https://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 70, Number 2, Pages 773–784 (2021)
DOI: 10.31801/cfsuasmas.768848
ISSN 1303–5991 E-ISSN 2618–6470

Received by the editors: July 15, 2020; Accepted: April 9, 2021

SOME PROPERTIES OF CONVOLUTION IN SYMMETRIC SPACES AND APPROXIMATE IDENTITY

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Abstract. This paper deals with the symmetric space of functions and its subspace where continuous functions are dense is considered. Main properties of convolution which plays a vital role in harmonic analysis, as in other areas of mathematics are established in this space. Following the classical case, it is proved that the convolution can be approximated by linear combinations of shifts in a subspace of the considered space. An approximate identity for the convolution is also considered in that subspace.

1. Introduction

Convolution operation plays a vital role in harmonic analysis, as in other areas of mathematics. This is mainly due to the fact that many key operators like Hilbert transform, Poisson integrals, Dirichlet integrals, different types of potentials including Riesz potential, singular integrals, etc are expressed in terms of convolution. Involved in the above operators, convolution operation plays a key role also in approximation theory. Therefore, to have knowledge of basic properties of convolution in various Banach function spaces is very important and useful in the study of the problems of harmonic analysis, approximation theory, theory of partial differential equations, etc.

Recent years have seen an increased interest in different function spaces, such as Lebesgue spaces with variable summability index, Orlicz spaces, Morrey spaces, grand-Lebesgue spaces, etc. Some problems of harmonic analysis and approximation theory have been considered in [1-14]. Basicity of the classical exponential system, as well as its perturbations in the subspaces of Morrey space of functions defined on $[-\pi, \pi]$ was investigated in [6, 11, 13, 15] by the method of boundary

2020 *Mathematics Subject Classification.* 46E30; 30E25.
Keywords and phrases. Symmetric spaces, convolution, approximate identity.
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value problems for analytic functions on a complex domain. In [12, 16, 23], an analogue of the classical Young inequality and some properties of the convolution of periodic functions belonging to Morrey type spaces have been obtained. In [12], it was proved that the convolution in the subspace of Morrey space can be approximated by finite linear combinations of shifts. In the same work, the validity of classical facts about approximate identities was also proved in Morrey space. Note that the spaces considered in the above works are all Banach function spaces (see, e.g., [17;18]). Moreover, all of them, except for Lebesgue spaces with variable summability index, are symmetric. Therefore, a question naturally arises: do the similar results hold for symmetric spaces? Some analogues of Young inequality for some symmetric spaces have been obtained in [20-22].

In this work, we consider a symmetric space of functions and its subspace where continuous functions are dense. We establish main properties of convolution in this space. We prove that the convolution can be approximated by the linear combinations of shifts in a subspace of this space. Approximate identity is also considered in that subspace.

2. Needful information

We will use the following standard notations and concepts. $R_+ = (0, +\infty)$; $\chi_M(\cdot)$ is the characteristic function of the set M; R is the set of real numbers; C is the complex plane; $\omega = \{z \in C : |z| < 1\}$ is a unit disk in C; $\gamma = \partial\omega$ is a unit circle; $\bar{M}$ is the closure of the set M with respect to appropriate norm; $(\bar{\cdot})$ is the complex conjugate. By $[X]$ we denote the algebra of linear bounded operators acting in a Banach space X.

We will need some concepts and facts from the theory of Banach function spaces (see e.g. [24;25]). Let $(R;\mu)$ be a measure space, and M^+ be the cone of μ-measurable functions on R whose values lie in $[0, +\infty]$. Denote the characteristic function of a μ-measurable subset E of R by χ_E .

Definition 1. *A mapping $\rho : M^+ \to [0, +\infty]$ is called a Banach function norm (or simply a function norm) if, for all $f, g, f_n, n \in N in\ M^+$, for all constants $a \geq 0$ and for all μ-measurable subsets $E \subset R$, the following properties hold:*

(P1)$\rho(f) = 0 \Leftrightarrow f = 0$ μ-a.e.; $\rho(af) = a\rho(f)$; $\rho(f + g) \leq \rho(f) + \rho(g)$;

(P2) $0 \leq g \leq f \mu$-a.e. $\Rightarrow \rho(g) \leq \rho(f)$;

(P3) $0 \leq f_n \uparrow f \mu$-a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$;

(P4) $\mu(E) < +\infty \Rightarrow \rho(\chi_E) < +\infty$;

(P5) $\mu(E) < +\infty \Rightarrow \int_E f d\mu \leq C_E \rho(f)$, for some constant $C_E : 0 < C_E < +\infty$ depending on E and ρ, but independent of f .

Let M denote the collection of all extended scalar-valued (real or complex) μ-measurable functions and $M_0 \subset M$ denote the subclass of functions that are finite μ-a.e.

Definition 2. *Let ρ be a function norm. The collection $X = X(\rho)$ of all functions f in M for which $\rho(|f|) < +\infty$ is called a Banach function space. For each $f \in X$, define $\|f\|_X = \rho(|f|)$.*

The following theorem is true.

Theorem 3. *Let ρ be a function norm and let $X = X(\rho)$ and $\|\cdot\|_X$ be as above. Then under the natural vector space operations, $(X; \|\cdot\|_X)$ is a normed linear space for which the inclusions*

$$M_s \subset X \subset M_0$$

hold, where M_s is the set of μ-simple functions. In particular, if $f_n \to f$ in X, then $f_n \to f$ in measure on sets of finite measure, and hence some subsequence converges pointwise μ-a.e. to f.

Let

$$\rho'(g) = \sup\left\{\int_\gamma f(\tau)\, g(\tau)\, |dt| : f \in M^+; \rho(f) \le 1\right\}, \forall g \in M^+.$$

A space

$$X' = \{g \in M : \rho'(|g|) < +\infty\},$$

is called an associate space (Kothe dual) of X.

The functions $f; g \in M_0$ are called equimeasurable if

$$|\{\tau \in \gamma : |f(\tau)| > \lambda\}| = |\{\tau \in \gamma : |g(\tau)| > \lambda\}|, \forall \lambda \ge 0.$$

Banach function norm $\rho : M^+ \to [0, \infty]$ is called rearrangement invariant if for arbitrary equimeasurable functions $f; g \in M_0^+$ the relation $\rho(f) = \rho(g)$ holds. In this case, Banach function space X with the norm $\|\cdot\|_X = \rho(|\cdot|)$ is said to be rearrangement invariant function space (r.i.s. for short). Classical Lebesgue, Orlicz, Lorentz, Lorentz-Orlicz spaces are r.i.s.

Definition 4. *Let X be a Banach function space. The closure of the set of simple functions M_s in X is denoted by X_b.*

To obtain our main results, we will significantly use the following fact from the monograph [17, p.13].

Recall that a closed linear subspace B of the dual space X^* of a Banach space X is said to be norm-fundamental if

$$\|f\|_X = \sup\{|L(f)| : L \in B \wedge \|L\|_{X^*} \le 1\}$$

for every $f \in X$. Thus, B is norm-fundamental if it contains sufficiently many functionals to reproduce the norm of every element of X. The following theorem is true.

Theorem 5. *The associate space X' of a Banach function space X is canonically isometrically isomorphic to a closed norm-fundamental subspace of the Banach space X^* of X, i.e.*

$$\|f\|_X = \sup_{g\in X'} \left\{ \left| \int_{-\pi}^{\pi} fg dt \right| : \|g\|_{X'} \le 1 \right\}, \forall f \in X.$$

In the sequel, we will assume that all the considered functions are defined on $[-\pi, \pi]$ and periodically continued to the whole real axis. By T_s we will denote the shift operator, i.e. $(T_s f)(x) = f(x+s)$, $\forall s; x \in (-\pi, \pi]$.

We will also use the following lemma of [17, p.157].

Lemma 6. *Let X be a r.i.s. on γ and X_b be the closure of simple functions in X. The following assertions are equivalent:*

(1) *X_b is the closure of continuous functions;*

(2) *translation is continuous in X_b, that is*

$$\lim_{s\to 0} \|T_s f - f\|_X = 0, \forall f \in X_b;$$

(3) *X_b is the closure in X of trigonometric polynomials.*

3. Main results

3.1. **Convolution.** Let X be a Banach function space with the norm $\|\cdot\|_X$ invariant with respect to shift on $[-\pi, \pi]$ (we will assume that the functions from X and X' are periodically continued to the whole axis R with period 2π). We will call such space a norm-invariant space for short. Let $f \in X$ and $g \in X'$. Consider the convolution

$$(f * g)(x) = \int_{-\pi}^{\pi} f(x-y)\, g(y)\, dy, x \in [-\pi, \pi].$$

As $X \subset L_1$ and $X' \subset L_1$, the existence of the convolution $(f * g)(x)$ a.e. $x \in [-\pi, \pi]$ is beyond any doubt. Applying Hölder's inequality, we obtain

$$|(f * g)(x)| \le \|f(x - \cdot)\|_X \|g\|_{X'} = \|f\|_X \|g\|_{X'}, \quad \text{a.e. } x \in (-\pi, \pi).$$

Consequently,

$$\|f * g\|_\infty \le \|f\|_X \|g\|_{X'}. \tag{1}$$

Let T_δ be a shift operator, i.e. $(T_\delta f)(x) = f(\delta + x)$, $x \in [-\pi, \pi]$. It is not difficult to see that

$$T_\delta (f * g)(x) = (f * g)(x+\delta) = \int_{-\pi}^{\pi} f(x+\delta-t)\, g(t) dt = (T_\delta f * g)(x).$$

In view of the periodicity of the functions f and g, we also have

$$T_\delta (f * g)(x) = \int_{-\pi}^{\pi} f(x+\delta-t)\, g(t)\, dt = |t - \delta = \tau| = \int_{-\pi-\delta}^{\pi-\delta} f(x-\tau)\, g(\delta+\tau)\, d\tau =$$

$$= \int_{-\pi}^{\pi} f(x-\tau)(T_\delta g)(\tau)\,d\tau = (f * T_\delta g)(x).$$

Denote by $X_s\,(X'_s)$ the subspace of functions from X (from X') whose shifts are continuous in X (in X'). Applying inequality (1), we obtain

$$\|T_\delta(f*g) - f*g\|_\infty = \|T_\delta f * g - f*g\|_\infty = \|(T_\delta f - f)*g\| \le \|T_\delta f - f\|_X \|g\|_{X'}.$$

Similarly we have

$$\|T_\delta(f*g) - f*g\|_\infty \le \|f\|_X \|T_\delta g - g\|_{X'}.$$

These relations directly imply the validity of the following theorem.

Theorem 7. *Let X be a norm-invariant Banach function space. Then*

$$\|f*g\|_\infty \le \|f\|_X \|g\|_{X'}, \forall f \in X, \forall g \in X'.$$

Moreover, the convolution operation $f*g$ is continuous in L_∞ if either $f \in X_s$ or $g \in X'_s$.

Let X be a norm-invariant Banach function space and $f : [-\pi,\pi] \to R$ be some simple function, i.e. let $[-\pi,\pi] = \bigcup_{k=1}^{r} E_k$ be some division of segment $[-\pi,\pi]$ and $f(x) = c_k$, $\forall x \in E_k$, $k = \overline{1,r}$. Take an arbitrary function $g \in X_b$ and consider

$$(f*g)(x) = \int_{-\pi}^{\pi} f(x-y)\,g(y)\,dy = \int_{-\pi}^{\pi} f(y)\,g(x-y)\,dy =$$

$$= \sum_{k=1}^{r} c_k \int_{E_k} g(x-y)\,dy, \forall x \in [-\pi,\pi].$$

Consequently,

$$\|f*g\|_X \le \sum_{k=1}^{r} |c_k| \left\| \int_{E_k} g(x-y)\,dy \right\|_X \le \sum_{k=1}^{r} |c_k| \int_{E_k} \|g(\cdot - y)\|_X\,dy =$$

$$= \sum_{k=1}^{r} |c_k| \|g\|_X \int_{E_k} 1dy = \|f\|_{L_1} \|g\|_X, \forall f \in S[-\pi,\pi],$$

where $S[-\pi,\pi]$ is a set of all simple functions on $[-\pi,\pi]$. So the following inequality holds:

$$\|f*g\|_X \le \|f\|_{L_1} \|g\|_X, \forall f \in S[-\pi,\pi]. \qquad (2)$$

Let $f \in L_1(-\pi,\pi)$ be an arbitrary function. Consider

$$\forall \{f_n\}_{n\in N} \subset S[-\pi,\pi] : \|f_n - f\|_{L_1} \to 0, n \to \infty.$$

Since $S[-\pi,\pi]$ is dense in $L_1(-\pi,\pi)$, the choice of such a sequence is always possible. Then it follows directly from the inequality (2) that the sequence $\{f_n * g\}_{n\in N}$ is fundamental in X. Assume

$$(f*g)_1 = \lim_{n\to\infty} f_n * g.$$

By virtue of inequality (2), the definition of $(f*g)_1$ does not depend on the choice of the sequence $\{f_n\}_{n\in N} \subset S[-\pi,\pi]$. On the other hand, it is clear that the sequence $\{f_n*g\}_{n\in N}$ converges to $(f*g)_1$ also in $L_1(-\pi,\pi)$. As $f;g\in L_1(-\pi,\pi)$, by classical facts (see, e.g., [19]), there exists a convolution $f*g$ and, moreover, $f_n*g\to f*g$, $n\to\infty$, in $L_1(-\pi,\pi)$. Then it is clear that $(f*g)_1(x)=(f*g)(x)$ a.e. $x\in[-\pi,\pi]$. So the following theorem is true.

Theorem 8. *Let X be a norm-invariant Banach function space and $f\in L_1(-\pi,\pi)\wedge g\in X$ be arbitrary functions. Then $f*g\in X$ and the following inequality holds:*

$$\|f*g\|_X \le \|f\|_{L_1}\|g\|_X, \forall f\in L_1(-\pi,\pi), \forall g\in X. \tag{3}$$

Denote by M the space of measures on $[-\pi,\pi]$, i.e. M contains a distribution $F\in D$ (D is a space of distributions on $[-\pi,\pi]$) satisfying the inequality

$$|F(u)|\le c\|u\|_\infty, \forall u\in C_0^\infty,$$

where C_0^∞ are infinitely differentiable functions with compact support on $T=(-\pi,\pi)$. Such measures are called Radon measures. It is known (see Riesz-Markov-Kakutani theorem for compact space) that every functional (distribution) can be represented as an integral with respect to the unique regular Borel measure μ on T:

$$F(u)=\mu(u)=\int_T u(x)\,d\mu(x).$$

M is a Banach space with respect to the norm

$$\|\mu\|_1=\sup\{|\mu(u)| : u\in C[-\pi,\pi], \|u\|_\infty\le 1\}.$$

For more details on these facts we refer the reader to [19].

Let $\mu\in M$ and $f,g\in C[-\pi,\pi]$ be arbitrary functions. Then, as shown in the monograph [19] (see p. 93), we have the relation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(\mu*f)\,g\,dx=\mu\left(\breve{f}*g\right),$$

where $\breve{f}(t)=f(-t)$. It directly follows

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(\mu*f)\,g\,dx\right|\le\|\mu\|_1\|f*g\|_\infty.$$

Applying Theorem 7, we obtain

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(\mu*f)\,g\,dx\right|\le\|\mu\|_1\|f\|_X\|g\|_{X'}. \tag{4}$$

Passing to the limit, we see that the inequality (4) holds for $\forall f\in X_b$ and $\forall g\in(X')_b$. So the following lemma is true.

Lemma 9. *Let X be a norm-invariant Banach function space and $\mu \in M$ be a Radon measure. Then the following inequality holds*

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (\mu * f)\, g dx \right| \leq \|\mu\|_1 \|f\|_X \|g\|_{X'}, \forall f \in X_b, \forall g \in (X')_b . \tag{5}$$

Now let's assume that X has an absolutely continuous norm. Then, as is known (see, e.g., [17], Theorem 4.1, p. 20), $X = X_b$ and $X' = X^*$ (X^* is a conjugate space). Lemma 9 and the inequality (5) imply that $\mu * f \in X$ and

$$\|\mu * f\|_X \leq \|\mu\|_1 \|f\|_X . \tag{6}$$

So the following theorem is true.

Theorem 10. *Let X be a Banach function space with absolutely continuous and invariant norm. Then for $\forall \mu \in M$ and $\forall f \in X$ the relation $\mu * f \in X$ and the inequality* (6) *hold.*

In the sequel, we will need some direct corollaries of Theorem 8. Let all conditions of this theorem hold. Then from the inequality (3) we obtain

$$\|f * g\|_X \leq C \|f\|_X \|g\|_X, \forall f \in X_b, \forall g \in X, \tag{7}$$

and

$$\|f * g\|_X \leq C \|f\|_{(X')_b} \|g\|_X, \forall f \in (X')_b, \forall g \in X, \tag{8}$$

where C is a constant depending only on X. As $L_1(-\pi, \pi)$ is dense in X_b (in $(X')_b$), these inequalities follow from (3) by passage to the limit. So the following statement is true.

Proposition 11. *Let X be a norm-invariant Banach function space. Then for $\forall f \in X_b$ (or $\forall f \in (X')_b$) and $\forall g \in X : f * g \in X$ and the inequalities* (7), (8) *hold.*

A question naturally arises: does Proposition 11 hold for $\forall f \in X$? It is absolutely clear that $\forall f; g \in X$ the convolution $f * g$ is defined like an element of the space $L_1(-\pi, \pi)$. Let $S' = \{\vartheta \in X' : \|\vartheta\|_{X'} \leq 1\}$. Then, by Theorem 5, we have

$$\|f * g\|_X = \sup_{\vartheta \in S'} \left| \int_{-\pi}^{\pi} (f * g)(x)\, \vartheta(x)\, dx \right| = \sup_{\vartheta \in S'} \left| \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x-t)\, g(t)\, \vartheta(x)\, dt dx \right| =$$

$$= \sup_{\vartheta \in S'} \left| \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x-t)\, \vartheta(x)\, dx g(t)\, dt \right| \leq \int_{-\pi}^{\pi} \sup_{\vartheta \in S'} \left| \int_{-\pi}^{\pi} f(x-t)\, \vartheta(x)\, dx \right| |g(t)|\, dt =$$

$$= \int_{-\pi}^{\pi} \|f(\cdot - t)\|_X |g(t)|\, dt = \|f\|_X \int_{-\pi}^{\pi} |g(t)|\, dt = \|f\|_X \|g\|_{L_1} .$$

So the following lemma is true.

Lemma 12. *Let X be a norm-invariant Banach function space. Then for $\forall f; g \in X$: $f * g$ belongs to X and*

$$\|f * g\|_X \leq \|f\|_X \|g\|_{L_1}, \forall f; g \in X,$$

The following main theorem follows direcly from this lemma.

Theorem 13. *Let X be a norm-invariant Banach function space. Then for $\forall f; g \in X$: $f * g \in X$ and*

$$\|f * g\|_X \le C \|f\|_X \|g\|_X, \forall f; g \in X,$$

where C is a constant independent of f and g.

3.2. **Approximation of convolution by shifts.** Let's prove the theorem below following the classical case.

Theorem 14. *Let X be a r.i.s. with Boyd indices $\alpha_X; \beta_X \in (0,1)$. Let $f \in L_1(-\pi,\pi)$ and $g \in E$, where E denotes any one of the spaces $C[-\pi,\pi]$ or X_b. Then the convolution $f * g$ in E can be approximated by finite linear combinations of shifts g, i.e. $\forall \varepsilon > 0$, $\exists \{a_k\}_1^n \subset [-\pi,\pi] \wedge \{\lambda_k\}_1^n \subset R$:*

$$\left\| f * g - \sum_{k=1}^{n} \lambda_k T_{a_k} g \right\|_E < \varepsilon.$$

Proof. The case of $E = C[-\pi,\pi]$ is known (see, e.g.,[19]). Consider the case of $E = X_b$. Following the classical scheme, as a subset S_0, such that the finite linear combinations of elements from S_0 are dense in L_1, we take a set of functions f, each of which coincides on $[-\pi,\pi]$ with the characteristic function of some interval $M = [a,b]$, $-\pi < a < b < \pi$, and continues further on periodically.

Let $\forall \varepsilon > 0$ be arbitrary. Let's divide M into a finite number of subintervals I_k of length $|I_k| < \delta$. Take $\forall a_k \in I_k$. Let $f(x) = \chi_M(x)$. We have

$$(f * g)(x) - \sum_k |I_k| g(x - a_k) = \int_{\bigcup_k I_k} g(x-y)\, dy -$$

$$- \sum_k \int_{I_k} g(x-a_k)\, dy = \sum_k \int_{I_k} [g(x-y) - g(x-a_k)]\, dy = \sum_k h_k(x),$$

where

$$h_k(x) = \int_{I_k} [g(x-y) - g(x-a_k)]\, dy.$$

Consequently

$$\left\| (f * g)(\cdot) - \sum_k |I_k| g(\cdot - a_k) \right\|_X \le \sum_k \|h_k\|_X.$$

We have

$$\|h_k\|_X = \sup_{\vartheta \in S^1} \left| \int_{-\pi}^{\pi} h_k(t)\vartheta(t)\, dt \right| = \sup_{\vartheta \in S'} \left| \int_{-\pi}^{\pi} \int_{I_k} [g(t-x) - g(t-a_k)]\, dx \vartheta(t)\, dt \right| =$$

$$= \sup_{\vartheta \in S'} \left| \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [g(t-x) - g(t-a_k)]\, \chi_{I_k}(x) \vartheta(t)\, dx dt \right| =$$

$$= \sup_{\vartheta \in S'} \left| \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left[g(t-x) - g(t-a_k) \right] \vartheta(t)\, dt \chi_{I_k}(x)\, dx \right| \leq$$

$$\leq \int_{-\pi}^{\pi} \sup_{\vartheta \in S'} \left| \int_{-\pi}^{\pi} \left[g(t-x) - g(t-a_k) \right] \vartheta(t)\, dt \chi_{I_k}(x)\, dx \right| =$$

$$= \int_{I_k} \| g(\cdot - x) - g(\cdot - a_k) \|_X \, dx.$$

So the following relation is valid

$$\| h_k \|_X \leq \int_{I_k} \| g(\cdot - x) - g(\cdot - a_k) \|_X \, dx. \tag{9}$$

In the sequel, we will assume that X is a r.i.s. with Boyd indies α_X; $\beta_X \in (0,1)$. Then it follows from Corollary 6.11 of [17] (see p. 165) that trigonometric polynomials are dense in X_b, and hence, by Lemma 6, the shifts are continuus in X_b. Therefore, for $\forall \varepsilon > 0$, $\exists \delta > 0$:

$$\| T_x g - T_{a_k} g \|_X < \varepsilon, \forall x \in I_k.$$

Considering this relation in (9), we obtain

$$\| h_k \|_X \leq |I_k| \, \varepsilon,$$

and hence

$$\left\| (f * g)(\cdot) - \sum_k |I_k| \, g(\cdot - a_k) \right\|_X \leq |I| \, \varepsilon \leq 2\pi\varepsilon.$$

Since $\sum_k |I_k| \, T_{a_k} g$ is a finite linear combination of shifts g, it is clear that $f * g \in \bar{V}_g$, where $\bar{V}_g$ is a closed linear subspace in E, generated by shifts $T_a g$ of the function g. Let $f \in L_1(-\pi, \pi)$ be an arbitrary element. Then for $\forall \varepsilon > 0$ there exist a partition of $[-\pi, \pi]$ into a finite number of intervals M_k and a number λ_k such that the inequality

$$\left\| f(\cdot) - \sum_k \lambda_k \chi_{M_k}(\cdot) \right\|_{L_1} < \varepsilon$$

holds. It follows directly from the previous result that $F * g \in \bar{V}_g$, where $F(\cdot) = \sum_k \lambda_k \chi_{M_k}(\cdot)$. Then there exists a finite linear combination of shifts $\sum_n \mu_n T_{a_n} g$ such that

$$\left\| F * g - \sum_n \mu_n T_{a_n} g \right\|_X < \varepsilon.$$

By Lemma 12, we obtain

$$\left\| f * g - \sum_n \mu_n T_{a_n} g \right\|_X \leq \| f * g - F * g \|_X + \left\| F * g - \sum_n \mu_n T_{a_n} g \right\|_X \leq$$

$$\leq \varepsilon + \| f - F \|_{L_1} \| g \|_X \leq \varepsilon \left(1 + \| g \|_X \right).$$

The arbitrariness of $\varepsilon > 0$ implies $f * g \in \bar{V}_g$. The theorem is proved. □

3.3. **Approximate identity.** Let's consider the approximate identities for convolutions in the space X_b. By the approximate identity (for convolution) we mean $\left\{K_n^{(\cdot)}\right\}_{n\in N} \subset L_1(-\pi,\pi)$ such that

$\alpha)\sup\limits_n \|K_n\|_{L_1} < +\infty$;

$\beta)\lim\limits_n \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1$;

$\gamma)\lim\limits_{n\to\infty}\int_{\delta\le|x|\le\pi} |K_n(x)\,dx| = 0\ ,\ \ \forall\delta\in(0,\pi)\,.$

The following theorem is true.

Theorem 15. *Let X be a r.i.s. with Boyd indices $\alpha_X;\beta_X \in (0,1)$ and $\{K_n\}_{n\in N}$ be an approximate identity. Then*

$$\lim_{n\to\infty}\|K_n * f - f\|_X = 0, \forall f\in X_b.$$

Proof. Take $\forall f\in X_b$. Let $\varepsilon > 0$ be an arbitrary number. It is clear that $\exists g\in C[-\pi,\pi]$:

$$\|f-g\|_X < \varepsilon.$$

We have

$$\|K_n * f - K_n * g\|_X \le \|K_n\|_{L_1}\|f-g\|_X \le M\varepsilon,$$

where $M = \sup\limits_n \|K_n\|_{L_1}$. As is known (see, e.g., [19]),

$$\lim_{n\to\infty}\|K_n * g - g\|_\infty = 0.$$

Then $\exists n_0\in N$:

$$\|K_n * g - g\|_\infty < \varepsilon, \forall n\ge n_0.$$

We have

$$\|K_n * g - g\|_X \le \varepsilon\|1\|_X = C\varepsilon, \forall n\ge n_0.$$

Hence

$$\|K_n * f - f\|_X \le \|K_n * f - K_n * g\|_X + \|K_n * g - g\|_X + \\ + \|g-f\|_X \le (M+C+1)\,\varepsilon, \forall n\ge n_0.$$

The theorem is proved. □

Authors Contribution Statement The authors confirm contribution to the paper as follows: data collection: Chingiz M. Hasimov; analysis and interpretation of results: Chingiz M. Hasimov, Javad A. Asadzadeh; draft manuscript preparation: Javad A. Asadzadeh; the proof of Theorem 14 belongs to Chingiz M. Hasimov. The proof of theorem 15 belongs to Javad A. Asadzadeh. All authors reviewed the results and approved the final version of the manuscript.

Declaration of Competing Interests The authors declare that they have no competing interests.

Acknowledgements The authors would like to express their deep gratitude to corresponding member of the National Academy of Sciences of Azerbaijan, Professor Bilal T.Bilalov for his attention to this work.

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