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ON EQUITABLE COLORING OF BOOK GRAPH FAMILIES

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Abstract. A proper vertex coloring of a graph is equitable if the sizes of color classes differ by atmost one. The notion of equitable coloring was introduced by Meyer in 1973. A proper $h-$colorable graph K is said to be equitably h-colorable if the vertex sets of K can be partioned into h independent color classes $V_1, V_2, ..., V_h$ such that the condition $||V_i| - |V_j|| \leq 1$ holds for all different pairs of i and j and the least integer h is known as equitable chromatic number of K. In this paper, we find the equitable coloring of book graph, middle, line and central graphs of book graph.

1. Introduction

The idea of equitable coloring was discovered by Meyer [4] in 1973. Hajmal and Szemeredi [3] proved that graph K with degree Δ is equitable h-colorable, if $h \geq \Delta + 1$. Later Equitable Coloring Conjecture for bipartite graphs was proved. Equitable vertex coloring of corona graphs is NP- hard.

The graphs considered here are simple. Vertex coloring is a particular case of Graph coloring. The collection of vertices receiving same color is known as color class. A proper $h-$colorable graph K is said to be *equitably $h-$colorable* if the vertex sets of K can be partitioned into h independent color classes $V_1, V_2, ..., V_h$ such that the condition $||V_i| - |V_j|| \leq 1$ holds for all different pairs of i and j [1]. And the least integer h is known as *equitable chromatic number* of K [1]. Here we found equitable coloring of book graph, middle, line and central graphs of book graph.

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2. Preliminaries

Line graph [2] of K, $L(K)$ is attained by considering the edges of K as the vertices of $L(K)$. The adjacency of any two vertices of $L(K)$ is a consequence of the corresponding adjacency of edges in K.

Middle graph [5] of K, $M(K)$ is attained by adding new vertex to all the edges of K. The adjacency of any two new vertices of $M(K)$ is a consequence of the corresponding adjacency of edges in K or adjacency of a vertex and an edge incident with it.

Central graph [6] of K, $C(K)$ is attained by the insertion of new vertex to all the edges of K and connecting any two new vertices of K which were previously non-adjacent.

The q-book graph is defined as the graph Cartesian product $S_{(q+1)} \times P_2$, where S_q is a star graph and P_2 is the path graph.

3. Results

3.1. On Equitable Coloring of Middle Graph of Book Graph.

- Order of $M(B_q)$ is $5q + 3$
- Number of incidents of $M(B_q)$ is $q^2 + 9q + 2$
- Maximum degree of $M(B_q)$ is $2(q + 1)$
- Minimum degree of $M(B_q)$ is 2

Algorithm A

Input: The value 'q' of B_q, for $q \geq 3$

Outcome: Equitably colored $V[M(B_q)]$

Procedure:

start

{

$V_a = \{g, h, z\}$;

$C(g) = C(h) = 1$;

$C(z) = q + 2$;

for $s = 1$ to q

{

$V_b = \{g_s, h_s\}$;

$C(g_s) = s$;

$C(h_s) = s$;

}

for $s = 1$ to q

{

$V_c = \{k_s, l_s\}$;

$C(k_s) = s + 1$;

$C(l_s) = s + 1$;

}

for $s = 1$ to q
{
$V_d = \{m_s\}$;
if s is *odd*
$C(m_s) = q + 1$;
else
$C(m_s) = q + 2$;
}
}
V $= V_a \cup V_b \cup V_c \cup V_d$
end

Theorem 3.1. *For any book graph $M(B_q)$ the equitable chromatic number,*

$$\boldsymbol{\chi_=[M(B_q)] = q + 2}, \forall q \geq 3$$

Proof. For $q \geq 3$, $V(B_q) = \{g, h, g_s, h_s : 1 \leq s \leq q\}$.
$V[M(B_q)] = \{g, h, z\} \cup \{g_s : 1 \leq s \leq q\} \cup \{h_s : 1 \leq s \leq q\} \cup \{k_s : 1 \leq s \leq q\} \cup \{l_s : 1 \leq s \leq q\} \cup \{m_s : 1 \leq s \leq q\}$, where z, k_s, l_s and m_s are the subdivision of the edges gh, gg_s, hh_s and g_sh_s respectively.

Let us consider $V[M(B_q)]$ and the color set C $= \{c_1, c_2, ..., c_{q+2}\}$. Assign the equitable coloring by Algorithm A. Therefore,

$$\chi_=[M(B_q)] \leq q + 2.$$

And since, there exists a maximal induced complete subgraph of order $q+2$ by the vertices z, g, k_s and therefore $\chi_=[M(B_q)] \geq q + 2$.

$c_1, c_2, ..., c_{q+2}$ are independent sets of $M(B_q)$. And $||c_i| - |c_j|| \leq 1$, for every different pair of i and j. Hence,

$$\boldsymbol{\chi_=[M(B_q)] = q + 2}.$$

□

3.2. **On Equitable Coloring of Central Graph of Book Graph. Features of Central Graph of Book Graph**

- Order of $C(B_q)$ is $5q + 3$
- Number of incidents of $C(B_q)$ is $2(q^2 + 3q + 1)$
- Maximum degree of $C(B_q)$ is $2q + 1$
- Minimum degree of $C(B_q)$ is 2

Algorithm B
Input: The value 'q' of B_q, for $q \geq 3$
Outcome: Equitably colored $V[C(B_q)]$
Procedure:
start

{
for $s = 1$ to q
$V_a = \{g, h, m_s\}$;
{
if $s = 1$ to 3
$C(g) = C(h) = C(m_s) = 1$;
else
$C(m_s) = s - 1$;
}
for $s = 1$ to q
{
$V_b = \{g_s, h_s\}$;
$C(g_s) = s + 1$;
$C(h_s) = s + 1$;
}
if q is *odd*
{
$V_c = \{k_s, l_s, z\}$;
for $s = 1$ to q
{
if $s = 1$ to $q - 1$
{
$C(k_s) = s + 2$;
$C(l_s) = s + 2$;
}
else
$C(z) = C(k_s) = C(l_s) = 2$;
}
else
{
for $s = 1$ to q
$V_c = \{k_s, l_s, z\}$;
$C(z) = 2$;
$C(k_s) = C(l_s) = q - s + 2$;
}
}
}
V $= V_a \cup V_b \cup V_c$
end

Theorem 3.2. *For any book graph $C(B_q)$ the equitable chromatic number,*

$$\boldsymbol{\chi_=[C(B_q)] = q + 1, \forall q \geq 3}$$

Proof. For $q \geq 3$,

$$V(B_q) = \{g, h, g_s, h_s : 1 \leq s \leq q\}.$$

$$\begin{aligned} V[C(B_q)] &= \{g, h, z\} \cup \{g_s : 1 \leq s \leq q\} \cup \{h_s : 1 \leq s \leq q\} \\ \cup\{k_s &: 1 \leq s \leq q\} \cup \{l_s : 1 \leq s \leq q\} \cup \{m_s : 1 \leq s \leq q\}, \end{aligned}$$

where z, k_s, l_s and m_s are the subdivision of the edges gh, gg_s, hh_s and g_sh_s respectively.

Let us consider $V[C(B_q)]$ and the color set C $= \{c_1, c_2, ..., c_{q+1}\}$. Assign the equitable coloring by Algorithm B. Therefore,

$$\chi_{=}[C(B_q)] \leq q + 1$$

And $\chi[C(B_q)] = q + 1$. That is, $\chi_{=}[C(B_q)] \geq \chi[C(B_q)] = q + 1$. Therefore,

$$\chi_{=}[C(B_q)] \geq q + 1.$$

$c_1, c_2, ..., c_{q+1}$ are independent sets of $C(B_q)$. And $||c_i| - |c_j|| \leq 1$, for every different pair of i and j. Thus,

$$\chi_{=}[C(B_q)] = q + 1.$$

□

3.3. **On Equitable Coloring of Line Graph of Book Graph.**

- Order of $L(B_q)$ is $3q + 1$
- Number of incidents of $L(B_q)$ is $q(q + 3)$
- Maximum degree of $L(B_q)$ is $2q$
- Minimum degree of $L(B_q)$ is 2

Algorithm C

Input: The value 'q' of B_q, for $q \geq 3$

Outcome: Equitably coloring $V[L(B_q)]$

Procedure:

begin

{

for $s = 1$ to q

{

$V_a = \{g, z\} \cup \{m_s\}$;

$C(m_s) = s$;

$C(z) = C(g) = 1$;

}

for $s = 1$ to q

{

$V_b = \{k_s, l_s\}$;

$C(k_s) = s + 1$;

$C(l_s) = s + 1$;

}

}

V = $V_a \cup V_b$
end

Theorem 3.3. *For any book graph $L(B_q)$ the equitable chromatic number,*

$$\boldsymbol{\chi_{=}[L(B_q)] = q+1, \forall q \geq 3}$$

Proof. For $q \geq 3$,

$$V(B_q) = \{g, h, g_s, h_s : 1 \leq s \leq q\}.$$

The edge set of B_q is $\{z, k_s, l_s, m_s : 1 \leq s \leq q\}$ where z be the edge corresponding to the vertices gh, each k_s be the edge corresponding to the vertex gg_s, each edge l_s be the edge corresponding to the vertex hh_s , each edge m_s be the edge corresponding to the vertex g_sh_s. By the definition of line graph, the edge set of line graph is converted into vertices of $L(B_q)$.

$$\begin{aligned} V[L(B_q)] &= \{z\} \cup \{k_s : 1 \leq s \leq q\} \cup \{l_s : 1 \leq s \leq q\} \\ \cup\{m_s &: 1 \leq s \leq q\}. \end{aligned}$$

Let us consider the $V[L(B_q)]$ and the color set $C = \{c_1, c_2, ..., c_{q+1}\}$. Assign the equitable coloring by Algorithm C. Therefore,

$$\chi_{=}[L(B_q)] \leq q+1$$

And since, there exists a maximal induced complete subgraph of order $q+1$ by the vertices z, k_s and therefore

$$\chi_{=}[L(B_q)] \geq q+1.$$

$c_1, c_2, ..., c_{q+1}$ are independent sets of $L(B_q)$. And $||c_i| - |c_j|| \leq 1$, for every different pair of i and j. Thus,

$$\chi_{=}[L(B_q)] = q+1.$$

□

3.4. **On Equitable Coloring of Book Graph. Features of Book Graph.**

- Order of B_q is $2(q+1)$
- Number of incidents of B_q is $3q+1$
- Maximum degree of B_q is $q+1$
- Minimum degree of B_q is 2

Algorithm D

Input: The value 'q' of B_q, for q ≥ 3
Outcome: Equitably colored $V(B_q)$
Procedure:
start
{
for $s = 1$ to q
{
$V_a = \{g_s, h\}$;
C(h)=1;

$C(g_s) = 1;$
}
for $s = 1$ to q
{
$V_b = \{g, h_s\}$;
$C(g) = 2;$
$C(h_s) = 2;$
}
}
V = $V_a \cup V_b$
end

Theorem 3.4. *For any book graph B_q the equitable chromatic number,*

$$\boldsymbol{\chi_=(B_q) = 2}, \forall q \geq 3.$$

Proof. For $n \geq 3$,

$$V(B_q) = \{g, h\} \cup \{g_s : 1 \leq s \leq q\} \cup \{h_s : 1 \leq s \leq q\}.$$

Let us consider the $V(B_q)$ and the color set C $= \{c_1, c_2\}$. Assign the equitable coloring by Algorithm D. Therefore,

$$\chi_=(B_q) \leq 2.$$

And since, there exists a maximal induced complete subgraph of order 2 in B_q (say path P_2). Therefore,

$$\chi_=(B_q) \geq 2$$

c_1, c_2 are independent sets of B_q. And $||c_i| - |c_j|| \leq 1$, for every different pair of i and j. Hence,

$$\boldsymbol{\chi_=(B_q) = 2}.$$

□

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