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# Subclasses of Bi-Univalent Functions Associated with $q-$Confluent Hypergeometric Distribution Based Upon the Horadam Polynomials

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**Abstract**

In this paper, we introduce new subclasses of analytic and bi-univalent functions connected with a $q$-confluent hypergeometric distribution by using the Horadam polynomials. Furthermore, we find estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses and obtain Fekete-Szegő problem for these subclasses.

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## 1. Introduction

In [23] Srivastava presented and motivated about brief expository overview of the classical $q$-analysis versus the so-called $(p, q)$-analysis with an obviously redundant additional parameter $p$. We also briefly consider several other families of such extensivelyand widely-investigated linear convolution operators as (for example) the Dziok–Srivastava, Srivastava–Wright and Srivastava–Attiya linear convolution operators, together with their extended and generalized versions. The theory of $(p, q)$-analysis has important role in many areas of mathematics and physics. Our usages here of the $q$-calculus and the fractional qcalculus in

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geometric function theory of complex analysis are believed to encourage and motivate significant further developments on these and other related topics (see Srivastava and Karlsson [24, pp. 350–351], Srivastava [21, 22]).
Let $\mathcal{A}$ denote the subclass of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \; z \in \Delta, \tag{1}$$

and, let the function $h \in \mathcal{A}$ is given by

$$h(z) := z + \sum_{k=2}^{\infty} \psi_k z^k \; z \in \Delta. \tag{2}$$

The *Hadamard (or convolution) product* of $f$ and $h$ is defined by

$$(f * h)(z) := z + \sum_{k=2}^{\infty} a_k \psi_k z^k, \; z \in \Delta.$$

**Definition 1.1.** *For $f, g \in \mathcal{A}$, we say that $f$* is subordinate to *$g$, written $f(z) \prec g(z)$, if there exists a* Schwarz function *$w$, which is analytic in $\Delta$, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \Delta$, such that $f(z) = g(w(z))$, $z \in \Delta$. Furthermore, if the function $g$ is univalent in $\Delta$, then we have the following equivalence (see [4, 16]):*

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).$$

The confluent hypergeometric function of the first kind is given by the power series

$$\begin{aligned} F(b;c;z) &= 1 + \frac{b}{c}z + \frac{b}{c}\frac{(b+1)}{(c+1)}\frac{z^2}{2!} + ... \\ &= \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k \ (1)_k} z^k, \quad (b \in \mathbb{C}, \; c \in \mathbb{C} \setminus \{0, -1, -2, ...\}), \end{aligned}$$

where $(b)_k$ is the Pochhammer symbol defined in terms of the Gamma function by

$$(b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = \begin{cases} 1, & \text{if } \; k = 0, \\ b(b+1)...(b+k-1), & \text{if } \; k \in \mathbb{N} = \{1, 2, ...\}. \end{cases}$$

is convergent for all finite values of $z$ (see [20]). It can be written otherwise

$$F(b;c;m) = \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k \ (1)_k} m^k, \quad (b \in \mathbb{C}, \; c \in \mathbb{C} \setminus \{0, -1, -2, ...\}),$$

is convergent for $b, c, m > 0$.
Very recently, Porwal and Kumar [19] introduced the confluent hypergeometric distribution (CHD) whose probability mass function is

$$P(k) = \frac{(b)_k}{(c)_k \ k! F(b;c;m)} m^k, \; (b, c, m > 0, \; k = 0, 1, 2, ...).$$

Porwal [18] introduced a series $\mathcal{I}(b;c;m;z)$ whose coefficients are probabilities of confluent hypergeometric distribution

$$\mathcal{I}(b;c;m;z) = z + \sum_{k=2}^{\infty} \frac{(b)_{k-1} \, m^{k-1}}{(c)_{k-1} \ (k-1)! F(b;c;m)} z^k, \; (b, c, m > 0), \tag{3}$$

and defined a linear operator $\Omega(b;c;m)f : \mathcal{A} \to \mathcal{A}$ as follows

$$\begin{aligned} \Omega(b;c;m)f(z) &= \mathcal{I}(b;c;m;z) * f(z) \\ &= z + \sum_{k=2}^{\infty} \frac{(b)_{k-1}\, m^{k-1}}{(c)_{k-1}\ (k-1)!F(b;c;m)} a_k\ z^k,\ (b,c,m>0)\,. \end{aligned}$$

Srivastava [23] made use of various operators of $q$-calculus and fractional $q$-calculus and recalling the definition and notations. The $q$-shifted factorial is defined for $\lambda, q \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ as follows

$$(\lambda;q)_k = \begin{cases} 1 & k = 0, \\ (1-\lambda)(1-\lambda q)\ldots\left(1-\lambda q^{k-1}\right) & k \in \mathbb{N}. \end{cases}$$

By using the $q$-gamma function $\Gamma_q(z)$, we get

$$\left(q^{\lambda};q\right)_k = \frac{(1-q)^k\ \Gamma_q(\lambda+k)}{\Gamma_q(\lambda)}, \qquad (k \in \mathbb{N}_0)\,,$$

where (see [8])

$$\Gamma_q(z) = (1-q)^{1-z}\,\frac{(q;q)_{\infty}}{(q^z;q)_{\infty}}, \qquad (|q|<1)\,.$$

Also, we note that

$$(\lambda;q)_{\infty} = \prod_{k=0}^{\infty}\left(1-\lambda q^k\right), \qquad (|q|<1)\,,$$

and, the $q$-gamma function $\Gamma_q(z)$ is known

$$\Gamma_q(z+1) = [z]_q\ \Gamma_q(z),$$

where $[k]_q$ denotes the basic $q$-number defined as follows

$$[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & k \in \mathbb{C}, \\ 1 + \sum\limits_{j=1}^{k-1} q^j, & k \in \mathbb{N}. \end{cases} \tag{4}$$

Using the definition formula (4) we have the next two products:
(i) For any non negative integer $k$, the *$q$-shifted factorial* is given by

$$[k]_q! := \begin{cases} 1, & \text{if} \quad k = 0, \\ \prod\limits_{n=1}^{k} [n]_q, & \text{if} \quad k \in \mathbb{N}. \end{cases}$$

(ii) For any positive number $r$, the *$q$-generalized Pochhammer symbol* is defined by

$$[r]_{q,k} := \begin{cases} 1, & \text{if} \quad k = 0, \\ \prod\limits_{n=r}^{r+k-1} [n]_q, & \text{if} \quad k \in \mathbb{N}. \end{cases}$$

It is known in terms of the classical (Euler's) gamma function $\Gamma(z)$, that

$$\Gamma_q(z) \to \Gamma(z) \qquad \text{as } q \to 1^-.$$

Also, we observe that

$$\lim_{q\to 1^-}\left\{\frac{\left(q^{\lambda};q\right)_k}{(1-q)^k}\right\}=(\lambda)_k\,.$$

For $0<q<1$, the *q-derivative operator* [13] (see also [1, 12]) for $\mathcal{I}(b;c;m;z)$ is defined by

$$\begin{aligned} D_q\left(\Omega(b;c;m)f(z)\right) &:= \frac{\Omega(b;c;m)f(z)-\Omega(b;c;m)f(qz)}{z(1-q)} \\ &= 1+\sum_{k=2}^{\infty}[k]_q\frac{(b)_{k-1}\,m^{k-1}}{(c)_{k-1}\;(k-1)!F(b;c;m)}a_k\;z^{k-1},\;(b,c,m>0,\;z\in\Delta)\,, \end{aligned}$$

where

$$[k]_q:=\frac{1-q^k}{1-q}=1+\sum_{j=1}^{k-1}q^j,\qquad [0,q]:=0. \tag{5}$$

For $\lambda>-1$ and $0<q<1$, we defined the linear operator $\mathcal{I}^{\lambda,q}(b;c;m)f:\mathcal{A}\to\mathcal{A}$ by

$$\mathcal{I}^{\lambda,q}(b;c;m)f(z)*\mathcal{N}_{q,\lambda+1}(z)=z\,D_q\left(\Omega(b;c;m)f(z)\right),\;z\in\Delta,$$

where the function $\mathcal{N}_{q,\lambda+1}$ is given by

$$\mathcal{N}_{q,\lambda+1}(z):=z+\sum_{k=2}^{\infty}\frac{[\lambda+1]_{q,k-1}}{[k-1]_q!}z^k,\;z\in\Delta.$$

A simple computation shows that

$$\mathcal{I}^{\lambda,q}(b;c;m)f(z):=z+\sum_{k=2}^{\infty}\psi_k a_k\;z^k\;\;(b,c,m>0,\;\lambda>-1,\;0<q<1,\;z\in\Delta). \tag{6}$$

where

$$\psi_k:=\frac{(b)_{k-1}m^{k-1}[k]_q!}{(c)_{k-1}\;(k-1)!F(b;c;m)[\lambda+1]_{q,k-1}}. \tag{7}$$

From the definition relation (6), we can easily verify that the next relations hold for all $f\in\mathcal{A}$:

(i) $[\lambda+1]_q\,\mathcal{I}^{\lambda,q}(b;c;m)f(z)=[\lambda]_q\mathcal{I}^{\lambda+1,q}(b;c;m)f(z)+q^{\lambda}\,z\;D_q\left(\mathcal{I}^{\lambda+1,q}(b;c;m)f(z)\right),\;z\in\Delta;$ (8)

(ii) $\mathcal{M}^{\lambda}(b;c;m)f(z):=\lim\limits_{q\to 1^-}\mathcal{I}^{\lambda,q}(b;c;m)f(z)=z+\sum\limits_{k=2}^{\infty}\frac{k(b)_{k-1}m^{k-1}}{(c)_{k-1}\;F(b;c;m)(\lambda+1)_{k-1}}a_kz^k,\quad z\in\Delta.$ (9)

**Remark 1.2.** *Putting $b=c$ in the operator $\mathcal{I}^{\lambda,q}(b;c;m)$, we obtain the q-analogue of Poisson operator $I_q^{\lambda,m}$ defined by El-Deeb et al. [7] as follows*

$$I_q^{\lambda,m}f(z):=z+\sum_{k=2}^{\infty}\frac{m^{k-1}}{(k-1)!}e^{-m}\cdot\frac{[k]_q!}{[\lambda+1]_{q,k-1}}a_k\;z^k,\;z\in\Delta. \tag{10}$$

**Remark 1.3.** *The Horadam polynomials $h_n(x)$ are defined by the following recurrence relation (see [10])*

$$h_n(x)=\rho x h_{n-1}(x)+\sigma h_{n-2}(x),\qquad\qquad (x\in\mathbb{R};\;n\in\mathbb{N}=\{1,2,3,...\})\,, \tag{11}$$

*with*

$$h_1(x)=\alpha\qquad \textit{and}\qquad h_2(x)=\beta x,$$

*for some real constants $\alpha,\beta,\rho$ and $\sigma$. The generating function of the Horadam polynomials $h_n(x)$ is given as follows (see [11])*

$$\Upsilon(x,z)=\sum_{n=1}^{\infty}h_n(x)z^{n-1}=\frac{\alpha+(\beta-\alpha\rho)xz}{1-\rho xz-\sigma z^2}. \tag{12}$$

**Remark 1.4.** *By selecting the particular values of $\alpha$, $\beta$, $\rho$ and $\sigma$, the Horadam polynomial $h_n(x)$ reduces to several known polynomials.*
*(i) Fibonacci polynomials $F_n(x)$. If $\alpha = \beta = \rho = \sigma = 1$;*
*(ii) Lucas polynomials $L_n(x)$. If $\alpha = 2$ and $\beta = \rho = \sigma = 1$;*
*(iii) Pell polynomials $P_n(x)$. If $\alpha = \sigma = 1$ and $b = \rho = 2$;*
*(iv) Pell-Lucas polynomials $Q_n(x)$. If $\alpha = \beta = \rho = 2$ and $\sigma = 1$;*
*(v) Chebyshev polynomials $T_n(x)$ of the first kind. If $\alpha = \beta = 1,\ \rho = 2$ and $\sigma = -1$;*
*(vi) Chebyshev polynomials $U_n(x)$ of the second kind. If $\alpha = 1,\ \beta = \rho = 2$ and $\sigma = -1$.*

These polynomials, the families of orthogonal polynomials and other special polynomials, as well as their extensions and generalizations, are potentially important in a variety of disciplines in many branches of science, especially in the mathematical, statistical and physical sciences. For more information associated with these polynomials (see [9, 10, 14, 15]).

The Koebe one-quarter theorem (see [5]) proves that the image of $\Delta$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Therefore, every function $f \in \mathcal{A}$ has an inverse $f^{-1}$ that satisfies

$$f^{-1}(f(z)) = z, \quad (z \in \Delta),$$

and

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \dots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\Delta$ if both $f$ and $f^{-1}$ are univalent in $\Delta$. Let $\Sigma$ denote the class of bi-univalent functions in $\Delta$ given by (1). Note that the following functions $f_1(z) = \frac{z}{1-z}$, $f_2(z) = \frac{1}{2}\log\frac{1+z}{1-z}$, $f_3(z) = -\log(1-z)$, with their corresponding inverses $g_1(w) = \frac{w}{1+w}$, $g_2(w) = \frac{e^{2w}-1}{e^{2w}+1}$, $g_3(w) = \frac{e^w - 1}{e^w}$, respectively, are elements of $\Sigma$ (see [6, 7, 25]). For a brief history and interesting examples in the class $\Sigma$ see, for example [2]. Brannan and Taha [3] (see also [25]) introduced certain subclasses of the bi-univalent functions class $\Sigma$ similar to the familiar subclasses $\mathcal{S}^*(\delta)$ and $\mathcal{K}(\delta)$ of starlike and convex functions of order $\delta$ $(0 \leq \delta < 1)$, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_\Sigma^*(\delta)$ of strongly bi-starlike functions of order $\delta$ $(0 < \delta \leq 1)$, if each of the following conditions is satisfied:

$$f \in \Sigma, \quad \text{with} \quad \left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\delta\pi}{2},\ z \in \Delta,$$

and

$$\left|\arg\frac{zg'(w)}{g(w)}\right| < \frac{\delta\pi}{2},\ w \in \Delta,$$

where the function $g$ is the analytic extension of $f^{-1}$ to $\Delta$, and is given by

$$g(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \dots,\ w \in \Delta. \tag{13}$$

The classes $S_\Sigma^*(\alpha)$ and $K_\Sigma(\alpha)$ of bi-starlike functions of order $\alpha$ and bi-convex functions of order $\alpha$ $(0 < \alpha \leq 1)$, corresponding to the function classes $S^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S_\Sigma^*(\alpha)$ and $K_\Sigma(\alpha)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [3] and [25]).

The object of the present paper is to introduce new subclasses of the function class $\Sigma$ involving the $q-$confluent Hypergeometric function connected with Horadam polynomials $h_n(x)$ that generalize the previous defined classes, and find estimates on the coefficients $|a_2|$, and $|a_3|$ for functions in these new subclasses of the function class $\Sigma$.

**Definition 1.5.** *Let $0 \leq \gamma \leq 1,\ \eta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\},\ b,c,m > 0,\ \lambda > -1,\ 0 < q < 1$ and $x \in \mathbb{R}$, then $f \in \Sigma$ is said to be in the class $\mathcal{K}_{\Sigma}^{\lambda,q}(\eta,\gamma,b,c,m,x)$ if the following conditions are satisfied:*

$$1 + \frac{1}{\eta}\left(\frac{\gamma z^2\ \left(\mathcal{I}^{\lambda,q}(b;c;m)f(z)\right)'' + z\left(\mathcal{I}^{\lambda,q}(b;c;m)f(z)\right)'}{\gamma z\left(\mathcal{I}^{\lambda,q}(b;c;m)f(z)\right)' + (1-\gamma)\mathcal{I}^{\lambda,q}(b;c;m)f(z)} - 1\right) \prec \Upsilon(x,z) + 1 - \alpha, \tag{14}$$

*and*

$$1 + \frac{1}{\eta}\left(\frac{\gamma w^2\ \left(\mathcal{I}^{\lambda,q}(b;c;m)g(w)\right)'' + w\left(\mathcal{I}^{\lambda,q}(b;c;m)g(w)\right)'}{\gamma w\left(\mathcal{I}^{\lambda,q}(b;c;m)g(w)\right)' + (1-\gamma)\mathcal{I}^{\lambda,q}(b;c;m)g(w)} - 1\right) \prec \Upsilon(x,w) + 1 - \alpha, \tag{15}$$

*where $\alpha$ is real constant and the function $g$ is the analytic extension of $f^{-1}$ to $\Delta$, and is given by (13).*

**Remark 1.6.** *(i) For $q \to 1^-$ we obtain that $\lim\limits_{q\to 1^-} \mathcal{K}_{\Sigma}^{\lambda,q}(\eta,\gamma,b,c,m,x) =: \mathcal{H}_{\Sigma}^{\lambda}(\eta,\gamma,b,c,m,x)$, where $M_{\Sigma}^{\lambda}(\eta,\gamma,b,c,m,x)$ represents the functions $f \in \Sigma$ that satisfies (14) and (15) for $\mathcal{I}^{\lambda,q}(b;c;m)$ replaced with $\mathcal{M}^{\lambda}(b;c;m)$ (see (9)).*

*(ii) For $b = c$, we obtain the class $R_{\Sigma}^{\lambda,q}(\eta,\gamma,m,x)$, that represents the functions $f \in \Sigma$ that satisfies (14) and (15) for $\mathcal{I}^{\lambda,q}(b;c;m)$ replaced with $I_q^{\lambda,m}$ (see (10)).*

**Lemma 1.7.** *[17, p. 172] If $w$ is a Schwarz function, so that $w(z) = \sum\limits_{k=1}^{\infty} p_k z^k$, $z \in \Delta$, then*

$$|p_1| \leq 1, \quad |p_k| \leq 1 - |p_1|^2,\ k \geq 1.$$

## 2. Coefficient bounds for the function class $\mathcal{K}_{\Sigma}^{\lambda,q}(\eta,\gamma,b,c,m,x)$

Unless otherwise mentioned, we shall assume in the reminder of this paper that $0 \leq \gamma \leq 1,\ \eta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\},\ b,c,m > 0,\ \lambda > -1,\ 0 < q < 1$ and $x \in \mathbb{R}$, the powers are understood as principle values.

**Theorem 2.1.** *Let the function $f$ given by (1) belongs to the class $\mathcal{K}_{\Sigma}^{\lambda,q}(\eta,\gamma,b,c,m,x)$, then*

$$|a_2| \leq \frac{|\beta\eta x|\sqrt{|\beta x|}}{\sqrt{\left|\left(\left[2\eta\beta^2x^2(1+2\gamma)\psi_3 - (\gamma+1)^2(\eta\beta+\rho)\psi_2^2\right]\beta x^2 - \sigma\alpha(\gamma+1)^2\psi_2^2\right)\right|}},$$

*and*

$$|a_3| \leq \frac{|\eta|\,|\beta x|}{2(2\gamma+1)\psi_3} + \frac{|\eta|^2(\beta x)^2}{(\gamma+1)^2\psi_2^2},$$

*where $\psi_k$, $k \in \{2,3\}$, are given by (7).*

*Proof.* Let $f \in \mathcal{K}_{\Sigma}^{\lambda,q}(\eta,\gamma,b,c,m,x)$. Then there exist $U$ and $V$, two analytic functions in $\Delta$ with $U(0) = V(0) = 0$, and $|U(z)| < 1$, $|V(w)| < 1$ for all $z,w \in \Delta$, given by

$$U(z) = \sum_{k=1}^{\infty} u_k z^k \text{ and } V(w) = \sum_{k=1}^{\infty} v_k w^k, \qquad z,w \in \Delta,$$

from Lemma 1.7 we have

$$|u_k| \leq 1 \text{ and } |v_k| \leq 1,\ k \in \mathbb{N}. \tag{16}$$

From (14) and (15), we have

$$\frac{1}{\eta}\left(\frac{\gamma z^2\ \left(\mathcal{I}^{\lambda,q}(b;c;m)f(z)\right)'' + z\left(\mathcal{I}^{\lambda,q}(b;c;m)f(z)\right)'}{\gamma z\left(\mathcal{I}^{\lambda,q}(b;c;m)f(z)\right)' + (1-\gamma)\mathcal{I}^{\lambda,q}(b;c;m)f(z)} - 1\right) = \Upsilon(x,U(z)) - \alpha, \tag{17}$$

and

$$\frac{1}{\eta}\left(\frac{\gamma w^2\ \left(\mathcal{I}^{\lambda,q}(b;c;m)g(w)\right)'' + w\left(\mathcal{I}^{\lambda,q}(b;c;m)g(w)\right)'}{\gamma w\left(\mathcal{I}^{\lambda,q}(b;c;m)g(w)\right)' + (1-\gamma)\mathcal{I}^{\lambda,q}(b;c;m)g(w)} - 1\right) = \Upsilon(x,V(w)) - \alpha. \tag{18}$$

Since

$$\begin{aligned}&\frac{1}{\eta}\left(\frac{\gamma z^2\ \left(\mathcal{I}^{\lambda,q}(b;c;m)f(z)\right)'' + z\left(\mathcal{I}^{\lambda,q}(b;c;m)f(z)\right)'}{\gamma z\left(\mathcal{I}^{\lambda,q}(b;c;m)f(z)\right)' + (1-\gamma)\mathcal{I}^{\lambda,q}(b;c;m)f(z)} - 1\right)\\ =\ &\frac{1}{\eta}\left[(\gamma+1)\psi_2 a_2 z + \left[2(2\gamma+1)\psi_3 a_3 - (\gamma+1)^2\psi_2^2 a_2^2\right]z^2 + \dots\right],\end{aligned}$$

$$\begin{aligned}&\frac{1}{\eta}\left(\frac{\gamma w^2\ \left(\mathcal{I}^{\lambda,q}(b;c;m)g(w)\right)'' + w\left(\mathcal{I}^{\lambda,q}(b;c;m)g(w)\right)'}{\gamma w\left(\mathcal{I}^{\lambda,q}(b;c;m)g(w)\right)' + (1-\gamma)\mathcal{I}^{\lambda,q}(b;c;m)g(w)} - 1\right)\\ =\ &\frac{1}{\eta}\left[-(\gamma+1)\psi_2 a_2 w + \left[2(2\gamma+1)\psi_3\left(2a_2^2 - a_3\right) - (\gamma+1)^2\psi_2^2 a_2^2\right]w^2 + \dots\right],\end{aligned}$$

and

$$\begin{aligned}\Upsilon(x,U(z)) - \alpha &= h_2(x)u_1 z + \left(h_2(x)u_2 + h_3(x)u_1^2\right)z^2 + \dots,\\ \Upsilon(x,V(w)) - \alpha &= h_2(x)v_1 w + \left(h_3(x)v_2 + h_3(x)v_1^2\right)w^2 + \dots.\end{aligned}$$

Now, equating the corresponding coefficients of $z$ and $w$ in (17) and (18), we get

$$\frac{(\gamma+1)}{\eta}\psi_2 a_2 = h_2(x)u_1, \tag{19}$$

$$\frac{1}{\eta}\left[2(1+2\gamma)\psi_3 a_3 - (\gamma+1)^2\psi_2^2 a_2^2\right] = h_2(x)u_2 + h_3(x)u_1^2, \tag{20}$$

$$-\frac{(\gamma+1)}{\eta}\psi_2 a_2 = h_2(x)v_1, \tag{21}$$

$$\frac{1}{\eta}\left[2(1+2\gamma)\psi_3\left(2a_2^2 - a_3\right) - (\gamma+1)^2\psi_1^2 a_2^2\right] = h_2(x)v_2 + h_3(x)v_1^2. \tag{22}$$

From (19) and (21), we obtain

$$u_1 = -v_1. \tag{23}$$

If we square (19) and (21), then adding the new relations wehave

$$\frac{2(\gamma+1)^2}{\eta^2}a_2^2\psi_2^2 = h_2^2(x)\left(u_1^2 + v_1^2\right), \tag{24}$$

adding (20) and (22) we have

$$\frac{2}{\eta}\left[2(1+2\gamma)\psi_3 - (\gamma+1)^2\psi_2^2\right]a_2^2 = h_2(x)\left(u_2+v_2\right) + h_3(x)\left(u_1^2+v_1^2\right).$$

We can rewrite (24) as

$$u_1^2 + v_1^2 = \frac{2(\gamma+1)^2}{\eta^2 h_2^2(x)}a_2^2\psi_2^2.$$

Using the above equation, we get

$$2\left[2\eta(1+2\gamma)h_2^2(x)\psi_3 - (\gamma+1)^2\left(\eta h_2^2(x) + h_3(x)\right)\psi_2^2\right]a_2^2 = \eta^2 h_2^3(x)\left(u_2+v_2\right),$$

it follows that

$$a_2^2 = \frac{\eta^2 h_2^3(x)\left(u_2+v_2\right)}{2\left[2\eta(1+2\gamma)h_2^2(x)\psi_3 - (\gamma+1)^2\left(\eta h_2^2(x) + h_3(x)\right)\psi_2^2\right]}. \tag{25}$$

Then taking the absolute value to the above equation and from (11) and (16), we obtain

$$|a_2| \leq \frac{|\eta|\,|\beta x|\sqrt{|\beta x|}}{\sqrt{\left|\left(\left[2\eta\beta^2x^2(1+2\gamma)\psi_3-(\gamma+1)^2\left(\eta\beta+\rho\right)\psi_1^2\right]\beta x^2-\sigma\alpha(\gamma+1)^2\psi_2^2\right)\right|}},$$

which gives the bound for $|a_2|$ as we asserted in our theorem.

Also to find the bound for $|a_3|$, if we subtract (22) from (20), we find that

$$\frac{4}{\eta}(1+2\gamma)\psi_3\left(a_3-a_2^2\right)=\left[h_2(x)\left(u_2-v_2\right)+h_3(x)\left(u_1^2-v_1^2\right)\right]. \tag{26}$$

Form (26), (23) and (24), we obtain

$$a_3=\frac{\eta h_2(x)\left(u_2-v_2\right)}{4(1+2\gamma)\psi_3}+\frac{\eta^2h_2^2(x)\left(u_1^2+v_1^2\right)}{2\left(\gamma+1\right)^2\psi_2^2}. \tag{27}$$

Using (11) and (16), we get

$$|a_3| \leq \frac{|\eta|\,|\beta x|}{2(2\gamma+1)\psi_3}+\frac{|\eta|^2\left(\beta x\right)^2}{\left(\gamma+1\right)^2\psi_2^2}.$$

□

Putting $q\to 1^-$ in Theorem 2.1 we obtain the following corollary:

**Corollary 2.2.** *If the function $f$ given by (1) belongs to the class $\mathcal{H}_\Sigma^\lambda(\eta,\gamma,b,c,m,x)$, and $\eta\in\mathbb{C}^*$, then*

$$|a_2| \leq \frac{|\beta\eta x|\sqrt{|\beta x|}}{\sqrt{\left|\left(\left[\frac{6\eta(\beta m x)^2(1+2\gamma)(b)_2}{(c)_2(\lambda+1)_2F(b;c;m)}-\frac{4(bm(\gamma+1))^2(\eta\beta+\rho)}{(c(\lambda+1)F(b;c;m))^2}\right]\beta x^2-\frac{4\sigma\alpha(bm(\gamma+1))^2}{(c(\lambda+1)F(b;c;m))^2}\right)\right|}},$$

*and*

$$|a_3| \leq \frac{|\eta|\,|\beta x|\,(c)_2\,(\lambda+1)_2\,F\left(b;c;m\right)}{6m^2(2\gamma+1)(b)_2}+\frac{|\eta|^2\left(\beta x c(\lambda+1)F\left(b;c;m\right)\right)^2}{4\left(bm(\gamma+1)\right)^2}.$$

Putting $b=c$ in Theorem 2.1 we obtain the following corollary:

**Corollary 2.3.** *If the function $f$ given by (1) belongs to the class $R_\Sigma^{\lambda,q}(\eta,\gamma,m,x)$, and $\eta\in\mathbb{C}^*$, then*

$$|a_2| \leq \frac{|\beta\eta x|\sqrt{|\beta x|}}{\sqrt{\left|\left(\left[\frac{\eta(m\beta x)^2(1+2\gamma)e^{-m}[3]_q!}{[\lambda+1]_{q,2}}-\frac{\left(me^{-m}\ [2]_q!(\gamma+1)\right)^2(\eta\beta+\rho)}{[\lambda+1]_q^2}\right]\beta x^2-\frac{\sigma\alpha\left(me^{-m}\ [2]_q!(\gamma+1)\right)^2}{[\lambda+1]_q^2}\right)\right|}},$$

*and*

$$|a_3| \leq \frac{|\eta|\,|\beta x|\,[\lambda+1]_{q,2}}{m^2(2\gamma+1)e^{-m}\ [3]_q!}+\frac{|\eta|^2\left(\beta x\ \ [\lambda+1]_q\right)^2}{\left(me^{-m}\ \ [2]_q!\,(\gamma+1)\right)^2}.$$

## 3. Fekete-Szegő problem for the function class $\mathcal{K}_\Sigma^{\lambda,q}(\eta,\gamma,b,c,m,x)$

**Theorem 3.1.** *If the function $f$ given by (1) belongs to the class $\mathcal{K}_\Sigma^{\lambda,q}(\eta,\gamma,b,c,m,x)$, and $\eta\in\mathbb{C}^*$, then*

$$\left|a_3-\mu a_2^2\right| \leq |\eta||\beta x|\left(|M+N|+|M-N|\right), \tag{28}$$

*where*

$$M=\frac{(1-\mu)\,\eta(\beta x)^2}{2\left[\left(2\eta(2\gamma+1)\psi_3-(\gamma+1)^2\left(\eta\beta-2\rho\right)\psi_2^2\right)\beta x^2-\left(\gamma+1\right)^2\psi_2^2\sigma\alpha\right]}, \tag{29}$$

*and*

$$N=\frac{1}{4(2\gamma+1)\psi_3},$$

*where* $\mu\in\mathbb{C}$, *and* $\psi_k$, $k\in\{2,3\}$, *are given by* (7).

*Proof.* If $f\in\mathcal{K}_{\Sigma}^{\lambda,q}(\eta,\gamma,b,c,m,x)$. As in the proof of Theorem 2.1, from (23) and (26), we have

$$a_3-a_2^2=\frac{\eta h_2\left(u_2-v_2\right)}{4(2\gamma+1)\psi_3}, \tag{30}$$

and multiplying (25) by $(1-\mu)$ we get

$$(1-\mu)\,a_2^2=\frac{(1-\mu)\,\eta^2h_2^3\left(u_2+v_2\right)}{2\left[\left(2\eta(2\gamma+1)\psi_3-\eta\psi_2^2(\lambda+1)^2\right)h_2^2-\left(\lambda+1\right)^2\psi_2^2h_3\right]}. \tag{31}$$

Summing (30) and (31) leads to

$$a_3-\mu a_2^2=\eta h_2\left[\left(M+N\right)u_2+\left(M-N\right)v_2\right], \tag{32}$$

where $M$ and $N$ are given by (29), and taking the absolute value of (32), from (16) we obtain the inequality (28). $\square$

**Remark 3.2.** *A simple computation shows that the inequality* $|M|\leq N$ *is equivalent to*

$$|\mu-1|\leq\left|\frac{2\eta\beta x\left[\left(2\eta(2\gamma+1)\psi_3-(\gamma+1)^2\left(\eta\beta-2\rho\right)\psi_2^2\right)\beta x^2-\left(\gamma+1\right)^2\psi_2^2\sigma\alpha\right]}{4\left(2\gamma+1\right)\eta^2\left(\rho\beta x^2+\alpha\sigma\right)\psi_3}\right|,$$

*therefore, from Theorem 3.1 we get the next result:*

*If the function* $f$ *given by* (1) *belongs to the class* $\mathcal{K}_{\Sigma}^{\lambda,q}(\eta,\gamma,b,c,m,x)$, *and* $\eta\in\mathbb{C}^*$, *then*

$$\left|a_3-\mu a_2^2\right|\leq\frac{|\eta|\,|\beta x|}{2(2\gamma+1)\psi_3},$$

*where* $\mu\in\mathbb{C}$, *with*

$$|\mu-1|\leq\left|\frac{2\eta\beta x\left[\left(2\eta(2\gamma+1)\psi_3-(\gamma+1)^2\left(\eta\beta-2\rho\right)\psi_2^2\right)\beta x^2-\left(\gamma+1\right)^2\psi_2^2\sigma\alpha\right]}{4\left(2\gamma+1\right)\eta^2\left(\rho\beta x^2+\alpha\sigma\right)\psi_3}\right|,$$

*and* $\psi_k$, $k\in\{2,3\}$, *are given by* (7).

Putting $q\to 1^-$ in Theorem 3.1 we obtain the following corollary:

**Corollary 3.3.** *If the function* $f$ *given by* (1) *belongs to the class* $\mathcal{H}_{\Sigma}^{\lambda}(\eta,\gamma,b,c,m,x)$, *and* $\eta\in\mathbb{C}^*$, *then*

$$\left|a_3-\mu a_2^2\right|\leq|\eta||\beta x|\left(|M+N|+|M-N|\right),$$

*where*

$$M=\frac{(1-\mu)\eta(\beta x)^2}{2\left[\left(\frac{6\eta m^2(2\gamma+1)(b)_2}{(c)_2(\lambda+1)_2F(b;c;m)}-4\left(\frac{bm(\gamma+1)}{c(\lambda+1)F(b;c;m)}\right)^2(\eta\beta-2\rho)\right)\beta x^2-\left(\frac{bm(\gamma+1)}{c(\lambda+1)F(b;c;m)}\right)^2\sigma\alpha\right]},$$

*and*

$$N=\frac{(c)_2\left(\lambda+1\right)_2F(b;c;m)}{12m^2(2\gamma+1)\left(b\right)_2},$$

*where* $\mu\in\mathbb{C}$.

Putting $b = c$ in Theorem 3.1 we obtain the following corollary:

**Corollary 3.4.** *If the function $f$ given by (1) belongs to the class $R_{\Sigma}^{\lambda,q}(\eta,\gamma,m,x)$, and $\eta \in \mathbb{C}^*$, then*

$$\left|a_3 - \mu a_2^2\right| \leq |\eta||\beta x| \left(|M+N| + |M-N|\right),$$

*where*

$$M = \frac{(1-\mu)\eta(\beta x)^2}{2\left[\left(\frac{\eta m^2(2\gamma+1)e^{-m}[3]_q!}{[\lambda+1]_{q,2}} - \left(\frac{me^{-m}(\gamma+1)[2]_q!}{[\lambda+1]_q}\right)^2(\eta\beta-2\rho)\right)\beta x^2 - \left(\frac{me^{-m}(\gamma+1)[2]_q!}{[\lambda+1]_q}\right)^2\sigma\alpha\right]},$$

*and*

$$N = \frac{[\lambda+1]_{q,2}}{2m^2(2\gamma+1)e^{-m}[3]_q!},$$

*where $\mu \in \mathbb{C}$.*

For $\eta = 1$ and $\gamma = 1$. Therefore, from Theorem 2.1 and Theorem 3.1

**Example 3.5.** *Let the function $f$ given by (1) belongs to the class $\mathcal{K}_{\Sigma}^{\lambda,q}(1,1,b,c,m,x)$, then*

$$|a_2| \leq \frac{|\beta x|\sqrt{|\beta x|}}{\sqrt{\left|\left(\left[6\beta^2x^2\psi_3 - 4(\beta+\rho)\psi_2^2\right]\beta x^2 - 4\sigma\alpha\psi_2^2\right)\right|}},$$

$$|a_3| \leq \frac{|\beta x|}{6\psi_3} + \frac{(\beta x)^2}{4\psi_2^2},$$

*and*

$$\left|a_3 - \mu a_2^2\right| \leq |\beta x| \left(|M+N| + |M-N|\right),$$

*with*

$$M = \frac{(1-\mu)(\beta x)^2}{2\left[\left(2\psi_3 - (\eta\beta-2\rho)\psi_2^2\right)\beta x^2 - \psi_2^2\sigma\alpha\right]} \quad \textit{and} \quad N = \frac{1}{12\psi_3},$$

*where $\psi_k$, $k \in \{2,3\}$, are given by (7).*

For $\eta = 1$ and $\gamma = 0$. Therefore, from Theorem 2.1 and Theorem 3.1

**Example 3.6.** *Let the function $f$ given by (1) belongs to the class $\mathcal{K}_{\Sigma}^{\lambda,q}(1,0,b,c,m,x)$, then*

$$|a_2| \leq \frac{|\beta x|\sqrt{|\beta x|}}{\sqrt{\left|\left(\left[2\beta^2x^2\psi_3 - (\beta+\rho)\psi_2^2\right]\beta x^2 - \sigma\alpha\psi_2^2\right)\right|}},$$

$$|a_3| \leq \frac{|\beta x|}{2\psi_3} + \frac{(\beta x)^2}{\psi_2^2},$$

*and*

$$\left|a_3 - \mu a_2^2\right| \leq |\beta x| \left(|M+N| + |M-N|\right),$$

*with*

$$M = \frac{(1-\mu)(\beta x)^2}{2\left[\left(6\psi_3 - 4(\eta\beta-2\rho)\psi_2^2\right)\beta x^2 - 4\psi_2^2\sigma\alpha\right]} \quad \textit{and} \quad N = \frac{1}{4\psi_3},$$

*where $\psi_k$, $k \in \{2,3\}$, are given by (7).*

For $\eta = \zeta\cos\theta e^{i\theta}$ $\left(0 < \zeta \leq 1,\ |\theta| < \frac{\pi}{2}\right)$. Therefore, from Theorem 2.1 and Theorem 3.1

**Example 3.7.** *Let the function $f$ given by* (1) *belongs to the class* $\mathcal{K}_{\Sigma}^{\lambda,q}\left(\zeta\cos\theta e^{i\theta},\gamma,b,c,m,x\right)$*, then*

$$|a_2| \leq \frac{|\beta x|\sqrt{|\beta x|}\zeta\cos\theta}{\sqrt{\left|\left(\left[2\zeta\cos\theta e^{i\theta}\beta^2x^2(1+2\gamma)\psi_3-(\gamma+1)^2\left(\beta\zeta\cos\theta e^{i\theta}+\rho\right)\psi_2^2\right]\beta x^2-\sigma\alpha(\gamma+1)^2\psi_2^2\right)\right|}},$$

$$|a_3| \leq \frac{|\beta x|\zeta\cos\theta}{2(2\gamma+1)\psi_3}+\frac{(\beta\zeta x\cos\theta)^2}{(\gamma+1)^2\psi_2^2},$$

*and*

$$\left|a_3-\mu a_2^2\right| \leq |\eta||\beta x|\left(|M+N|+|M-N|\right),$$

*where*

$$M=\frac{(1-\mu)(\beta x)^2\zeta\cos\theta e^{i\theta}}{2\left[\left(2\zeta\cos\theta e^{i\theta}(2\gamma+1)\psi_3-(\gamma+1)^2\left(\beta\zeta\cos\theta e^{i\theta}-2\rho\right)\psi_2^2\right)\beta x^2-(\gamma+1)^2\psi_2^2\sigma\alpha\right]},\quad N=\frac{1}{4(2\gamma+1)\psi_3},$$

*where $\psi_k$, $k\in\{2,3\}$, are given by* (7).

**Remark 3.8.** *We mention that all the above estimations for the coefficients $|a_2|$, $|a_3|$, and Fekete-Szegő problem for the function class $\mathcal{K}_{\Sigma}^{\lambda,q}(\eta,\gamma,b,c,m,x)$ are not sharp. To find the sharp upper bounds for the above functionals remains an interesting open problem, as well as those for $|a_n|$, $n\geq 4$.*

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