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Symmetry in Complex Sasakian Manifolds

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Abstract

In this paper, we give some results on complex Sasakian manifolds. In addition, we introduce a complex $\eta-$Einstein Sasakian manifold. We study on conformal curvature tensor, concircular curvature tensor, projective curvature tensor and conharmonic curvature tensor on complex Sasakian manifolds. Moreover, we examine such manifolds under the symmetry conditions with related to special curvature tensors like as conformal curvature tensor, concircular curvature tensor, projective curvature tensor and conharmonic curvature tensor. Furthermore, we present some properties of these curvature tensors for a complex Sasakian manifold.

Keywords: *complex contact manifold, complex Sasakian manifold, symmetry conditions*
2010 Mathematics Subject Classification: *53C15, 53C25, 53D10.*

1. Introduction

The Riemannian geometry of complex contact manifolds has been studied since the late 1970s. Ishihara and Konishi proved the existence of complex almost contact structure on a complex contact metric manifolds and they presented the normality of this structure [5]. In 2000 Korkmaz [6] defined a new notion for normality of complex almost contact metric manifolds. The difference between these two normality notion is to be Kählerian of almost complex structure. From this difference, the normality notion of Korkmaz has been studied by many researchers [3, 2, 9, 10, 12]. Also, Korkmaz proved that a complex Heisenberg group is an example of normal complex contact metric manifolds [6].

A complex Sasakian manifold is a normal complex contact metric manifold has globally defined complex contact form [4]. The normality of Ishihara-Konishi, also do not satisfy complex Sasakian structure [11]. In [13] presented authors studied on some curvature properties of complex Sasakian manifolds

In this paper we work on complex Sasakian manifolds under some symmetry conditions with related to conformal curvature tensor, concircular curvature tensor, projective curvature tensor and conharmonic curvature tensor. We present some properties of these curvature tensors for a complex Sasakian manifold.

2. Preliminaries

In this section we give fundamental facts on complex contact manifolds. For details we refer the reader to the Chapter 12 in [1].

Definition 2.1. *Let N be a complex manifold of odd complex dimension $2p+1$ covered by an open covering $\mathscr{C}=\{\mathscr{A}_i\}$ consisting of coordinate neighborhoods. If there is a holomorphic 1-form η_i on each $\mathscr{A}_i \in \mathscr{C}$ in such a way that for any $\mathscr{A}_i, \mathscr{A}_j \in \mathscr{C}$ and for a holomorphic function f_{ij} on $\mathscr{A}_i \cap \mathscr{A}_j \neq \emptyset$*

$$\eta_i \wedge (d\eta_j)^p \neq 0 \text{ in } \mathscr{A}_i,$$

$$\eta_i = f_{ij}\eta_j, \ \mathscr{A}_i \cap \mathscr{A}_j \neq \emptyset,$$

then the set $\{(\eta_i, \mathscr{A}_i) \mid \mathscr{A}_i \in \mathscr{C}\}$ of local structures is called complex contact structure and with this structure N is called a complex contact manifold.

η_i is called a complex contact form on A_i. The kernel of complex contact form determines a non-integrable distribution H_i by the equation $\eta_i = 0$ such as

$$H_i = \{X_P : \eta_i(X_P) = 0, X_P \in T_P N\},$$

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and a holomorphic vector field ξ_i is defined by $\eta_i(\xi_i)=1$, and a complex line bundle is defined by $E_i = Span\{\xi_i\}$.
Let $T^c(N)$ be a complexification of tangent bundle of (N,J,η_i) and let define vector fields

$$U_i=\xi_i+\bar{\xi}_i \qquad V_i=-i(\xi_i+\bar{\xi}_i)$$

and 1-forms

$$u_i=\frac{1}{2}(\eta_i+\bar{\eta}_i) \qquad v_i=\frac{1}{2}i(\eta_i-\bar{\eta}_i).$$

Therefore we get

1. $V_i=-JU_i$ and $v_i=u_i\circ J$
2. $U_i=JV_i$ and $u_i=-v_i\circ J$
3. $u_i(U_i)=v_i(V_i)=1$ and $u_i(V_i)=v_i(V_i)=0$.

The complexified H_i and E_i is defined by

$$\begin{aligned} {H_i}^c &= \{W\in T^c(N)|u(W)=v(W)=0\} \\ {E_i}^c &= Span\{U,V\}. \end{aligned}$$

We use notation $\mathscr{H}$ and $\mathscr{V}$ for the union of ${H_i}^c$ and E_i^c respectively. $\mathscr{H}$ is called horizontal distribution and $\mathscr{V}$ is called vertical distribution and we can write $TN=\mathscr{H}\oplus\mathscr{V}$.

Definition 2.2. *Let N be a complex manifold with complex structure J, Hermitian metric g and $\mathscr{C}=\{\mathscr{A}_i\}$ be open covering of N with coordinate neighbourhoods $\{\mathscr{A}_i\}$. If N satisfies the following two conditions then it is called a complex almost contact metric manifold:*
1. In each $\mathscr{A}_i$ there exist 1-forms u_i and $v_i=u_i\circ J$, with dual vector fields U_i and $V_i=-JU_i$ and $(1,1)$ tensor fields G_i and $H_i=G_iJ$ such that

$$H_i^2=G_i^2=-I+u_i\otimes U_i+v_i\otimes V_i$$

$$G_iJ=-JG_i,\quad GU_i=0,\quad g(X,G_iY)=-g(G_iX,Y).$$

2.On $\mathscr{A}_i\cap\mathscr{A}_j\neq\emptyset$ we have

$$\begin{aligned} u_j &= au_i-bv_i,\quad v_j=bu_i+av_i, \\ G_j &= aG_i-bH_i,\quad H_j=bG_i+aH_i \end{aligned}$$

where a and b are functions on $\mathscr{U}_i\cap\mathscr{U}_j$ with $a^2+b^2=1$ [5].

Ishihara and Konishi [5] studied on normality of complex almost contact metric manifolds. They defined local tensors

$$\begin{aligned} S(X,Y) &= [G,G](X,Y)+2g(X,GY)U-2g(X,HY)V \\ &\quad +2(v(Y)HX-v(X)HY)+\sigma(GY)HX \\ &\quad -\sigma(GX)HY+\sigma(X)GHY-\sigma(Y)GHX, \\ T(X,Y) &= [H,H](X,Y)-2g(X,GY)U+2g(X,HY)V \\ &\quad +2(u(Y)GX-u(X)GY)+\sigma(HX)GY \\ &\quad -\sigma(HY)GX+\sigma(X)GHY-\sigma(Y)GHX \end{aligned}$$

for all $X,Y\in\Gamma(TN)$.

Definition 2.3. *[6] A complex almost contact metric manifold is normal if*

- $S(X,Y)=T(X,Y)=0$ *for all X,Y in $\mathscr{H}$,*
- $S(X,U)=T(X,V)=0$ *for all X.*

Definition 2.4. *Let (N,G,H,J,U,V,u,v,g) be a normal complex contact metric manifold and $\eta=u-iv$ is globally defined. If the fundamental $2-$ forms $\widetilde{G}$ and $\widetilde{H}$ are defined by*

$$\widetilde{G}(X,Y)=du(X,Y) \text{ and } \widetilde{H}(X,Y)=dv(X,Y)$$

then N is called complex Sasakian manifold.

The curvature properties of complex Sasakian manifolds are given as follow [13];

$$R(X,U)U=X+u(X)U+v(X)V \tag{2.1}$$

$$R(X,V)V=X-u(X)U-v(X)V \tag{2.2}$$

$$\begin{aligned} R(X,Y)U &= v(X)JY-v(Y)JX+2v(X)u(Y)V-2v(Y)u(X)V \\ &\quad +u(Y)X-u(X)Y-2g(JX,Y)V \end{aligned} \tag{2.3}$$

$$\begin{aligned} R(X,Y)V &= 3u(X)JY-3u(Y)JX-2u(X)v(Y)U+2u(Y)v(X)U \\ &\quad +v(Y)X-v(X)Y+2g(JX,Y)U \end{aligned} \tag{2.4}$$

$$\begin{aligned} R(X,U)Y &= -2v(Y)v(X)U+2u(Y)v(X)V-g(Y,X)U \\ &\quad +u(Y)X+g(JY,X)V \end{aligned} \tag{2.5}$$

$$\rho(U,U)=\rho(V,V)=4p,\ \ \rho(U,V)=0. \tag{2.6}$$

Definition 2.5. *Let (N,G,H,U,V,u,v,g) be a complex Sasakian manifold. If for α and β are smooth functions on N the Ricci tensor satisfies*

$$\rho = \alpha g + \beta(u \otimes u + v \otimes v)$$

then N is called a complex $\eta-$Einstein Sasakian manifold.

The conformal curvature tensor, the concircular curvature tensor, the projective curvature tensor and the conharmonic curvature tensor are defined as follow on a complex Sasakian manifold;

$$\begin{aligned}
\mathscr{C}(X,Y)Z &= R(X,Y)Z + \frac{\tau}{(4n+1)4n}(g(Y,Z)X - g(X,Z)Y) \\
&+ \frac{1}{4n}(g(X,Z)QY - g(Y,Z)QX + \rho(X,Z)Y - \rho(Y,Z)X) \quad (2.7)\\
\mathscr{Z}(X,Y)Z &= R(X,Y)Z - \frac{\tau}{(4n+2)(4n+1)}[g(Y,Z)X - g(X,Z)Y] \quad (2.8)\\
\mathscr{P}(X,Y)Z &= R(X,Y)Z - \frac{1}{4n+1}[\rho(Y,Z)X - \rho(X,Z)Y] \quad (2.9)\\
\mathscr{K}(X,Y)Z &= R(X,Y)Z - \frac{1}{4n}[\rho(Y,Z)X - \rho(X,Z)Y + g(Y,Z)QX \\
&- g(X,Z)QY] \quad (2.10)
\end{aligned}$$

where $X,Y,Z \in \Gamma(TN)$, Q is Ricci operator, ρ is Ricci tensor and τ is scalar curvature of N.

3. Certain Symmetry Conditions on Complex Sasakian Manifolds

Let's take a $(1,3)-$tensor field $\mathscr{T}_1$ on a complex Sasakian manifold N. Then, we have

$$\begin{aligned}
(\mathscr{T}_1(X,Y).\mathscr{T}_2)(T,U)W &= \mathscr{T}_1(X,Y)\mathscr{T}_2(T,U)W - \mathscr{T}_2(\mathscr{T}_1(X,Y)T,U)W \\
&-\mathscr{T}_2(T,\mathscr{T}_1(X,Y)U)W - \mathscr{T}_2(T,U)\mathscr{T}_1(X,Y)W. \quad (3.1)
\end{aligned}$$

If $R.\mathscr{T} = 0$ then the manifold is called $\mathscr{T}-$semi symmetric. For an Riemannian manifold if $R.R = 0$ then the manifold is called locally symmetric or semi-symmetric. In this section we present some results on the complex Sasakian manifold under certain symmetry conditions.

Theorem 3.1. *Let N be a complex Sasakian manifold which is satifying $R(X,U).R = 0$ or $R(X,V).R = 0$. Then the sectional curvature of M is equal to $1 - g^2(JY,X)$.*

Proof. Let N be a complex Sasakian manifold which is satifying $R(X,U).R = 0$. Then by using (2.5) in (3.1) we obtain

$$(R(X,U)R)(X,Y)Y = -g(X,R(X,Y)Y) + g(X,X)g(Y,Y) - g^2(JY,X) - g^2(Y,X).$$

Thus the sectional curvature is given by

$$\begin{aligned}
K(X,Y) &= g(X,X)g(Y,Y) - g^2(JY,X) - g^2(Y,X) \\
&= 1 - g^2(JY,X)
\end{aligned}$$

for unit, mutually orthogonal horizontal vector fields X,Y on N. By following similar steps one can obtain same results for $R(X,V).R = 0$. □

Corollary 3.1. *Let N be a complex Sasakian manifold which is satisfying $R(X,U).R = 0$ or $R(X,V).R = 0$. If $g(X,JY) = 0$ for unit, mutually orthogonal horizontal vector fields X,Y on N then sectional curvature is 1.*

On a complex Sasakian manifold the curvature tensors have the following properties;

$$\mathscr{P}(U,W)U = -\frac{1}{4n+1}W \tag{3.2}$$

$$\mathscr{P}(Y,W)U = -2g(JY,W)V \tag{3.3}$$

$$\mathscr{P}(U,V)U = \frac{4n}{4n+1}V \tag{3.4}$$

$$\mathscr{P}(U,QU)U = -\frac{1}{4n+1}QU - \frac{1}{4n+1}\rho(QU,U)U \tag{3.5}$$

$$\mathscr{P}(U,W)Y = -g(JY,W)V + g(Y,W)U - \frac{1}{4n+1}\rho(W,Y)U \tag{3.6}$$

$$\mathscr{Z}(U,Y)W = -g(JW,Y)V + \frac{(4n+2)(4n+1)-\tau}{(4n+2)(4n+1)}g(Y,W)U \tag{3.7}$$

$$\mathscr{Z}(U,Y)U = \frac{\tau-(4n+2)(4n+1)}{(4n+2)(4n+1)}Y \tag{3.8}$$

$$\mathscr{K}(U,W)U = -\frac{QW}{4n} \tag{3.9}$$

$$\mathscr{K}(U,QU)U = -QU - \frac{1}{4n}\rho(QU,U)U + \frac{1}{4n}Q(QU) \tag{3.10}$$

$$\mathscr{K}(U,V)U = V + \frac{QV}{4n} \tag{3.11}$$

$$\mathscr{K}(Y,W)U = -2g(JY,W)V \tag{3.12}$$

$$\mathscr{K}(U,W)Y = -g(JY,W)V + g(Y,W)U - \frac{1}{4n}[\rho(W,Y)U + g(W,Y)QU] \tag{3.13}$$

$$\begin{aligned}\mathscr{C}(U,Y)QW = R(U,Y)QW &+ \frac{\tau}{(4n+1)4n}(g(Y,QW)U - g(U,QW)Y) \\ &+ \frac{1}{4n}(g(U,QW)QY - g(Y,QW)QU + \rho(U,QW)Y \\ &- \rho(Y,QW)U)\end{aligned} \tag{3.14}$$

$$\begin{aligned}\mathscr{C}(U,Y)W = -g(JW,Y)V + g(W,Y)U &+ \frac{\tau}{(4n+1)4n}g(Y,W)U \\ &+ \frac{1}{4n}(-g(Y,W)QU - \rho(Y,W)U)\end{aligned} \tag{3.15}$$

$$\mathscr{C}(U,Y)U = -\frac{\tau}{(4n+1)4n}Y + \frac{1}{4n}Q \tag{3.16}$$

for all $Y,W \in \Gamma(TN)$.

Theorem 3.2. *Let N be a complex Sasakian manifold which is satisfies $R(U,Y)\mathscr{P} = 0$ or $R(V,Y)\mathscr{P} = 0$ for a vector fields Y on N. Then N is a complex η-Einstein Sasakian manifold.*

Proof. Let N be a complex Sasakian manifold and $R(U,Y).\mathscr{P} = 0$ for all $Y \in \Gamma(TN)$. From (3.1) we obtain

$R(U,Y)\mathscr{P}(U,W)U - \mathscr{P}(R(U,Y)U),W) - \mathscr{P}(U,R(U,Y)W)U$
$-\mathscr{P}(U,W)R(U,Y)U = 0.$

By taking inner product above equation with U and using (2.1), (2.3), (2.5) and equalities from (3.2) to (3.6), we get

$g(Y,W)(-4n) = \rho(Y,W) \quad , \quad Y,W \in \Gamma(\mathscr{H}).$

By consider $Y = Y_0 + u(Y)U + v(Y)V$, $W = W_0 + u(W)U + v(W)V$ we obtain

$\rho(Y,W) = -4ng(Y,W) + 8n(u(Y)u(W) + v(Y)v(W)).$

Thus N is a complex η-Einstein Sasakian manifold. By following same steps one can obtain same result for $R(V,Y)\mathscr{P} = 0$ condition. □

Theorem 3.3. *Let N be a complex Sasakian manifold which is satisfies $\mathscr{Z}(U,Y)\mathscr{P} = 0$ or $\mathscr{Z}(V,Y)\mathscr{P} = 0$ for a vector fields Y on N. Then, N is a complex η-Einstein Sasakian manifold.*

Proof. Let N be a complex Sasakian manifold and $\mathscr{Z}(U,Y).\mathscr{P} = 0$ for all $Y \in \Gamma(TN)$. From (3.1) we obtain

$\mathscr{Z}(U,Y)\mathscr{P}(U,W)U - \mathscr{P}(\mathscr{Z}(U,Y)U),W) - \mathscr{P}(U,\mathscr{Z}(U,Y)W)U$
$-\mathscr{P}(U,W)\mathscr{Z}(U,Y)U = 0.$

By taking inner product of above equation with U and using (2.1),(2.3),(2.5), and equalities from (3.2) to (3.6), we get

$g(Y,W)(-4n) = \rho(Y,W) \quad , \quad Y,W \in \Gamma(\mathscr{H})$

By considering $Y = Y_0 + u(Y)U + v(Y)V$, $W = W_0 + u(W)U + v(W)V$ we obtain

$\rho(Y,W) = -4ng(Y,W) + 8n(u(Y)u(W) + v(Y)v(W))$

Thus N is a complex η-Einstein Sasakian manifold. By following same steps one can obtain same result for $\mathscr{Z}(V,Y)\mathscr{P} = 0$ condition. □

Theorem 3.4. *Let N be a complex Sasakian manifold which satisfies $R(U,Y)\mathscr{K}=0$ or $R(V,Y)\mathscr{K}=0$ for a vector fields Y on N. Then, N is a complex η-Einstein Sasakian manifold.*

Proof. Let N be a complex Sasakian manifold and $R(U,Y).\mathscr{K}=0$ for all $Y\in\Gamma(TN)$. From (3.1), we obtain

$$R(U,Y)\mathscr{K}(U,W)U-\mathscr{K}(R(U,Y)U),W)-\mathscr{K}(U,R(U,Y)W)U$$
$$-\mathscr{K}(U,W)R(U,Y)U=0.$$

By taking inner product of above equation with U and using (2.1), (2.3),(2.5) and from (3.9) to (3.13), we get

$$g(Y,W)=-\rho(Y,W)\quad,\quad X,Y\in\mathscr{H}.$$

By considering $Y=Y_0+u(Y)U+v(Y)V$, $W=W_0+u(W)U+v(W)V$, we obtain

$$\rho(Y,W)=-g(Y,W)-(1+4n)u(Y)u(W)+v(Y)v(W).$$

Thus N is a complex η-Einstein Sasakian manifold. By following same steps one can obtain same result for $R(V,Y).\mathscr{P}=0$ condition. □

Theorem 3.5. *Let N be a complex Sasakian manifold which satisfies $\mathscr{K}(U,Y).\mathscr{P}=0$ or $\mathscr{K}(V,Y).\mathscr{P}=0$ for a vector fields Y on N. Then N is Ricci flat or a complex η-Einstein Sasakian manifold.*

Proof. Let N be a complex Sasakian manifold and $\mathscr{K}(U,Y).\mathscr{P}=0$ for all $Y\in\Gamma(TN)$. From (3.1) we obtain

$$\mathscr{K}(U,Y)\mathscr{P}(U,W)U-\mathscr{P}(\mathscr{K}(U,Y)U),W)-\mathscr{P}(U,\mathscr{K}(U,Y)W)U$$
$$-\mathscr{P}(U,W)\mathscr{K}(U,Y)U=0.$$

By taking inner product above equation with U and using (2.1),(2.3),(2.5), equalities from (3.2) to (3.6) and from (3.9) to (3.13), we get

$$\frac{1}{4n.(4n+1)}\rho(W,Y)-\frac{1}{4n}g(Y,W)+\frac{1}{4n.(4n+1)}\rho(W,QY)=0$$

and thus

$$(Q^2-4nQ)Y=0,$$
$$Q(QY-4nY)=0.$$

From this equation if $Q=0$ then N is Ricci flat. If $QY=4nY$ then

$$\rho(Y,W)=4ng(Y,W)\quad,\quad Y,W\in\Gamma(\mathscr{H}).$$

By consider $Y=Y_0+u(Y)U+v(Y)V$, $W=W_0+u(W)U+v(W)V$ we obtain

$$\rho(Y,W)=-g(Y,W)-(1+4n)u(Y)u(W)+v(Y)v(W).$$

Thus, N is a complex η-Einstein Sasakian manifold. By following same steps one can obtain same result for $\mathscr{K}(V,Y).\mathscr{P}=0$ condition. □

Theorem 3.6. *On a complex Sasakian manifold satisfying $\mathscr{K}.\mathscr{K}=0$,we have $Q^2=I$.*

Proof. From the definition under the condition $\mathscr{K}.\mathscr{K}=0$ we have

$$\mathscr{K}(X,Y)\mathscr{K}(Z,W)T-\mathscr{K}(\mathscr{K}(X,Y)Z),W)-\mathscr{K}(Z,\mathscr{K}(X,Y)W)T$$
$$-\mathscr{K}(Z,W)\mathscr{K}(X,Y)T=0$$

for all X,Y,Z,W,T vector fields on N. With setting $X=Z=T=U$ and by taking inner product with U we get

$$u(\mathscr{K}(U,Y)\mathscr{K}(U,W)U)u(-\mathscr{K}(\mathscr{K}(U,Y)U),W)-u(\mathscr{K}(U,\mathscr{K}(U,Y)W)U)$$
$$-u(\mathscr{K}(U,W)\mathscr{K}(U,Y)U)=0. \tag{3.17}$$

Thus, by using equalities from (3.9) to (3.13) we obtain

$$g(Y,W)=(1-\frac{1}{(4n)^2})\rho(Y,QW)+\frac{1}{(4n)^2}\rho(W,QY)$$

and thus we get

$$g(Y,W)=g(QY,QW)$$

which provides

$$g(Q^2Y-Y,W)=0.$$

Thus we get $Q^2=I$. □

Theorem 3.7. *On a complex Sasakian manifold satisfying $\mathscr{C}.\mathscr{K}=0$, we have*

$$\rho(JY,W)=2ng(Y,W)+\frac{\tau}{4n+1}g(JY,W).$$

Proof. Let N be a complex Sasakian satisfying $\mathscr{C}.\mathscr{K}=0$. Then we have

$$\mathscr{C}(X,Y)\mathscr{K}(Z,W)T-\mathscr{K}(\mathscr{C}(X,Y)Z),W)-\mathscr{K}(Z,\mathscr{C}(X,Y)W)T-\mathscr{K}(Z,W)\mathscr{C}(X,Y)T=0$$

for all vector fields X,Y,Z,W,T on N. By setting $X=Z=T=U$ and taking inner product with U, we get

$$u(\mathscr{C}(U,Y)\mathscr{K}(U,W)U)-u(\mathscr{K}(\mathscr{C}(U,Y)U),W)-u(\mathscr{K}(U,\mathscr{C}(U,Y)W)U)$$
$$-u(\mathscr{K}(U,W)\mathscr{C}(U,Y)U)=0.$$

Using equalities from (3.9) to (3.13) and (3.14) to (3.16), we obtain

$$\rho(JY,W)=2ng(Y,W)+\frac{\tau}{4n+1}g(JY,W).$$

□

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