}{}_{\exp}\odot\left[\left(f(x\oplus h,y\oplus k)\ominus f(x\oplus h,y)\right)\ominus\left(f(x,y\oplus k)\ominus f(x,y)\right)\right]\\ &= \frac{1}{k}{}_{\exp}\odot\frac{\partial^{BG}}{\partial x^{BG}}\left(f(x\oplus\lambda_3\odot h,y\oplus k)\ominus f(x\oplus\lambda_3\odot h,y)\right)\\ &= \frac{\partial^{BG}}{\partial x^{BG}}\left(\frac{f(x\oplus\lambda_3\odot h,y\oplus k)\ominus f(x\oplus\lambda_3\odot h,y)}{k}{}_{\exp}\right)\\ &= \frac{\partial^{BG}}{\partial x^{BG}}\left(\frac{f(x\oplus\lambda_3\odot h,y\oplus\lambda_4\odot k)\ominus f(x\oplus\lambda_3\odot h,y\oplus\lambda_4\odot k)}{k}{}_{\exp}\right)\\ &= \frac{\partial^{BG}}{\partial x^{BG}}\frac{\partial^{BG}}{\partial y^{BG}}f(x\oplus\lambda_3\odot h,y\oplus\lambda_4\odot k)\end{aligned}$$

for some $1<_{\exp}\lambda_1,\lambda_2,\lambda_3,\lambda_4<_{\exp}e$ which all of them depend on x,y,h,k. Therefore,

$$\frac{\partial^{BG}}{\partial y^{BG}}\frac{\partial^{BG}}{\partial x^{BG}}f(x\oplus\lambda_2\odot h,y\oplus\lambda_1\odot k)=\frac{\partial^{BG}}{\partial x^{BG}}\frac{\partial^{BG}}{\partial y^{BG}}f(x\oplus\lambda_3\odot h,y\oplus\lambda_4\odot k)$$

for all h and k. Taking the BG-limit $h\to 1$ and $k\to 1$ and using the assumed BG-continuity of both partial derivatives, it gives $\frac{\partial^{BG}}{\partial y^{BG}}\frac{\partial^{BG}}{\partial x^{BG}}f(x,y)=\frac{\partial^{BG}}{\partial x^{BG}}\frac{\partial^{BG}}{\partial y^{BG}}f(x,y)$.

Theorem 8. *(BG-Leibniz Formula)* Let A,I be $\exp$-open set and f be a BG-continuous function on $A\times I$ into $\mathbb{R}_{\exp}$. If f_x^{BG} exists and is BG-continuous on $A\times I$, $u(x),v(x)$ are BG-continuously differentiable functions of A into I, then

$$\frac{d^{BG}}{dx^{BG}}\left({}_{BG}\int_{u(x)}^{v(x)}f(x,t)dt^{BG}\right)={}_{BG}\int_{u(x)}^{v(x)}\left(f_x^{BG}(x,t)\right)dt^{BG}\oplus\left[f(x,v(x))\odot v_x^{BG}\right]\ominus\left[f(x,u(x))\odot u_x^{BG}\right].$$

Proof. Take $f(x,t)=\frac{\partial^{BG}}{\partial t^{BG}}F(x,t)=F_t^{BG}(x,t)$. Hence we find

$${}_{BG}\int_{u(x)}^{v(x)}f(x,t)dt^{BG}={}_{BG}\int_{u(x)}^{v(x)}\frac{\partial^{BG}}{\partial t^{BG}}F(x,t)dt^{BG}=F(x,v(x))\ominus F(x,u(x)).$$

Therefore, we obtain

$$\frac{d^{BG}}{dx^{BG}}\left({}_{BG}\int_{u(x)}^{v(x)}f(x,t)dt^{BG}\right)=\frac{d^{BG}}{dx^{BG}}\left(F(x,v(x))\ominus F(x,u(x))\right)=\frac{d^{BG}}{dx^{BG}}F(x,v(x))\ominus\frac{d^{BG}}{dx^{BG}}F(x,u(x)) \qquad (1)$$

by using the properties of BG- derivative. We can write as

$$\frac{d^{BG}}{dx^{BG}}F(x,u(x))=F_x^{BG}(x,u(x))\oplus F_{u(x)}^{BG}(x,u(x))\odot u_x^{BG} \qquad (2)$$

and

$$\frac{d^{BG}}{dx^{BG}}F\left(x,v\left(x\right)\right)=F_x^{BG}\left(x,v\left(x\right)\right)\oplus F_{v(x)}^{BG}\left(x,v\left(x\right)\right)\odot v_x^{BG} \tag{3}$$

from BG-chain rule. Hence by using Theorem 7, we get

$$\begin{aligned}\frac{d^{BG}}{dx^{BG}}\left({}_{BG}\int_{u(x)}^{v(x)} f\left(x,t\right)dt^{BG}\right)&=F_x^{BG}\left(x,v\left(x\right)\right)\ominus F_x^{BG}\left(x,u\left(x\right)\right)\oplus\left[F_{v(x)}^{BG}\left(x,v\left(x\right)\right)\odot v_x^{BG}\right]\ominus\left[F_{u(x)}^{BG}\left(x,u\left(x\right)\right)\odot u_x^{BG}\right]\\&={}_{BG}\int_{u(x)}^{v(x)}\left(\frac{\partial^{BG}}{\partial t^{BG}}F_x^{BG}\left(x,t\right)\right)dt^{BG}\oplus\left[f\left(x,v\left(x\right)\right)\odot v_x^{BG}\right]\ominus\left[f\left(x,u\left(x\right)\right)\odot u_x^{BG}\right]\\&={}_{BG}\int_{u(x)}^{v(x)}\left(\frac{\partial^{BG}}{\partial x^{BG}}F_t^{BG}\left(x,t\right)\right)dt^{BG}\oplus\left[f\left(x,v\left(x\right)\right)\odot v_x^{BG}\right]\ominus\left[f\left(x,u\left(x\right)\right)\odot u_x^{BG}\right]\\&={}_{BG}\int_{u(x)}^{v(x)}\left(f_x^{BG}\left(x,t\right)\right)dt^{BG}\oplus\left[f\left(x,v\left(x\right)\right)\odot v_x^{BG}\right]\ominus\left[f\left(x,u\left(x\right)\right)\odot u_x^{BG}\right]\end{aligned}$$

from the expressions (1), (2) and (3).

Example 2. Show that the BG-Volterra integral equation $u\left(x\right)=\sin x\oplus{}_{BG}\int_1^x e^x\odot u\left(t\right)dt^{BG}$ can be transformed to BG-differential equation.

Solution. If we consider the equation and differentiate it by using BG-Leibniz formula, we obtain

$$\frac{d^{BG}}{dx^{BG}}u\left(x\right)=\frac{d^{BG}}{dx^{BG}}\sin x\oplus\frac{d^{BG}}{dx^{BG}}\left({}_{BG}\int_1^x u\left(t\right)\odot e^x dt^{BG}\right)$$

$$u^{BG}\left(x\right)=e^{x\cot x}\oplus{}_{BG}\int_1^x\frac{\partial^{BG}}{\partial x^{BG}}\left(u\left(t\right)\odot e^x\right)dt^{BG}\oplus\left(u\left(x\right)\odot e^x\odot x_x^{BG}\right)\ominus\left(u\left(1\right)\odot e^x\odot 1_x^{BG}\right)$$

$$u^{BG}\left(x\right)=e^{x\cot x}\oplus{}_{BG}\int_1^x e dt^{BG}\oplus\left(u\left(x\right)\odot e^x\right)$$

$$u^{BG}\left(x\right)=e^{x\cot x}\oplus x\oplus\left(u\left(x\right)\odot e^x\right).$$

Thus the BG-Volterra integral equation is equivalent to the BG-differential equaiton $u^{BG}\left(x\right)\ominus\left(u\left(x\right)\odot e^x\right)=e^{x\cot x}\oplus x$.

3.1.2. The Conversion of the BG -Linear Diferential Equations to BG -Volterra Integral Equations

In this part, we prove that it is converted to BG-Volterra integral equations by defining BG- linear differential equation with constant coefficients and variable.

Definition 12. The equation of the form

$$y^{n_{BG}}\oplus a_1\left(x\right)\odot y^{(n-1)_{BG}}\oplus\cdots\oplus a_{n-1}\left(x\right)\odot y^{BG}\oplus a_n\left(x\right)\odot y=f\left(x\right)$$

where f is a bipositive function, is called n^{th} order BG-linear differential equation. If the coefficients $a_n\left(x\right)$ are constants, then the equation is called as BG- linear differential equation with constant coefficients; if not it is called BG-linear differential equation with variable coefficients.

Theorem 9. If n is a positive integer and $a\in\mathbb{R}_{\exp}$ with $x\geq_{\exp} a$, then we have

$${}_{BG}\int_a^x\cdots(n)\cdots{}_{BG}\int_a^x u\left(t\right)dt^{BG}\cdots dt^{BG}=\frac{e}{\left(n-1\right)!_{\exp}}\,{}_{\exp}\odot\,{}_{BG}\int_a^x\left(x\ominus t\right)^{(n-1)_{exp}}\odot u\left(t\right)dt^{BG}$$

Proof. Take

$$I_n = {}_{BG}\int_a^x (x \ominus t)^{(n-1)_{\exp}} \odot u(t)\, dt^{BG} . \tag{4}$$

If it is taken $F(x,t) = (x \ominus t)^{(n-1)_{\exp}} \odot u(t)$, we write

$$\frac{d^{BG} I_n}{dx^{BG}} = \frac{d^{BG}}{dx^{BG}}\left({}_{BG}\int_a^x F(x,t)\, dt^{BG} \right)$$

$$= {}_{BG}\int_a^x \frac{\partial^{BG}}{\partial x^{BG}}\left(F(x,t)\right) dt^{BG} \oplus \left(F(x,x) \odot x_x^{BG}\right) \ominus \left(F(x,a) \odot a_x^{BG}\right)$$

$$= {}_{BG}\int_a^x e^{x\frac{F_x'(x,t)}{F(x,t)}}\, dt^{BG} \tag{5}$$

by using BG-Leibniz rule. Since $F(x,t) = (x \ominus t)^{(n-1)_{\exp}} \odot u(t) = \left[\left(\frac{x}{t}\right)^{\left(\ln\left(\frac{x}{t}\right)\right)^{(n-2)}} \right]^{\ln u(t)} = \left(\frac{x}{t}\right)^{\left(\ln\left(\frac{x}{t}\right)\right)^{(n-2)} \ln u(t)}$,

$F_x'(x,t) = \frac{1}{x}(n-1) F(x,t) \ln u(t) \left(\ln\left(\frac{x}{t}\right)\right)^{(n-2)}$. Therefore we find

$$\frac{d^{BG} I_n}{dx^{BG}} = {}_{BG}\int_a^x e^{x\frac{F_x'(x,t)}{F(x,t)}}\, dt^{BG} = {}_{BG}\int_a^x e^{(n-1)\ln u(t)\left(\ln\left(\frac{x}{t}\right)\right)^{(n-2)}}\, dt^{BG} = {}_{BG}\int_a^x \left(e^{\left(\ln\left(\frac{x}{t}\right)\right)^{(n-2)}} \right)^{\ln u(t)^{(n-1)}} dt^{BG} = {}_{BG}\int_a^x e^{\left(\ln\left(\frac{x}{t}\right)\right)^{(n-2)}} \odot u(t)^{(n-1)}\, dt^{BG}$$

$$= {}_{BG}\int_a^x e^{\left(\ln\left(\frac{x}{t}\right)\right)^{(n-2)}} \odot e^{(n-1)} \odot u(t)\, dt^{BG} = {}_{BG}\int_a^x \left(e^{\ln\left(\frac{x}{t}\right)} \right)^{\left(\ln\left(\frac{x}{t}\right)\right)^{(n-3)}} \odot e^{(n-1)} \odot u(t)\, dt^{BG}$$

$$= {}_{BG}\int_a^x e^{(n-1)} \odot (x \ominus t)^{(n-3)_{\exp}} \odot u(t)\, dt^{BG} = \left({}_{BG}\int_a^x (x \ominus t)^{(n-3)_{\exp}} \odot u(t)\, dt^{BG} \right)^{(n-1)}$$

$$= \left(I_{n-1}\right)^{(n-1)} = e^{n-1} \odot I_{n-1}$$

from the equation (5). Hence we get

$$\frac{d^{BG} I_n}{dx^{BG}} = \left({}_{BG}\int_a^x (x \ominus t)^{(n-3)_{\exp}} \odot u(t)\, dt^{BG} \right)^{(n-1)} = e^{n-1} \odot I_{n-1} \tag{6}$$

for $n > 1$. Since $I_1 = {}_{BG}\int_a^x u(t)\, dt^{BG}$ for $n = 1$, then we write

$$\frac{d^{BG} I_1}{dx^{BG}} = \frac{d^{BG}}{dx^{BG}}\left({}_{BG}\int_a^x u(t)\, dt^{BG} \right) = u(x). \tag{7}$$

If it is taken BG-derivative of the equation (6) by using BG-Leibniz formula, then

$$\frac{d^{2_{BG}} I_n}{dx^{2_{BG}}} = \frac{d^{BG}}{dx^{BG}}\left({}_{BG}\int_a^x e^{(n-1)} \odot (x \ominus t)^{(n-3)_{\exp}} \odot u(t)\, dt^{BG} \right) = {}_{BG}\int_a^x \frac{\partial^{BG}}{\partial x^{BG}}\left(e^{(n-1)} \odot (x \ominus t)^{(n-3)_{\exp}} \odot u(t) \right) dt^{BG}$$

$$= {}_{BG}\int_a^x e^{(n-1)(n-2)\ln\left(\frac{x}{t}\right)^{(n-3)} \ln u(t)}\, dt^{BG} = {}_{BG}\int_a^x e^{(n-1)} \odot e^{(n-2)} \odot (x \ominus t)^{(n-3)_{\exp}} \odot u(t)\, dt^{BG} = e^{(n-1)} \odot e^{(n-2)} \odot I_{n-2}.$$

By proceeding similarly, we get

$$\frac{d^{(n-1)_{BG}} I_n}{dx^{(n-1)_{BG}}} = e^{(n-1)} \odot e^{(n-2)} \odot \cdots \odot e^{1} \odot I_1 = e^{(n-1)!} \odot I_1 = (n-1)!_{\exp} \odot I_1 .$$

Hence we write

$$\frac{d^{n_{BG}} I_n}{dx^{n_{BG}}} = (n-1)!_{\exp} \odot u(x)$$

from the equation (7). Now, we will take BG-integral by considering the above relations. From the equation (7), $I_1(x) = {}_{BG}\int_a^x u(x_1)\,dx_1^{BG}$. Also, we have

$$I_2(x) = e^1 \odot {}_{BG}\int_a^x I_1(x_1)\,dx_1^{BG} = e \odot {}_{BG}\int_a^x {}_{BG}\int_a^{x_1} u(x_1)\,dx_1^{BG} dx_2^{BG}$$

where x_1 and x_2 are parameters. By proceeding likewise, we get

$$I_n(x) = e^{(n-1)!} \odot {}_{BG}\int_a^x {}_{BG}\int_a^{x_n} \dots {}_{BG}\int_a^{x_3} {}_{BG}\int_a^{x_2} u(x_1)\,dx_1^{BG} dx_2^{BG} \dots dx_{n-1}^{BG} dx_n^{BG}$$

$$= (n-1)!_{\exp} \odot {}_{BG}\int_a^x {}_{BG}\int_a^{x_n} \dots {}_{BG}\int_a^{x_3} {}_{BG}\int_a^{x_2} u(x_1)\,dx_1^{BG} dx_2^{BG} \dots dx_{n-1}^{BG} dx_n^{BG}$$

where x_1, x_2, $\dots, x_n$ are parameters. If we write the equation (4) instead of the statement I_n, then we find

$${}_{BG}\int_a^x (x \ominus t)^{(n-1)_{\exp}} \odot u(t)\,dt^{BG} = (n-1)!_{\exp} \odot {}_{BG}\int_a^x {}_{BG}\int_a^{x_n} \dots {}_{BG}\int_a^{x_3} {}_{BG}\int_a^{x_2} u(x_1)\,dx_1^{BG} dx_2^{BG} \dots dx_{n-1}^{BG} dx_n^{BG} \;.$$

If it is taken $x = x_1 = x_2 = \dots = x_n$, then we obtain

$${}_{BG}\int_a^x (x \ominus t)^{(n-1)_{\exp}} \odot u(t)\,dt^{BG} = (n-1)!_{\exp} \odot {}_{BG}\int_a^x \dots (n) \dots {}_{BG}\int_a^x u(t)\,dt^{BG} \dots dt^{BG} \;.$$

Therefore, we get

$${}_{BG}\int_a^x \dots (n) \dots {}_{BG}\int_a^x u(t)\,dt^{BG} \dots dt^{BG} = \frac{e}{(n-1)!_{\exp}}\,{}_{\exp}\!\odot\, {}_{BG}\int_a^x (x \ominus t)^{(n-1)_{\exp}} \odot u(t)\,dt^{BG}$$

and this completes the proof.

Let n^{th} order BG-linear differential equation

$$y^{n_{BG}} \oplus a_1(x) \odot y^{(n-1)_{BG}} \oplus \dots \oplus a_{n-1}(x) \odot y^{BG} \oplus a_n(x) \odot y = f(x) \tag{8}$$

given with the initial conditions

$$y(1) = c_0, y^{BG}(1) = c_1, \dots, y^{(n-1)_{BG}}(1) = c_{n-1} \tag{9}$$

This n^{th} order BG-linear differential equation can be reduced to a BG-Volterra integral equation. Hence the solution of (8)-(9) may be reduced to a solution of some BG-Volterra integral equation.

Taking $y^{n_{BG}} = u(x)$, we can write $\frac{d^{BG}}{dx^{BG}} y^{(n-1)_{BG}}(x) = u(x)$. By BG-integrating both sides of this equality,

$${}_{BG}\int_1^x d^{BG}\left(y^{(n-1)_{BG}}\right) = {}_{BG}\int_1^x u(t)\,dt^{BG}$$

$$y^{(n-1)_{BG}}(x) \ominus y^{(n-1)_{BG}}(1) = {}_{BG}\int_1^x u(t)\,dt^{BG}$$

$$y^{(n-1)_{BG}}(x) = c_{n-1} \oplus {}_{BG}\int_1^x u(t)\,dt^{BG}$$

By proceeding similarly, we find

$$ {}_{BG}\int_1^x d^{BG}\left(y^{(n-2)_{BG}}\right) = {}_{BG}\int_1^x \left(c_{n-1} \oplus {}_{BG}\int_1^x u(t)\,dt^{BG}\right) dt^{BG} $$

$$ y^{(n-2)_{BG}}(x) \ominus y^{(n-2)_{BG}}(1) = {}_{BG}\int_1^x c_{n-1} dt^{BG} \oplus {}_{BG}\int_1^x {}_{BG}\int_1^x u(t)\,dt^{BG} dt^{BG} $$

$$ y^{(n-2)_{BG}}(x) = c_{n-2} \oplus c_{n-1} \odot x \oplus {}_{BG}\int_1^x {}_{BG}\int_1^x u(t)\,dt^{BG} dt^{BG} $$

$\vdots$

$$ y^{BG} = c_1 \oplus c_2 \odot x \oplus \cdots \oplus c_{n-1} \odot x^{(n-2)_{\exp}} \oplus {}_{BG}\int_1^x \cdots (n-1) \cdots {}_{BG}\int_1^x u(t)\,dt^{BG} \cdots dt^{BG}. $$

Therefore, we obtain,

$$ y = c_0 \oplus c_1 \odot x \oplus c_2 \odot x^{2_{\exp}} \oplus \cdots \oplus c_{n-1} \odot x^{(n-1)_{\exp}} \oplus {}_{BG}\int_1^x \cdots (n) \cdots {}_{BG}\int_1^x u(t)\,dt^{BG} \cdots dt^{BG} \ . $$

If we take into account the above expressions, the BG-linear differential equation is written as follows

$$ u(x) \oplus a_1(x) \odot \left(c_{n-1} \oplus {}_{BG}\int_1^x u(t)\,dt^{BG} \right) \oplus a_2(x) \odot \left(c_{n-2} \oplus c_{n-1} \odot x \oplus {}_{BG}\int_1^x {}_{BG}\int_1^x u(t)\,dt^{BG} dt^{BG} \right) \oplus \cdots $$

$$ \oplus a_n(x) \odot \left(c_0 \oplus c_1 \odot x \oplus c_2 \odot x^{2_{\exp}} \oplus \cdots \oplus c_{n-1} \odot x^{(n-1)_{\exp}} \oplus {}_{BG}\int_1^x \cdots (n) \cdots {}_{BG}\int_1^x u(t)\,dt^{BG} \cdots dt^{BG} \right) = f(x) $$

$$ u(x) \oplus a_1(x) \odot {}_{BG}\int_1^x u(t)\,dt^{BG} \oplus a_2(x) \odot {}_{BG}\int_1^x {}_{BG}\int_1^x u(t)\,dt^{BG} dt^{BG} \oplus \cdots \oplus a_n(x) \odot {}_{BG}\int_1^x \cdots (n) \cdots {}_{BG}\int_1^x u(t)\,dt^{BG} \cdots dt^{BG} = $$

$$ f(x) \ominus \Big[\left(a_1(x) \oplus a_2(x) \odot x \cdots \oplus a_n(x) \odot x^{(n-1)_{\exp}} \right) \odot c_{n-1} \oplus \left(a_2(x) \oplus a_3(x) \odot x \oplus \cdots \oplus a_n(x) \odot x^{(n-2)_{\exp}} \right) \odot c_{n-2} \oplus $$

$$ \cdots \oplus a_n(x) \odot c_0 \Big] \tag{10} $$

If we set

$$ a_1(x) \oplus a_2(x) \odot x \cdots \oplus a_n(x) \odot x^{(n-1)_{\exp}} = f_{n-1}(x) $$

$$ a_2(x) \oplus a_3(x) \odot x \oplus \cdots \oplus a_n(x) \odot x^{(n-2)_{\exp}} = f_{n-2}(x) $$

$\vdots$

$$ a_n(x) = f_0(x) $$

and

$$ F(x) = f(x) \ominus f_{n-1}(x) \odot c_{n-1} \ominus f_{n-2}(x) \odot c_{n-2} \ominus \cdots \ominus f_0(x) \odot c_0 $$

Then, one can see that the equation (10) is in the following form:

$$ u(x) \oplus a_1(x) \odot {}_{BG}\int_1^x u(t)\,dt^{BG} \oplus a_2(x) \odot {}_{BG}\int_1^x {}_{BG}\int_1^x u(t)\,dt^{BG} dt^{BG} \oplus \cdots \oplus a_n(x) \odot {}_{BG}\int_1^x \cdots (n) \cdots {}_{BG}\int_1^x u(t)\,dt^{BG} \cdots dt^{BG} $$

$$ = F(x). $$

By using Theorem 9, we get

$$ u(x) \oplus a_1(x) \odot {}_{BG}\int_1^x u(t)\,dt^{BG} \oplus a_2(x) \odot \frac{e}{1!_{\exp}}\,{}_{\exp}\odot\, {}_{BG}\int_a^x (x \ominus t)^{1_{exp}} \odot u(t)\,dt^{BG} \oplus \cdots \oplus a_n(x) \odot \frac{e}{(n-1)!_{\exp}}\,{}_{\exp}\odot\, {}_{BG}\int_a^x (x \ominus t)^{(n-1)_{exp}} \odot u(t)\,dt^{BG} $$

$$ = F(x). $$

If we edit this equation as

$$ u(x) \oplus {}_{BG}\int_1^x u(t) \odot \left[a_1(x) \oplus \frac{e}{1!_{\exp}}\,{}_{\exp}\odot\, a_2(x) \odot (x \ominus t)^{1_{exp}} \oplus \cdots \oplus \frac{e}{(n-1)!_{\exp}}\,{}_{\exp}\odot\, (x \ominus t)^{(n-1)_{exp}} \odot a_n(x) \right] dt^{BG} = F(x) $$

and set

$$ K(x,t) = a_1(x) \oplus \frac{e}{1!_{\exp}}\,{}_{\exp}\odot\, a_2(x) \odot (x \ominus t)^{1_{exp}} \oplus \cdots \oplus \frac{e}{(n-1)!_{\exp}}\,{}_{\exp}\odot\, (x \ominus t)^{(n-1)_{exp}} \odot a_n(x) $$

as the kernel function, then the equation (8) is turned into

$$u(x) \oplus {}_{BG}\int_1^x u(t) \odot K(x,t)\, dt^{BG} = F(x)$$

which is a BG-Volterra integral equation of the second kind.

Example 3. Find BG-Volterra integral equation corresponding to the BG-differential $y^{2_{BG}} \oplus x \odot y^{BG} \oplus y = e^2$ with the initial conditions $y(1) = e$, $y^{BG}(1) = 1$.

Solution. Let $\frac{d^{2_{BG}}}{dx^{2_{BG}}} = y^{2_{BG}} = u(x)$. Since $\frac{d^{2_{BG}}}{dx^{2_{BG}}} = \frac{d^{BG}}{dx^{BG}} y^{BG} = u(x)$, then we write

$${}_{BG}\int_1^x d^{BG}\left(y^{BG}\right) = {}_{BG}\int_1^x u(t)\, dt^{BG}$$

$$y^{BG}(x) \ominus y^{BG}(1) = {}_{BG}\int_1^x u(t)\, dt^{BG}$$

$$y^{BG}(x) = {}_{BG}\int_1^x u(t)\, dt^{BG}$$

Therefore, we find

$${}_{BG}\int_1^x d^{BG} y = {}_{BG}\int_1^x {}_{BG}\int_1^x u(t)\, dt^{BG} dt^{BG}$$

$$y(x) \ominus y(1) = \frac{e}{1!_{\exp}}{}_{\exp}\odot\, {}_{BG}\int_1^x (x \ominus t) \odot u(t)\, dt^{BG}$$

$$y(x) = e \oplus \left(e \odot {}_{BG}\int_1^x (x \ominus t) \odot u(t)\, dt^{BG} \right)$$

If we replace the findings above into the given BG-differential equation, we obtain

$$u(x) \oplus x \odot {}_{BG}\int_1^x u(t)\, dt^{BG} \oplus e \oplus \left(e \odot {}_{BG}\int_1^x (x \ominus t) \odot u(t)\, dt^{BG} \right) = e^2$$

From this, we get BG-Volterra integral equation as $u(x) = e \oplus {}_{BG}\int_1^x \left(x^2 \ominus t\right) \odot u(t)\, dt^{BG}$.

4. Conclusion

In this paper, the Volterra integral equations are defined in the sense of bigeometric calculus by using the concept of bigeometric integral. The Leibniz formula is proved in bigeometric calculus and aid of this the bigeometric Volterra integral equations are converted to bigeometric differential equations. By defining the bigeometric linear differential equations with constant coefficients and variable coefficients, they are converted to bigeometric Volterra equations is proved.

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