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(α,β)-CUTS AND INVERSE (α,β)-CUTS IN BIPOLAR FUZZY SOFT SETS

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Abstract. Bipolar fuzzy soft set theory, which is a very useful hybrid set in decision making problems, is a mathematical model that has been emphasized especially recently. In this paper, the concepts of (α,β)-cuts, first type semi-strong (α,β)-cuts, second type semi-strong (α,β)-cuts, strong (α,β)-cuts, inverse (α,β)-cuts, first type semi-weak inverse (α,β)-cuts, second type semi-weak inverse (α,β)-cuts and weak inverse (α,β)-cuts of bipolar fuzzy soft sets were introduced together with some of their properties. In addition, some distinctive properties between (α,β)-cuts and inverse (α,β)-cuts were established. Moreover, some related theorems were formulated and proved. It is further demonstrated that both (α,β)-cuts and inverse (α,β)-cuts of bipolar fuzzy soft sets were useful tools in decision making.

1. INTRODUCTION

Many mathematical models have been introduced to the literature in order to express the uncertainty problems encountered in the most accurate way. For example; the fuzzy sets put forward by Zadeh [1] is a theory that allows the abandonment of strict rules in classical mathematics in expressing uncertainty. After this theory was introduced, the theories of fuzzy sets and fuzzy systems developed rapidly. As is well known, the cut set (or level set) of fuzzy set [1] is an important concept in theory of fuzzy sets and systems, which plays a significant role in fuzzy algebra [7,8], fuzzy reasoning [9, 10], fuzzy measure [11, 12, 13] and so on. The cut set allows us to express fuzzy sets as classical sets. Based on the cut sets, the decomposition theorems and representation theorems can be established [14]. The cut sets on fuzzy sets are described in [15] by using the neighborhood relations between fuzzy point and fuzzy set. It is pointed out that there are four kinds of definitions of cut

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sets on fuzzy sets, each of which has similar properties.

Fuzzy set is a type of important mathematical structure to represent a collection of objects whose boundary is vague. There are several types of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets [16], interval-valued fuzzy sets [17], vague sets [18], etc. Bipolar-valued fuzzy set is another an extension of fuzzy set whose membership degree range is different from the above extensions. In 2000, Lee [19] initiated an extension of fuzzy set named bipolar-valued fuzzy set. Bipolar-valued fuzzy sets membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. In a bipolar-valued fuzzy set, the membership degree 0 indicate that elements are irrelevant to the corresponding property, the membership degrees on $(0, 1]$ assigne that elements some what satisfy the property, and the membership degrees on $[-1, 0)$ assigne that elements somewhat satisfy the implicit counterproperty [19]. However, it was not practical to express an uncertainty problem using fuzzy sets and its extensions.

Realizing the inadequacy of fuzzy set theory and extensions in expressing uncertainty problems, Molodsov [2] thought that this deficiency was due to the lack of a parameterization tool. Therefore, he [2] proposed the soft set theory in 1999 and gave some relevant features. Such theory is a general mathematical tool for dealing with uncertain, fuzzy, not clearly defined objects. Especially with the introduction of soft sets to the literature, the construction of hybrid set types has accelerated. This is due to the easy and practical applicability of the parameter tool. It is also because the hybrid set is more successful in expressing uncertainty, as it retains the properties of the set types that compose it. One of these hybrid sets is the bipolar fuzzy soft set, a combination of bipolar fuzzy set and soft set provided by Abdullah et al. [20]. As another example, the bipolar soft set with applications in decision making popularized by Shabir et al. [4] and discussed exhaustively by Karaaslan et al. [21] are another hybrid set model. This mathematical approach has managed to attract the attention of researchers since it was built with the contribution of a parameterization tool to this theory by addressing bipolar fuzzy sets, which is an effective generalization of fuzzy sets. In addition, we can easily say that the studies with hybrid cluster models introduced for the solution of uncertainty problems are increasing day by day [22, 23, 24, 27, 28].

In this paper, the concepts of (α, β)-cuts, first type semi-strong (α, β)-cuts, second type semi-strong (α, β)-cuts, strong (α, β)-cuts for bipolar fuzzy soft sets were introduced and some of their properties were examined. Moreover, the concepts of inverse (α, β)-cuts, first type semi-weak inverse (α, β)-cuts, second type semi-weak inverse (α, β)-cuts and weak inverse (α, β)-cuts for bipolar fuzzy soft sets were identified and some of their distinctive features were investigated. Thanks to these cuts, bipolar fuzzy soft sets can be expressed as bipolar soft sets, which in turn can assist us in the decision making process. In addition, related examples are given in the paper in order to better understand this situation.

Throughout this study, let $U = \{u_1, u_2, ..., u_m\}$ be a non-empty universe set and $E = \{x_1, x_2, ..., x_n\}$ be a set of parameters. Also, let $P(U)$ denote the power set of U and $A \subseteq E$.

2. PRELIMINARIES

Here, we remind some basic information from the literature for subsequent use.

2.1. **Fuzzy Sets.** It is possible to express definite expressions in classical mathematics with values of 0 ("false") and 1 ("true"). However, in real life this situation may not always be possible. For example; the FS theory (Zadeh 1965) put forward to present human thoughts expresses this situations in the interval $[0, 1]$ with the help of membership functions for better outcome. Zadeh expressed this set theory as follows,

Definition 1. *[1] A FS X over U is a set defined by a function μ_X representing a mapping*

$$\mu_X : U \to [0, 1]$$

μ_X is called the membership function of X, and the value $\mu_X(u)$ is called the grade of membership of $u \in U$. The value represents the degree of u belonging to the FS X. Thus, a FS X over U can be represented as follows:

$$X = \{(u, \mu_X(u)) : \mu_X(u) \in [0, 1], u \in \mathcal{U}\}$$

State that the set of all the FSs over U will be denoted by $F(U)$.

With Zadeh's [1] min-max system, FS union, intersection, and complement operations are defined below.

The union of two FSs M and N is a FS in U, denoted by $M \cup N$, whose membership grade is $\mu_{M\cup N}(u) = \mu_M(u) \vee \mu_N(u) = max\{\mu_M(u), \mu_N(u)\}$ for each $u \in U$. So

$$M \cup N = \Big\{\Big(u, \mu_{M\cup N}(u)\Big) : \mu_{M\cup N}(u) = max\{\mu_M(u), \mu_N(u)\}, \forall u \in U\Big\}.$$

The intersection of two FSs M and N is a FS in U, denoted by $M \cap N$, whose membership grade is $\mu_{M\cap N}(u) = \mu_M(u) \wedge \mu_N(u) = min\{\mu_M(u), \mu_N(u)\}$ for each $u \in U$. So

$$M \cap N = \Big\{\Big(u, \mu_{M\cap N}(u)\Big) : \mu_{M\cap N}(u) = min\{\mu_M(u), \mu_N(u)\}, \forall u \in U\Big\}.$$

Let D be a FS defined over U. Then its complement, denoted by D^c, is defined in terms of membership grade as $\mu_{D^c}(u) = 1 - \mu_D(u)$ for each $u \in U$.

$$D^c = \Big\{\Big(u, \mu_{D^c}(u)\Big) : u \in U\Big\}.$$

Definition 2. *[1] Let $X \in F(U)$ and $\alpha \in [0,1]$. Then the non-fuzzy set (or crisp set) $X_\alpha = \{u \in U : \mu_X(u) \geq \alpha\}$ is called the α-cut or α-level set of X.*

If the weak inequality $\geq$ is replaced by the strict inequality $>$, the it is called the strong α-cut of X, denoted by X_{α^+}. That is, $X_{\alpha^+} = \{u \in U : \mu_X(u) > \alpha\}$.

Definition 3. *[3] Let $X \in F(U)$ and $\alpha \in [0,1]$. Then the non-fuzzy set $X_\alpha^{-1} = \{u \in U : \mu_X(u) < \alpha\}$ is called an inverse α-cut or inverse α-level set of X.*

If the strict inequality $<$ is replaced by the weak inequality $\leq$, the it is called the weak inverse α-cut of X, denoted by $X_{\alpha^-}^{-1}$. That is, $X_{\alpha^-}^{-1} = \{u \in U : \mu_X(u) \leq \alpha\}$.

2.2. **Bipolar Fuzzy Sets.**

Definition 4. *[25, 26] Let U be any nonempty set. Then a bipolar fuzzy set, is an object of the form*

$$\chi = \{(u, <\mu_\chi^+(u), \mu_\chi^-(u)>) : u \in U\}$$

and $\mu_\chi^+ : U \to [0,1]$ and $\mu_\chi^- : U \to [-1,0]$, $\mu_\chi^+(u)$ is a positive material and $\mu_\chi^-(u)$ is a negative material of $u \in U$. For simplicity, we donate the bipolar fuzzy set as $\chi =< \mu_\chi^+, \mu_\chi^- >$ in its place of $\chi = \{(u, <\mu_\chi^+(u), \mu_\chi^-(u)>) : u \in U\}$.

Definition 5. *[25, 26] Let $\chi_1 =< \mu_{\chi_1}^+, \mu_{\chi_1}^- >$ and $\chi_2 =< \mu_{\chi_2}^+, \mu_{\chi_2}^- >$ be two bipolar fuzzy sets, on U. Then we define the following operations.*
(i) $\chi_1^c = \{< 1 - \mu_{\chi_1}^+(u), -1 - \mu_{\chi_1}^-(u) >\}$,
(ii) $\chi_1 \cup \chi_2 =< max(\mu_{\chi_1}^+(u), \mu_{\chi_2}^+(u)), min(\mu_{\chi_1}^-(u), \mu_{\chi_2}^-(u)) >$,
(iii) $\chi_1 \cap \chi_2 =< min(\mu_{\chi_1}^+(u), \mu_{\chi_2}^+(u)), max(\mu_{\chi_1}^-(u), \mu_{\chi_2}^-(u)) >$.

2.3. **Soft Sets and Bipolar Soft Sets.**

Definition 6. *[2] Let U be an initial universe, E be the set of parameters, $A \subset E$ and $P(U)$ is the power set of U. Then (F, A) is called a soft set, where $F : A \to P(U)$.*

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\epsilon \in A$, $F(\epsilon)$ may be considered as the set of ϵ-approximate elements of the soft set (F, A), or as the set of ϵ-approximate elements of the soft set.

Definition 7. *[5] Let $E = \{x_1, x_2, ..., x_n\}$ be a set of parameters. The NOT set of E denoted by $\neg E$ is defined by $\neg E = \{\neg x_1, \neg x_2, ..., \neg x_n\}$ where, $\neg x_i =$ not x_i for all i.*

Definition 8. *[4] A triplet (F, G, A) is called a bipolar soft set over U, where F and G are mappings, given by $F : A \to P(U)$ and $G : \neg A \to P(U)$ such that $F(x) \cap G(\neg x) = \emptyset$ (Empty Set) for all $x \in A$.*

Definition 9. *[6] Let (F, G, A) be a BSS over U. The presentation of*

$$(F, G, A) = \{(x, F(x), G(\neg x)) : x \in A \subseteq E, \neg x \in \neg A \subseteq \neg E \text{ and } F(x), G(\neg x) \in P(U)\}$$

is said to be a short expansion of BSS (F, G, A).

Example 10. *Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the set of five cars under consideration and $A = \{x_1 = Expensive, x_2 = Modern\ Technology, x_3 = Comfortable, x_4 = Fast\} \subseteq E$ be the set of parameters. Then*

$\neg A = \{x_1 = Cheap, x_2 = Classic\ Technology,\ x_3 = Not\ Comfortable, x_4 = Slow\} \subseteq \neg E$.

Suppose that a BSS (F, G, A) is given as follows.

$$F(x_1) = \{u_2, u_4\},\quad F(x_2) = \{u_1, u_4, u_5\},\quad F(x_3) = \{u_1, u_3, u_4\},\quad F(x_4) = \{u_3, u_5\},$$

$$G(\neg x_1) = \{u_1, u_5\},\quad G(\neg x_2) = \{u_2, u_3\},\quad G(\neg x_3) = \{u_5\},\quad G(\neg x_4) = \{u_2, u_4\}.$$

Then the short expansion of BSS (F, G, A) is denoted by

$$(F, G, A) = \left\{ \begin{array}{c} (x_1, \{u_2, u_4\}, \{u_1, u_5\}), (x_2, \{u_1, u_4, u_5\}, \{u_2, u_3\}), \\ (x_3, \{u_1, u_3, u_4\}, \{u_5\}), (x_4, \{u_3, u_5\}, \{u_2, u_4\}) \end{array} \right\}$$

Definition 11. *[4] For two bipolar soft sets (F, G, A) and (F_1, G_1, B) over a universe U, we say that (F, G, A) is a bipolar soft subset of (F_1, G_1, B), if,*
(1) $A \subseteq B$ and
(2) $F(e) \subseteq F_1(e)$ and $G_1(\neg x) \subseteq G(\neg x)$ for all $x \in A$.
This relationship is denoted by $(F, G, A)\tilde{\subseteq}(F_1, G_1, B)$. Similarly (F, G, A) is said to be a bipolar soft superset of (F_1, G_1, B), if (F_1, G_1, B) is a bipolar soft subset of (F, G, A). We denote it by $(F, G, A)\tilde{\supseteq}(F_1, G_1, B)$.

Definition 12. *[4] Two bipolar soft sets (F, G, A) and (F_1, G_1, B) over a universe U are said to be equal if (F, G, A) is a bipolar soft subset of (F_1, G_1, B) and (F_1, G_1, B) is a bipolar soft subset of (F, G, A).*

Definition 13. *[4] The complement of a bipolar soft set (F, G, A) is denoted by $(F, G, A)^c$ and is defined by $(F, G, A)^c = (F^c, G^c, A)$ where F^c and G^c are mappings given by $F^c(x) = G(\neg x)$ and $G^c(\neg x) = F(x)$ for all $x \in A$.*

Definition 14. *[4] Extended Union of two bipolar soft sets (F, G, A) and (F_1, G_1, B) over the common universe U is the bipolar soft set (H, I, C) over U, where $C = A \cup B$ and for all $x \in C$,*

$$H(x) = \begin{cases} F(x) & \text{if } x \in A - B \\ F_1(x) & \text{if } x \in B - A \\ F(x) \cup F_1(x) & \text{if } x \in A \cap B \end{cases}$$

$$I(\neg x) = \begin{cases} G(\neg x) & \text{if } \neg x \in (\neg A) - (\neg B) \\ G_1(\neg x) & \text{if } \neg x \in (\neg B) - (\neg A) \\ G(\neg x) \cap G_1(\neg x) & \text{if } \neg x \in (\neg A) \cap (\neg B) \end{cases}$$

We denote it by $(F, G, A)\tilde{\cup}(F_1, G_1, B) = (H, I, C)$.

Definition 15. *[4] Extended Intersection of two bipolar soft sets (F, G, A) and (F_1, G_1, B) over the common universe U is the bipolar soft set (H, I, C) over U,* *where $C = A \cup B$ and for all $x \in C$,*

$$H(x) = \begin{cases} F(x) & \text{if } x \in A - B \\ F_1(x) & \text{if } x \in B - A \\ F(x) \cap F_1(x) & \text{if } x \in A \cap B \end{cases}$$

$$I(\neg x) = \begin{cases} G(x) & \text{if } x \in (\neg A) - (\neg B) \\ G_1(x) & \text{if } x \in (\neg B) - (\neg A) \\ G(x) \cup G_1(x) & \text{if } x \in (\neg A) \cap (\neg B) \end{cases}$$

We denote it by $(F, G, A)\tilde{\cap}(F_1, G_1, B) = (H, I, C)$.

Definition 16. *[4] Restricted Union of two bipolar soft sets (F, G, A) and (F_1, G_1, B) over the common universe U is the bipolar soft set (H, I, C), where $C = A \cap B$ is non-empty and for all $x \in C$*

$$H(x) = F(x) \cup G(x) \quad \text{and} \quad I(\neg x) = F_1(\neg x) \cap G_1(\neg x)$$

We denote it by $(F, G, A) \cup_{\mathfrak{R}} (F_1, G_1, B) = (H, I, C)$.

Definition 17. *[4] Restricted Intersection of two bipolar soft sets (F, G, A) and (F_1, G_1, B) over the common universe U is the bipolar soft set (H, I, C), where $C = A \cap B$ is non-empty and for all $x \in C$*

$$H(x) = F(x) \cap G(x) \quad \text{and} \quad I(\neg x) = F_1(\neg x) \cup G_1(\neg x)$$

We denote it by $(F, G, A) \cap_{\mathfrak{R}} (F_1, G_1, B) = (H, I, C)$.

2.4. **Bipolar Fuzzy Soft Sets.**

Definition 18. *[20] Define $f : A \to BF^U$, where BF^U is the collection of all bipolar fuzzy subsets of U. Then (f, A), denoted by f_A, is said to be a bipolar fuzzy soft set over a universe U. It is defined by*

$$f_A = \left\{ \left(u, \mu^+_{(f_A)_x}(u), \mu^-_{(f_A)_x}(u) \right) : \forall u \in U, x \in A \right\}$$

Example 19. *Let $U = \{u_1, u_2, u_3, u_4\}$ be the set of four computers under consideration and $A = \{x_1 = Modern\ Technology, x_2 = Cost, x_3 = Fast\} \subseteq E$ be the set of parameters. Then,*

$$f_A = \left\{ \begin{array}{c} f(x_1) = \left\{ \begin{array}{c} (u_1, 0.45, -0.2), (u_2, 0.6, -0.43), \\ (u_3, 0.7, -0.35), (u_4, 0.55, -0.25) \end{array} \right\}, \\ f(x_2) = \left\{ \begin{array}{c} (u_1, 0.34, -0.65), (u_2, 0.32, -0.22), \\ (u_3, 0.48, -0.24), (u_4, 0.64, -0.8) \end{array} \right\}, \\ f(x_3) = \left\{ \begin{array}{c} (u_1, 0.9, -0.15), (u_2, 0.72, -0.34), \\ (u_3, 0.34, -0.56), (u_4, 0.24, -0.87) \end{array} \right\} \end{array} \right\}$$

Definition 20. *[20] Let U be a universe and E a set of attributes. Then, (U, E) is the collection of all bipolar fuzzy soft sets on U with attributes from E and is said to be bipolar fuzzy soft class.*

Definition 21. *[20] Let f_A and g_B be two bipolar fuzzy soft sets over a common universe U. We say that f_A is a bipolar fuzzy soft subset of g_B, if*
(i) $A \subseteq B$ and
(ii) For all $x \in A$, $f(x)$ is a bipolar fuzzy subset of $g(x)$. We write $f_A \widehat{\subseteq} g_B$.

Moreover, we say that f_A and g_B are bipolar fuzzy soft equal sets if f_A is a bipolar fuzzy soft subset of g_B and g_B is a bipolar fuzzy soft subset of f_A.

Definition 22. *[20] The complement of a bipolar fuzzy soft set f_A is denoted ${f_A}^c$ and is defined by ${f_A}^c = \Big\{ \Big(u, 1 - \mu^{+}_{(f_A)_x}(u), -1 - \mu^{-}_{(f_A)_x}(u) \Big) : \forall u \in U, x \in A \Big\}$.*

It should be noted that $1 - f(x)$ denotes the fuzzy complement of $f(x)$ for $x \in A$.

Definition 23. *[20] Let f_A and g_B be two bipolar fuzzy soft sets over a common universe U. Then*
(i) The union of bipolar fuzzy soft sets f_A and g_B is defined as the bipolar fuzzy soft set $h_C = f_A \widehat{\cup} g_B$ over U, where $C = A \cup B$, $h : C \to BF^U$ and

$$h(e) = \begin{cases} f(x) & \text{if } x \in A \setminus B \\ g(x) & \text{if } x \in B \setminus A \\ f(x) \cup g(x) & \text{if } x \in A \cap B \end{cases}$$

for all $x \in C$.
(ii) The restricted union of bipolar fuzzy soft sets f_A and g_B is defined as the bipolar fuzzy soft set $h_C = f_A \widehat{\cup}_{\mathcal{R}} g_B$ over U, where $C = A \cap B \neq \emptyset$, $h : C \to BF^U$ and $h(x) = f(x) \cup g(x)$ for all $x \in C$.
(iii) The extended intersection of bipolar fuzzy soft sets f_A and g_B is defined as the bipolar fuzzy soft set $h_C = f_A \widehat{\cap} g_B$ over U, where $C = A \cup B$, $h : C \to BF^U$ and

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \setminus B \\ g(x) & \text{if } x \in B \setminus A \\ f(x) \cap g(x) & \text{if } x \in A \cap B \end{cases}$$

for all $x \in C$.
(iv) The restricted intersection of bipolar fuzzy soft sets f_A and g_B is defined as the bipolar fuzzy soft set $h_C = f_A \widehat{\cap}_{\mathcal{R}} g_B$ over U, where $C = A \cap B \neq \emptyset$, $h : C \to BF^U$ and $h(x) = f(x) \cap g(x)$ for all $x \in C$.

3. (α, β)-cuts and its Properties in Bipolar Fuzzy Soft Sets

In this section, the concepts of (α, β)-cuts and strong (α, β)-cuts of BFSSs were introduced together with some of their properties.

Definition 24. *Let f_A be a BFSS over U and $\alpha \in [0, 1]$, $\beta \in [-1, 0]$. Then the (α, β)-cut or (α, β)-level BSS of f_A denoted by $[f_A]_{(\alpha,\beta)}$ is defined as*

$$[f_A]_{(\alpha,\beta)} = \Big\{ \Big(x, \widehat{F}^{(\alpha,\beta)}_{[f_A]}(x), \widehat{G}^{(\alpha,\beta)}_{[f_A]}(\neg x) \Big) : x \in A \subseteq E, \neg x \in \neg A \subseteq \neg E \Big\}$$

where

$$\widehat{F}^{(\alpha,\beta)}_{[f_A]}(x)=\left\{u:\left[\mu^{+}_{[f_A]_x}(u)\geq\alpha\right]\wedge\left[\mu^{-}_{[f_A]_x}(u)\leq\beta\right]\wedge\left[\mu^{+}_{[f_A]_x}(u)\geq\left|\mu^{-}_{[f_A]_x}(u)\right|\right]\right\},$$

$$\widehat{G}^{(\alpha,\beta)}_{[f_A]}(\neg x)=\left\{u:\left[\mu^{+}_{[f_A]_x}(u)\geq\alpha\right]\wedge\left[\mu^{-}_{[f_A]_x}(u)\leq\beta\right]\wedge\left[\mu^{+}_{[f_A]_x}(u)<\left|\mu^{-}_{[f_A]_x}(u)\right|\right]\right\}.$$

The first type semi-strong (α,β)-cut, denoted by $[f_A]_{(\alpha^+,\beta)}$ is defined as

$$[f_A]_{(\alpha^+,\beta)}=\left\{\left(x,\widehat{F}^{(\alpha^+,\beta)}_{[f_A]}(x),\widehat{G}^{(\alpha^+,\beta)}_{[f_A]}(\neg x)\right):x\in A\subseteq E,\neg x\in\neg A\subseteq\neg E\right\}$$

where

$$\widehat{F}^{(\alpha^+,\beta)}_{[f_A]}(x)=\left\{u:\left[\mu^{+}_{[f_A]_x}(u)>\alpha\right]\wedge\left[\mu^{-}_{[f_A]_x}(u)\leq\beta\right]\wedge\left[\mu^{+}_{[f_A]_x}(u)\geq\left|\mu^{-}_{[f_A]_x}(u)\right|\right]\right\},$$

$$\widehat{G}^{(\alpha^+,\beta)}_{[f_A]}(\neg x)=\left\{u:\left[\mu^{+}_{[f_A]_x}(u)>\alpha\right]\wedge\left[\mu^{-}_{[f_A]_x}(u)\leq\beta\right]\wedge\left[\mu^{+}_{[f_A]_x}(u)<\left|\mu^{-}_{[f_A]_x}(u)\right|\right]\right\}.$$

The second type semi-strong (α,β)-cut, denoted by $[f_A]_{(\alpha,\beta^+)}$ is defined as

$$[f_A]_{(\alpha,\beta^+)}=\left\{\left(x,\widehat{F}^{(\alpha,\beta^+)}_{[f_A]}(x),\widehat{G}^{(\alpha,\beta^+)}_{[f_A]}(\neg x)\right):x\in A\subseteq E,\neg x\in\neg A\subseteq\neg E\right\}$$

where

$$\widehat{F}^{(\alpha,\beta^+)}_{[f_A]}(x)=\left\{u:\left[\mu^{+}_{[f_A]_x}(u)\geq\alpha\right]\wedge\left[\mu^{-}_{[f_A]_x}(u)<\beta\right]\wedge\left[\mu^{+}_{[f_A]_x}(u)\geq\left|\mu^{-}_{[f_A]_x}(u)\right|\right]\right\},$$

$$\widehat{G}^{(\alpha,\beta^+)}_{[f_A]}(\neg x)=\left\{u:\left[\mu^{+}_{[f_A]_x}(u)\geq\alpha\right]\wedge\left[\mu^{-}_{[f_A]_x}(u)<\beta\right]\wedge\left[\mu^{+}_{[f_A]_x}(u)<\left|\mu^{-}_{[f_A]_x}(u)\right|\right]\right\}.$$

The strong (α,β)-cut, denoted by $[f_A]_{(\alpha^+,\beta^+)}$ is defined as

$$[f_A]_{(\alpha^+,\beta^+)}=\left\{\left(x,\widehat{F}^{(\alpha^+,\beta^+)}_{[f_A]}(x),\widehat{G}^{(\alpha^+,\beta^+)}_{[f_A]}(\neg x)\right):x\in A\subseteq E,\neg x\in\neg A\subseteq\neg E\right\}$$

where

$$\widehat{F}^{(\alpha^+,\beta^+)}_{[f_A]}(x)=\left\{u:\left[\mu^{+}_{[f_A]_x}(u)>\alpha\right]\wedge\left[\mu^{-}_{[f_A]_x}(u)<\beta\right]\wedge\left[\mu^{+}_{[f_A]_x}(u)\geq\left|\mu^{-}_{[f_A]_x}(u)\right|\right]\right\},$$

$$\widehat{G}^{(\alpha^+,\beta^+)}_{[f_A]}(\neg x)=\left\{u:\left[\mu^{+}_{[f_A]_x}(u)>\alpha\right]\wedge\left[\mu^{-}_{[f_A]_x}(u)<\beta\right]\wedge\left[\mu^{+}_{[f_A]_x}(u)<\left|\mu^{-}_{[f_A]_x}(u)\right|\right]\right\}.$$

Example 25. *Let $U=\{u_1,u_2,u_3\}$, $A=\{x_1,x_2,x_3\}\subseteq E$ and BFSS f_A over U be*

$$f_A=\left\{\begin{array}{c} f(x_1)=\{\ (u_1,0.56,-0.42),(u_2,0.75,-0.5),(u_3,0.5,-0.3)\ \},\\ f(x_2)=\{\ (u_1,0.8,-0.15),(u_2,0.4,-0.56),(u_3,0.64,-0.15)\ \},\\ f(x_3)=\{\ (u_1,0.35,-0.6),(u_2,0.1,-0.5),(u_3,0.56,-0.2)\ \}\end{array}\right\}$$

For example; let U be the supplier firms that apply to become a supplier of a pharmaceutical company and $A\subseteq E$ is the set of parameters that the company wants from the supplier. This BFSS is represented in tabular form as follows:

TABLE 1. Representation of BFSS f_A

$U\setminus E$	$Experienced=x_1$	$Cheap=x_2$	$Fast=x_3$
u_1	$<0.56,-0.42>$	$<0.8,-0.15>$	$<0.35,-0.6>$
u_2	$<0.75,-0.5>$	$<0.4,-0.56>$	$<0.64,-0.15>$
u_3	$<0.35,-0.6>$	$<0.1,-0.5>$	$<0.56,-0.2>$

Example 26. *Then if* $\alpha = 0.56$ *and* $\beta = -0.5$*, we have*

$$[f_A]_{(0.56,-0.5)} = \{(x_1, \{u_1, u_2\}, \{u_3\}), (x_2, \{u_1\}, \{u_2, u_3\}), (x_3, \{u_2, u_3\}, \{u_1\})\},$$
$$[f_A]_{(0.56^+,-0.5)} = \{(x_1, \{u_2\}, \{u_3\}), (x_2, \{u_1\}, \{u_2, u_3\}), (x_3, \{u_2\}, \{u_1\})\},$$
$$[f_A]_{(0.56,-0.5^+)} = \{(x_1, \{u_1, u_2\}, \{u_3\}), (x_2, \{u_1\}, \{u_2\}), (x_3, \{u_2, u_3\}, \{u_1\})\}$$

and

$$[f_A]_{(0.56^+,-0.5^+)} = \{(x_1, \{u_2\}, \{u_3\}), (x_2, \{u_1\}, \{u_2\}), (x_3, \{u_2\}, \{u_1\})\}.$$

Then if $\alpha = 0.35$ *and* $\beta = -0.6$*, we have*

$$[f_A]_{(0.35,-0.6)} = \{(x_1, \{u_1, u_2\}, \{u_3\}), (x_2, \{u_1, u_2\}, \{\}), (x_3, \{u_2, u_3\}, \{u_1\})\},$$
$$[f_A]_{(0.35^+,-0.6)} = \{(x_1, \{u_1, u_2\}, \{u_3\}), (x_2, \{u_1, u_2\}, \{\}), (x_3, \{u_2, u_3\}, \{u_1\})\},$$
$$[f_A]_{(0.35,-0.6^+)} = \{(x_1, \{u_1, u_2\}, \{\}), (x_2, \{u_1, u_2\}, \{\}), (x_3, \{u_2, u_3\}, \{\})\}$$

and

$$[f_A]_{(0.35^+,-0.6^+)} = \{(x_1, \{u_1, u_2\}, \{\}), (x_2, \{u_1, u_2\}, \{\}), (x_3, \{u_2, u_3\}, \{\})\}.$$

Remark 27. (α, β)*-cut can be use to make a decision. For example, let's assume that the pharmaceutical company will consider the most suitable supplier firm as the firm that provides the most number of parameters under* (α, β)*. For this, the mapping* $\Theta_{[f_A]_{(\alpha,\beta)}}$ *is defined by* $\Theta_{[f_A]_{(\alpha,\beta)}} : U \to [-n, n]$ *for all* $u_i \in U$ *as follows: (*$1 \le i \le s(E) = n$ *and* $1 \le j \le s(U) = m$*)*

$$\Theta_{[f_A]_{(\alpha,\beta)}}(u_i) = \sum_{j=1}^{n} \Upsilon^{ij}_{[f_A]_{(\alpha,\beta)}} \tag{1}$$

$$\Upsilon^{ij}_{[f_A]_{(\alpha,\beta)}} = \begin{cases} 1, & \text{if } u_i \in \widehat{F}^{(\alpha,\beta)}_{[f_A]}(x_j) \\ -1, & \text{if } u_i \in \widehat{G}^{(\alpha,\beta)}_{[f_A]}(x_j) \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Here, the value $\Theta_{[f_A]_{(\alpha,\beta)}}(u_i)$ *is called the "total score" for the objects and the greater the total score of an object, the more recommended it is to select that object. Under these conditions, the calculation of the total scores for* $\alpha = 0.56$ *and* $\beta = -0.5$ *given in Example 25 is as follows;*

$$\Theta_{[f_A]_{(0.56,-0.5)}}(u_1) = \Upsilon^{11}_{[f_A]_{(0.56,-0.5)}} + \Upsilon^{12}_{[f_A]_{(0.56,-0.5)}} + \Upsilon^{13}_{[f_A]_{(0.56,-0.5)}} = 1+1+(-1) = 1,$$
$$\Theta_{[f_A]_{(0.56,-0.5)}}(u_2) = 1, \quad \Theta_{[f_A]_{(0.56,-0.5)}}(u_3) = -1.$$

Similarly, for $\alpha = 0.35$ *and* $\beta = -0.6$

$$\Theta_{[f_A]_{(0.35,-0.6)}}(u_1) = 1, \quad \Theta_{[f_A]_{(0.35,-0.6)}}(u_2) = 3, \quad \Theta_{[f_A]_{(0.35,-0.6)}}(u_3) = 2.$$

As can be seen, it is not possible to choose the best element for $\alpha = 0.56$ *and* $\beta = -0.5$*, because there are two supplier firms that have the highest total score. However, the total scores calculated for* $\alpha = 0.35$ *and* $\beta = -0.6$ *indicate that the most suitable supplier firm for the pharmaceutical company is* u_2*.*

Proposition 28. *Let $\alpha \in [0,1]$, $\beta \in [-1,0]$ and f_A, g_B be BFSSs over U, the following properties hold:*
(i) $[f_A]_{(\alpha^+,\beta^+)} \widetilde{\subseteq} [f_A]_{(\alpha^+,\beta)} \widetilde{\subseteq} [f_A]_{(\alpha,\beta)}$ *and* $[f_A]_{(\alpha^+,\beta^+)} \widetilde{\subseteq} [f_A]_{(\alpha,\beta^+)} \widetilde{\subseteq} [f_A]_{(\alpha,\beta)}$.
(ii) $[f_A]_{(\alpha^+,\beta)} \widetilde{\cap} [f_A]_{(\alpha,\beta^+)} = [f_A]_{(\alpha^+,\beta^+)}$.
(iii) If $\alpha_1 \leq \alpha_2$ *and* $\beta_1 \geq \beta_2$, *then* $[f_A]_{(\alpha_2,\beta_2)} \widetilde{\subseteq} [f_A]_{(\alpha_1,\beta_1)}$.
(iv) $[f_A \widehat{\cup} g_B]_{(\alpha,\beta)} = [f_A]_{(\alpha,\beta)} \widetilde{\cup} [g_B]_{(\alpha,\beta)}$.
(v) $[f_A \widehat{\cap} g_B]_{(\alpha,\beta)} = [f_A]_{(\alpha,\beta)} \widetilde{\cap} [g_B]_{(\alpha,\beta)}$.

Proof. (i) Let $(x,\{u_i\},\{u_j\}) \in [f_A]_{(\alpha^+,\beta^+)}$
$\Rightarrow \left[\mu^+_{[f_A]_x}(u_i) > \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_i) < \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_i) \geq \left|\mu^-_{[f_A]_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]_x}(u_j) > \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_j) < \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_j) < \left|\mu^-_{[f_A]_x}(u_j)\right|\right]$, $\forall x \in A$
$\Rightarrow \left[\mu^+_{[f_A]_x}(u_i) > \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_i) \leq \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_i) \geq \left|\mu^-_{[f_A]_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]_x}(u_j) > \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_j) \leq \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_j) < \left|\mu^-_{[f_A]_x}(u_j)\right|\right]$, $\forall x \in A$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A]_{(\alpha^+,\beta)}$
Therefore $[f_A]_{(\alpha^+,\beta^+)} \widetilde{\subseteq} [f_A]_{(\alpha^+,\beta)}$. Similarity, for $(x,\{u_i\},\{u_j\}) \in [f_A]_{(\alpha^+,\beta)}$
$\Rightarrow \left[\mu^+_{[f_A]_x}(u_i) > \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_i) \leq \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_i) \geq \left|\mu^-_{[f_A]_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]_x}(u_j) > \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_j) \leq \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_j) < \left|\mu^-_{[f_A]_x}(u_j)\right|\right]$, $\forall x \in A$
$\Rightarrow \left[\mu^+_{[f_A]_x}(u_i) \geq \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_i) \leq \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_i) \geq \left|\mu^-_{[f_A]_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]_x}(u_j) \geq \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_j) \leq \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_j) < \left|\mu^-_{[f_A]_x}(u_j)\right|\right]$, $\forall x \in A$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A]_{(\alpha,\beta)}$
Therefore $[f_A]_{(\alpha^+,\beta)} \widetilde{\subseteq} [f_A]_{(\alpha^+,\beta^+)}$. It is proved similarly in the other part.
(ii) Straighforward.
(iii) It is clear from Definition 11 and Definition 24.
(iv) Let $(x,\{u_i\},\{u_j\}) \in [f_A \widehat{\cup} g_B]_{(\alpha,\beta)}$
$\Rightarrow \left[\mu^+_{[f_A \widehat{\cup} g_B]_x}(u_i) \geq \alpha\right] \wedge \left[\mu^-_{[f_A \widehat{\cup} g_B]_x}(u_i) \leq \beta\right] \wedge \left[\mu^+_{[f_A \widehat{\cup} g_B]_x}(u_i) \geq \left|\mu^-_{[f_A \widehat{\cup} g_B]_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A \widehat{\cup} g_B]_x}(u_j) \geq \alpha\right] \wedge \left[\mu^-_{[f_A \widehat{\cup} g_B]_x}(u_j) \leq \beta\right] \wedge \left[\mu^+_{[f_A \widehat{\cup} g_B]_x}(u_j) < \left|\mu^-_{[f_A \widehat{\cup} g_B]_x}(u_j)\right|\right]$, $\forall x \in A \cup B$
$\Rightarrow \Bigg[\left[\mu^+_{[f_A]_x}(u_i) \geq \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_i) \leq \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_i) \geq \left|\mu^-_{[f_A]_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]_x}(u_j) \geq \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_j) \leq \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_j) < \left|\mu^-_{[f_A]_x}(u_j)\right|\right], \forall x \in A\Bigg]$ or $\Bigg[\left[\mu^+_{[g_B]_x}(u_i) \geq \alpha\right] \wedge \left[\mu^-_{[g_B]_x}(u_i) \leq \beta\right] \wedge \left[\mu^+_{[g_B]_x}(u_i) \geq \left|\mu^-_{[g_B]_x}(u_i)\right|\right]$ and $\left[\mu^+_{[g_B]_x}(u_j) \geq \alpha\right] \wedge \left[\mu^-_{[g_B]_x}(u_j) \leq \beta\right] \wedge \left[\mu^+_{[g_B]_x}(u_j) < \left|\mu^-_{[g_B]_x}(u_j)\right|\right], \forall x \in B\Bigg]$

$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A]_{(\alpha,\beta)}$ or $(x,\{u_i\},\{u_j\}) \in [g_B]_{(\alpha,\beta)}$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A]_{(\alpha,\beta)} \widetilde{\cup} [g_B]_{(\alpha,\beta)}$
Therefore, $[f_A \widehat{\cup} g_B]_{(\alpha,\beta)} \widetilde{\subset} [f_A]_{(\alpha,\beta)} \widetilde{\cup} [g_B]_{(\alpha,\beta)}$.

Conversely, suppose $(x,\{u_i\},\{u_j\}) \in [f_A]_{(\alpha,\beta)} \widetilde{\cup} [g_B]_{(\alpha,\beta)}$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A]_{(\alpha,\beta)}$ or $(x,\{u_i\},\{u_j\}) \in [g_B]_{(\alpha,\beta)}$
$\Rightarrow \Big[\Big[\mu^{+}_{[f_A]_x}(u_i) \geq \alpha\Big] \wedge \Big[\mu^{-}_{[f_A]_x}(u_i) \leq \beta\Big] \wedge \Big[\mu^{+}_{[f_A]_x}(u_i) \geq \Big|\mu^{-}_{[f_A]_x}(u_i)\Big|\Big]$ and $\Big[\mu^{+}_{[f_A]_x}(u_j) \geq \alpha\Big] \wedge \Big[\mu^{-}_{[f_A]_x}(u_j) \leq \beta\Big] \wedge \Big[\mu^{+}_{[f_A]_x}(u_j) < \Big|\mu^{-}_{[f_A]_x}(u_j)\Big|\Big], \forall x \in A\Big]$ or $\Big[\Big[\mu^{+}_{[g_B]_x}(u_i) \geq \alpha\Big] \wedge \Big[\mu^{-}_{[g_B]_x}(u_i) \leq \beta\Big] \wedge \Big[\mu^{+}_{[g_B]_x}(u_i) \geq \Big|\mu^{-}_{[g_B]_x}(u_i)\Big|\Big]$ and $\Big[\mu^{+}_{[g_B]_x}(u_j) \geq \alpha\Big] \wedge \Big[\mu^{-}_{[g_B]_x}(u_j) \leq \beta\Big] \wedge \Big[\mu^{+}_{[g_B]_x}(u_j) < \Big|\mu^{-}_{[g_B]_x}(u_j)\Big|\Big], \forall x \in B\Big]$
$\Rightarrow \Big[\mu^{+}_{[f_A \widehat{\cup} g_B]_x}(u_i) \geq \alpha\Big] \wedge \Big[\mu^{-}_{[f_A \widehat{\cup} g_B]_x}(u_i) \leq \beta\Big] \wedge \Big[\mu^{+}_{[f_A \widehat{\cup} g_B]_x}(u_i) \geq \Big|\mu^{-}_{[f_A \widehat{\cup} g_B]_x}(u_i)\Big|\Big]$ and $\Big[\mu^{+}_{[f_A \widehat{\cup} g_B]_x}(u_j) \geq \alpha\Big] \wedge \Big[\mu^{-}_{[f_A \widehat{\cup} g_B]_x}(u_j) \leq \beta\Big] \wedge \Big[\mu^{+}_{[f_A \widehat{\cup} g_B]_x}(u_j) < \Big|\mu^{-}_{[f_A \widehat{\cup} g_B]_x}(u_j)\Big|\Big]$, $\forall x \in A \cup B$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A \widehat{\cup} g_B]_{(\alpha,\beta)}$
Therefore, $[f_A]_{(\alpha,\beta)} \widetilde{\cup} [g_B]_{(\alpha,\beta)} \widetilde{\subset} [f_A \widehat{\cup} g_B]_{(\alpha,\beta)}$. Thus $[f_A \widehat{\cup} g_B]_{(\alpha,\beta)} = [f_A]_{(\alpha,\beta)} \widetilde{\cup} [g_B]_{(\alpha,\beta)}$.

(v) It is proved similar to step (iv). □

4. Inverse (α,β)-cuts and its Properties in Bipolar Fuzzy Soft Sets

In this section, the concepts of inverse (α,β)-cuts and weak inverse (α,β)-cuts of BFSSs were introduced together with some of their properties.

Definition 29. *Let f_A be a BFSS over U and $\alpha \in [0,1]$, $\beta \in [-1,0]$. Then the inverse (α,β)-cut or inverse (α,β)-level BSS of f_A denoted by $[f_A]^{-1}_{(\alpha,\beta)}$ is defined as*

$$[f_A]^{-1}_{(\alpha,\beta)} = \Big\{\Big(x, \widehat{F}^{(\alpha,\beta)}_{[f_A]^{-1}}(x), \widehat{G}^{(\alpha,\beta)}_{[f_A]^{-1}}(\neg x)\Big) : x \in A \subseteq E, \neg x \in \neg A \subseteq \neg E\Big\}$$

where

$$\widehat{F}^{(\alpha,\beta)}_{[f_A]^{-1}}(x) = \Big\{u : \Big[\mu^{+}_{[f_A]^{-1}_x}(u) < \alpha\Big] \wedge \Big[\mu^{-}_{[f_A]^{-1}_x}(u) > \beta\Big] \wedge \Big[\mu^{+}_{[f_A]^{-1}_x}(u) \geq \Big|\mu^{-}_{[f_A]^{-1}_x}(u)\Big|\Big]\Big\},$$

$$\widehat{G}^{(\alpha,\beta)}_{[f_A]^{-1}_x}(\neg x) = \Big\{u : \Big[\mu^{+}_{[f_A]^{-1}_x}(u) < \alpha\Big] \wedge \Big[\mu^{-}_{[f_A]^{-1}_x}(u) > \beta\Big] \wedge \Big[\mu^{+}_{[f_A]^{-1}_x}(u) < \Big|\mu^{-}_{[f_A]^{-1}_x}(u)\Big|\Big]\Big\}.$$

The first type semi-weak inverse (α,β)-cut, denoted by $[f_A]^{-1}_{(\alpha^-,\beta)}$ is defined as

$$[f_A]^{-1}_{(\alpha^-,\beta)} = \Big\{\Big(x, \widehat{F}^{(\alpha^-,\beta)}_{[f_A]^{-1}}(x), \widehat{G}^{(\alpha^-,\beta)}_{[f_A]^{-1}}(\neg x)\Big) : x \in A \subseteq E, \neg x \in \neg A \subseteq \neg E\Big\}$$

where

$\widehat{F}^{(\alpha^-,\beta)}_{[f_A]^{-1}}(x) = \Big\{u : \Big[\mu^+_{[f_A]^{-1}_x}(u) \le \alpha\Big] \wedge \Big[\mu^-_{[f_A]^{-1}_x}(u) > \beta\Big] \wedge \Big[\mu^+_{[f_A]^{-1}_x}(u) \ge \Big|\mu^-_{[f_A]^{-1}_x}(u)\Big|\Big]\Big\}$,

$\widehat{G}^{(\alpha^-,\beta)}_{[f_A]^{-1}}(\neg x) = \Big\{u : \Big[\mu^+_{[f_A]^{-1}_x}(u) \le \alpha\Big] \wedge \Big[\mu^-_{[f_A]^{-1}_x}(u) > \beta\Big] \wedge \Big[\mu^+_{[f_A]^{-1}_x}(u) < \Big|\mu^-_{[f_A]^{-1}_x}(u)\Big|\Big]\Big\}$.

The second type semi-weak inverse (α,β)-cut, denoted by $[f_A]^{-1}_{(\alpha,\beta^-)}$ is defined as

$$[f_A]^{-1}_{(\alpha,\beta^-)} = \Big\{\Big(x, \widehat{F}^{(\alpha,\beta^-)}_{[f_A]^{-1}}(x), \widehat{G}^{(\alpha,\beta^-)}_{[f_A]^{-1}}(\neg x)\Big) : x \in A \subseteq E, \neg x \in \neg A \subseteq \neg E\Big\}$$

where

$\widehat{F}^{(\alpha,\beta^-)}_{[f_A]^{-1}}(x) = \Big\{u : \Big[\mu^+_{[f_A]^{-1}_x}(u) < \alpha\Big] \wedge \Big[\mu^-_{[f_A]^{-1}_x}(u) \ge \beta\Big] \wedge \Big[\mu^+_{[f_A]^{-1}_x}(u) \ge \Big|\mu^-_{[f_A]^{-1}_x}(u)\Big|\Big]\Big\}$,

$\widehat{G}^{(\alpha,\beta^-)}_{[f_A]^{-1}}(\neg x) = \Big\{u : \Big[\mu^+_{[f_A]^{-1}_x}(u) < \alpha\Big] \wedge \Big[\mu^-_{[f_A]^{-1}_x}(u) \ge \beta\Big] \wedge \Big[\mu^+_{[f_A]^{-1}_x}(u) < \Big|\mu^-_{[f_A]^{-1}_x}(u)\Big|\Big]\Big\}$.

The weak inverse (α,β)-cut, denoted by $[f_A]^{-1}_{(\alpha^-,\beta^-)}$ is defined as

$$[f_A]^{-1}_{(\alpha^-,\beta^-)} = \Big\{\Big(x, \widehat{F}^{(\alpha^-,\beta^-)}_{[f_A]^{-1}}(x), \widehat{G}^{(\alpha^-,\beta^-)}_{[f_A]^{-1}}(\neg x)\Big) : x \in A \subseteq E, \neg x \in \neg A \subseteq \neg E\Big\}$$

where

$\widehat{F}^{(\alpha^-,\beta^-)}_{[f_A]^{-1}}(x) = \Big\{u : \Big[\mu^+_{[f_A]^{-1}_x}(u) \le \alpha\Big] \wedge \Big[\mu^-_{[f_A]^{-1}_x}(u) \ge \beta\Big] \wedge \Big[\mu^+_{[f_A]^{-1}_x}(u) \ge \Big|\mu^-_{[f_A]^{-1}_x}(u)\Big|\Big]\Big\}$,

$\widehat{G}^{(\alpha^-,\beta^-)}_{[f_A]^{-1}}(\neg x) = \Big\{u : \Big[\mu^+_{[f_A]^{-1}_x}(u) \le \alpha\Big] \wedge \Big[\mu^-_{[f_A]^{-1}_x}(u) \ge \beta\Big] \wedge \Big[\mu^+_{[f_A]^{-1}_x}(u) < \Big|\mu^-_{[f_A]^{-1}_x}(u)\Big|\Big]\Big\}$.

Example 30. *Consider the BFSS f_A as given in Example 25.*
Then if $\alpha = 0.56$ and $\beta = -0.5$, we have

$$[f_A]^{-1}_{(0.56,-0.5)} = \{(x_1,\{u_3\},\{u_1\}),(x_2,\{u_2,u_3\},\{u_1\}),(x_3,\{u_1\},\{u_2,u_3\})\},$$

$$[f_A]^{-1}_{(0.56^-,-0.5)} = \{(x_1,\{u_1,u_3\},\{\}),(x_2,\{u_2,u_3\},\{u_1\}),(x_3,\{u_1,u_3\},\{u_2\})\},$$

$$[f_A]^{-1}_{(0.56,-0.5^-)} = \{(x_1,\{u_3\},\{u_1,u_2\}),(x_2,\{u_2\},\{u_1,u_3\}),(x_3,\{u_1\},\{u_2,u_3\})\}$$

and

$$[f_A]^{-1}_{(0.56^-,-0.5^-)} = \{(x_1,\{u_1,u_3\},\{u_2\}),(x_2,\{u_2\},\{u_1,u_3\}),(x_3,\{u_1,u_3\},\{u_2\})\}.$$

Then if $\alpha = 0.35$ and $\beta = -0.6$, we have

$$[f_A]^{-1}_{(0.35,-0.6)} = \{(x_1,\{\},\{u_1,u_2\}),(x_2,\{\},\{u_1,u_2,u_3\}),(x_3,\{\},\{u_2,u_3\})\},$$

$$[f_A]^{-1}_{(0.35^-,-0.6)} = \{(x_1,\{u_3\},\{u_1,u_2\}),(x_2,\{\},\{u_1,u_2,u_3\}),(x_3,\{u_1\},\{u_2,u_3\})\},$$

$$[f_A]^{-1}_{(0.35,-0.6^-)} = \{(x_1,\{\},\{u_1,u_2,u_3\}),(x_2,\{\},\{u_1,u_2,u_3\}),(x_3,\{\},\{u_1,u_2,u_3\})\}$$

and

$$[f_A]^{-1}_{(0.35^-,-0.6^-)} = \{(x_1,\{\},\{u_1,u_2,u_3\}),(x_2,\{\},\{u_1,u_2,u_3\}),(x_3,\{\},\{u_1,u_2,u_3\})\}.$$

Remark 31. *Inverse (α,β)-cut can be use to know the most unfavorable selection. For example, let's assume that the pharmaceutical company will consider the unsuitable supplier firm as the firm that provides the least number of parameters under inverse (α,β). For this, let's create a similar mapping given in Remark 27 and the mapping $\Theta^{-1}_{[f_A]_{(\alpha,\beta)}}$ is defined by $\Theta^{-1}_{[f_A]_{(\alpha,\beta)}} : U \to [-n,n]$ for all $u_i \in U$ as follows: ($1 \le i \le s(E) = n$ and $1 \le j \le s(U) = m$)*

$$\Theta^{-1}_{[f_A]_{(\alpha,\beta)}}(u_i) = \sum_{j=1}^{n} \Upsilon^{ij}_{[f_A]^{-1}_{(\alpha,\beta)}} \tag{3}$$

$$\Upsilon^{ij}_{[f_A]^{-1}_{(\alpha,\beta)}} = \begin{cases} 1, & \text{if } u_i \in \widehat{F}^{(\alpha,\beta)}_{[f_A]^{-1}}(x_j) \\ -1, & \text{if } u_i \in \widehat{G}^{(\alpha,\beta)}_{[f_A]^{-1}}(x_j) \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Here, the value $\Theta_{[f_A]^{-1}_{(\alpha,\beta)}}(u_i)$ is called the "inverse total score" for the objects and the smaller the inverse total score of an object, the more recommended it is not to select that object. Under these conditions, the calculation of the total scores for $\alpha = 0.56$ and $\beta = -0.5$ given in Example 30 is as follows;

$$\begin{aligned} \Theta^{-1}_{[f_A]_{(0.56,-0.5)}}(u_1) &= \Upsilon^{11}_{[f_A]^{-1}_{(0.56,-0.5)}} + \Upsilon^{12}_{[f_A]^{-1}_{(0.56,-0.5)}} + \Upsilon^{13}_{[f_A]^{-1}_{(0.56,-0.5)}} \\ &= (-1) + (-1) + 1 = -1, \end{aligned}$$

$$\Theta^{-1}_{[f_A]_{(0.56,-0.5)}}(u_2) = 0, \quad \Theta^{-1}_{[f_A]_{(0.56,-0.5)}}(u_3) = 1.$$

Similarly, for $\alpha = 0.35$ and $\beta = -0.6$

$$\Theta^{-1}_{[f_A]_{(0.35,-0.6)}}(u_1) = -2, \quad \Theta^{-1}_{[f_A]_{(0.35,-0.6)}}(u_2) = -3, \quad \Theta^{-1}_{[f_A]_{(0.35,-0.6)}}(u_3) = -2.$$

As can be seen, the inverse total scores calculated for $\alpha = 0.35$ and $\beta = -0.6$ indicate that the unsuitable supplier firm for the pharmaceutical company is u_2. Moreover, the inverse total scores calculated for $\alpha = 0.56$ and $\beta = -0.5$ indicate that the unsuitable supplier firm for the pharmaceutical company is u_1. It means that the unsuitable object can change for the selected inverse (α,β)-cuts. In this case, we should pay attention to the selection of inverse (α,β)-cuts in order for the decision making process to function properly.

Remark 32. *Items (iv) and (v) given in Proposition 28 are not generally correct for inverse (α,β)-cuts. For this, let's examine Example 33 and 34:*

Example 33. $\big[$***Counter Example for** (iv):*$\big]$
Let $U = \{u_1, u_2, u_3\}$, $A = \{x_1, x_2, x_3\} \subseteq E$, $B = \{x_2, x_3, x_4\} \subseteq E$ and BFSS f_A, g_B over U be

$$f_A = \left\{ \begin{array}{c} f(x_1) = \left\{ \ (u_1, 0.56, -0.42), (u_2, 0.75, -0.5), (u_3, 0.5, -0.3) \ \right\}, \\ f(x_2) = \left\{ \ (u_1, 0.8, -0.15), (u_2, 0.4, -0.56), (u_3, 0.64, -0.15) \ \right\}, \\ f(x_3) = \left\{ \ (u_1, 0.35, -0.6), (u_2, 0.1, -0.5), (u_3, 0.56, -0.2) \ \right\} \end{array} \right\}$$

$$g_B=\left\{\begin{array}{c} g(x_2)=\left\{\ (u_1,0.64,-0.2),(u_2,0.57,-0.55),(u_3,0.6,-0.65)\ \right\}, \\ g(x_3)=\left\{\ (u_1,0.51,-0.24),(u_2,0.7,-0.2),(u_3,0.7,-0.52)\ \right\}, \\ g(x_4)=\left\{\ (u_1,0.18,-0.62),(u_2,0.33,-0.6),(u_3,0.5,-0.3)\ \right\} \end{array}\right\}$$

Then $h_C=f_A\widehat{\cup}g_B$, *where* $C=A\cup B=\{e_1,e_2,e_3,e_4\}$

$$h_C=\left\{\begin{array}{c} h(x_1)=\left\{\ (u_1,0.56,-0.42),(u_2,0.75,-0.5),(u_3,0.5,-0.3)\ \right\}, \\ h(x_2)=\left\{\ (u_1,0.8,-0.2),(u_2,0.57,-0.56),(u_3,0.64,-0.65)\ \right\}, \\ h(x_3)=\left\{\ (u_1,0.51,-0.6),(u_2,0.7,-0.5),(u_3,0.7,-0.52)\ \right\}, \\ h(x_4)=\left\{\ (u_1,0.18,-0.62),(u_2,0.33,-0.6),(u_3,0.5,-0.3)\ \right\} \end{array}\right\}.$$

Then

$$[f_A]^{-1}_{(0.56,-0.5)}=\{(x_1,\{u_3\},\{u_1\}),(x_2,\{u_2,u_3\},\{u_1\}),(x_3,\{u_1\},\{u_2,u_3\})\},$$

$$[g_B]^{-1}_{(0.56,-0.5)}=\{(x_2,\{\},\{u_1\}),(x_3,\{u_1\},\{u_2\}),(x_4,\{u_1,u_2,u_3\},\{\})\},$$

and

$$[h_C]^{-1}_{(0.56,-0.5)}=\{(x_1,\{u_3\},\{u_1\}),(x_2,\{\},\{u_1\}),(x_3,\{u_1\},\{\}),(x_4,\{u_1,u_2,u_3\},\{\})\}.$$

Also,

$$\begin{aligned}[f_A]^{-1}_{(0.56,-0.5)}\widetilde{\cup}[g_B]^{-1}_{(0.56,-0.5)}=\{&(x_1,\{u_3\},\{u_1\}),(x_2,\{u_2,u_3\},\{u_1\}),\\ &(x_3,\{u_1\},\{u_2,u_3\}),(x_4,\{u_1,u_2,u_3\},\{\})\}.\end{aligned}$$

Thus $[f_A]^{-1}_{(0.56,-0.5)}\widetilde{\cup}[g_B]^{-1}_{(0.56,-0.5)}\neq[f_A\widehat{\cup}g_B]^{-1}_{(0.56,-0.5)}$.

Example 34. $\big[$***Counter Example for*** (v)*:*$\big]$
Consider the BFSS f_A *and* g_B *as given in Example 33. In this case,* $h_C=f_A\widehat{\cap}g_B$, *where* $C=A\cap B=\{e_2,e_3\}$

$$h_C=\left\{\begin{array}{c} h(x_2)=\left\{\ (u_1,0.64,-0.15),(u_2,0.4,-0.55),(u_3,0.6,-0.15)\ \right\}, \\ h(x_3)=\left\{\ (u_1,0.51,-0.15),(u_2,0.4,-0.2),(u_3,0.64,-0.15)\ \right\} \end{array}\right\}.$$

Then

$$[h_C]^{-1}_{(0.56,-0.5)}=\{(x_2,\{u_2\},\{u_1,u_3\}),(x_3,\{u_1,u_2\},\{u_3\})\}$$

and

$$\begin{aligned}[f_A]^{-1}_{(0.56,-0.5)}\widetilde{\cap}[g_B]^{-1}_{(0.56,-0.5)} = \{&(x_1,\{u_3\},\{u_1\}),(x_2,\{\},\{u_1\}),\\ &(x_3,\{u_1\},\{u_2\}),(x_4,\{u_1,u_2,u_3\},\{\})\}.\end{aligned}$$

Thus $[f_A]^{-1}_{(0.56,-0.5)}\widetilde{\cap}[g_B]^{-1}_{(0.56,-0.5)}\neq[f_A\widehat{\cap}g_B]^{-1}_{(0.56,-0.5)}$.

Proposition 35. *Let* $\alpha\in[0,1]$, $\beta\in[-1,0]$ *and* f_A, g_B *be BFSSs over* U, *the following properties hold:*
(i) $[f_A]^{-1}_{(\alpha,\beta)}\widetilde{\subseteq}[f_A]^{-1}_{(\alpha^-,\beta)}\widetilde{\subseteq}[f_A]^{-1}_{(\alpha^-,\beta^-)}$ *and* $[f_A]^{-1}_{(\alpha,\beta)}\widetilde{\subseteq}[f_A]^{-1}_{(\alpha,\beta^-)}\widetilde{\subseteq}[f_A]^{-1}_{(\alpha^-,\beta^-)}$.
(ii) $[f_A]^{-1}_{(\alpha^-,\beta)}\widetilde{\cap}[f_A]^{-1}_{(\alpha,\beta^-)}=[f_A]^{-1}_{(\alpha^-,\beta^-)}$.
(iii) If $\alpha_1\leq\alpha_2$ *and* $\beta_1\geq\beta_2$, *then* $[f_A]^{-1}_{(\alpha_1,\beta_1)}\widetilde{\subseteq}[f_A]^{-1}_{(\alpha_2,\beta_2)}$.

(iv) $[f_A \widehat{\cup} g_B]^{-1}_{(\alpha,\beta)} \widetilde{\subset} [f_A]^{-1}_{(\alpha,\beta)} \widetilde{\cup} [g_B]^{-1}_{(\alpha,\beta)}$.
(v) $[f_A]^{-1}_{(\alpha,\beta)} \widetilde{\cap} [g_B]^{-1}_{(\alpha,\beta)} \widetilde{\subset} [f_A \widehat{\cap} g_B]^{-1}_{(\alpha,\beta)}$.

Proof. (i) Let $(x,\{u_i\},\{u_j\}) \in [f_A]^{-1}_{(\alpha,\beta)}$
$\Rightarrow \left[\mu^+_{[f_A]^{-1}_x}(u_i) < \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_i) > \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_i) \geq \left|\mu^-_{[f_A]^{-1}_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]^{-1}_x}(u_j) < \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_j) > \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_j) < \left|\mu^-_{[f_A]^{-1}_x}(u_j)\right|\right]$, $\forall x \in A$
$\Rightarrow \left[\mu^+_{[f_A]^{-1}_x}(u_i) \leq \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_i) > \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_i) \geq \left|\mu^-_{[f_A]^{-1}_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]^{-1}_x}(u_j) \leq \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_j) > \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_j) < \left|\mu^-_{[f_A]^{-1}_x}(u_j)\right|\right]$, $\forall x \in A$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A]^{-1}_{(\alpha^-,\beta)}$
Therefore $[f_A]^{-1}_{(\alpha,\beta)} \widetilde{\subseteq} [f_A]^{-1}_{(\alpha^-,\beta)}$. Similarity, for $(x,\{u_i\},\{u_j\}) \in [f_A]^{-1}_{(\alpha^-,\beta)}$
$\Rightarrow \left[\mu^+_{[f_A]^{-1}_x}(u_i) \leq \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_i) > \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_i) \geq \left|\mu^-_{[f_A]^{-1}_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]^{-1}_x}(u_j) \leq \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_j) > \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_j) < \left|\mu^-_{[f_A]^{-1}_x}(u_j)\right|\right]$, $\forall x \in A$
$\Rightarrow \left[\mu^+_{[f_A]^{-1}_x}(u_i) \leq \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_i) \geq \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_i) \geq \left|\mu^-_{[f_A]^{-1}_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]^{-1}_x}(u_j) \leq \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_j) \geq \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_j) < \left|\mu^-_{[f_A]^{-1}_x}(u_j)\right|\right]$, $\forall x \in A$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A]^{-1}_{(\alpha^-,\beta^-)}$
Therefore $[f_A]^{-1}_{(\alpha^-,\beta)} \widetilde{\subseteq} [f_A]^{-1}_{(\alpha^-,\beta^-)}$. It is proved similarly in the other part.
(ii) Straighforward.
(iii) It is clear from Definition 11 and Definition 29.
(iv) Let $(x,\{u_i\},\{u_j\}) \in [f_A \widehat{\cup} g_B]^{-1}_{(\alpha,\beta)}$
$\Rightarrow \left[\mu^+_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_i) < \alpha\right] \wedge \left[\mu^-_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_i) > \beta\right] \wedge \left[\mu^+_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_i) \geq \left|\mu^-_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_j) < \alpha\right] \wedge \left[\mu^-_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_j) > \beta\right] \wedge \left[\mu^+_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_j) < \left|\mu^-_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_j)\right|\right]$, $\forall x \in A \cup B$
$\Rightarrow \Bigg[\left[\mu^+_{[f_A]^{-1}_x}(u_i) < \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_i) > \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_i) \geq \left|\mu^-_{[f_A]^{-1}_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]^{-1}_x}(u_j) < \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_j) > \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_j) < \left|\mu^-_{[f_A]^{-1}_x}(u_j)\right|\right], \forall x \in A\Bigg]$
or $\Bigg[\left[\mu^+_{[g_B]^{-1}_x}(u_i) < \alpha\right] \wedge \left[\mu^-_{[g_B]^{-1}_x}(u_i) > \beta\right] \wedge \left[\mu^+_{[g_B]^{-1}_x}(u_i) \geq \left|\mu^-_{[g_B]^{-1}_x}(u_i)\right|\right]$ and $\left[\mu^+_{[g_B]^{-1}_x}(u_j) < \alpha\right] \wedge \left[\mu^-_{[g_B]^{-1}_x}(u_j) > \beta\right] \wedge \left[\mu^+_{[g_B]^{-1}_x}(u_j) < \left|\mu^-_{[g_B]^{-1}_x}(u_j)\right|\right], \forall x \in B\Bigg]$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A]^{-1}_{(\alpha,\beta)}$ or $(x,\{u_i\},\{u_j\}) \in [g_B]^{-1}_{(\alpha,\beta)}$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A]^{-1}_{(\alpha,\beta)} \widetilde{\cup} [g_B]^{-1}_{(\alpha,\beta)}$
Therefore, $[f_A \widehat{\cup} g_B]^{-1}_{(\alpha,\beta)} \widetilde{\subset} [f_A]^{-1}_{(\alpha,\beta)} \widetilde{\cup} [g_B]^{-1}_{(\alpha,\beta)}$.

(v) Let $(x,\{u_i\},\{u_j\}) \in [f_A]^{-1}_{(\alpha,\beta)} \widetilde{\cap} [g_B]^{-1}_{(\alpha,\beta)}$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A]^{-1}_{(\alpha,\beta)}$ and $(x,\{u_i\},\{u_j\}) \in [g_B]^{-1}_{(\alpha,\beta)}$

$$\Rightarrow \Bigg[\Big[\mu^{+}_{[f_A]^{-1}_x}(u_i) < \alpha\Big] \wedge \Big[\mu^{-}_{[f_A]^{-1}_x}(u_i) > \beta\Big] \wedge \Big[\mu^{+}_{[f_A]^{-1}_x}(u_i) \geq \Big|\mu^{-}_{[f_A]^{-1}_x}(u_i)\Big|\Big] \text{ and }$$

$$\Big[\mu^{+}_{[f_A]^{-1}_x}(u_j) < \alpha\Big] \wedge \Big[\mu^{-}_{[f_A]^{-1}_x}(u_j) > \beta\Big] \wedge \Big[\mu^{+}_{[f_A]^{-1}_x}(u_j) < \Big|\mu^{-}_{[f_A]^{-1}_x}(u_j)\Big|\Big], \forall x \in A\Bigg]$$

$$\text{and } \Bigg[\Big[\mu^{+}_{[g_B]^{-1}_x}(u_i) < \alpha\Big] \wedge \Big[\mu^{-}_{[g_B]^{-1}_x}(u_i) > \beta\Big] \wedge \Big[\mu^{+}_{[g_B]^{-1}_x}(u_i) \geq \Big|\mu^{-}_{[g_B]^{-1}_x}(u_i)\Big|\Big] \text{ and }$$

$$\Big[\mu^{+}_{[g_B]^{-1}_x}(u_j) < \alpha\Big] \wedge \Big[\mu^{-}_{[g_B]^{-1}_x}(u_j) > \beta\Big] \wedge \Big[\mu^{+}_{[g_B]^{-1}_x}(u_j) < \Big|\mu^{-}_{[g_B]^{-1}_x}(u_j)\Big|\Big], \forall x \in B\Bigg]$$

$$\Rightarrow \Big[\mu^{+}_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_i) < \alpha\Big] \wedge \Big[\mu^{-}_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_i) > \beta\Big] \wedge \Big[\mu^{+}_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_i) \geq \Big|\mu^{-}_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_i)\Big|\Big]$$
$$\text{and } \Big[\mu^{+}_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_j) < \alpha\Big] \wedge \Big[\mu^{-}_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_j) > \beta\Big] \wedge \Big[\mu^{+}_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_j) < \Big|\mu^{-}_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_j)\Big|\Big],$$
$\forall x \in A \cup B$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A \widehat{\cup} g_B]^{-1}_{(\alpha,\beta)}$
Therefore, $[f_A]^{-1}_{(\alpha,\beta)} \widetilde{\cap} [g_B]^{-1}_{(\alpha,\beta)} \widetilde{\subseteq} [f_A \widehat{\cap} g_B]^{-1}_{(\alpha,\beta)}$. □

5. Results and Conclusion

The concepts of (α,β)-cut, first type semi-strong (α,β)-cut, second type semi-strong (α,β)-cut, strong (α,β)-cut, inverse (α,β)-cut, first type semi-weak inverse (α,β)-cut, second type semi-weak inverse (α,β)-cut and weak inverse (α,β)-cut for bipolar fuzzy soft sets were introduced and their applications were highlighted. It is shown that (α,β)-cut of bipolar fuzzy soft sets can be used to determine the best choice while inverse (α,β)-cuts of bipolar fuzzy soft sets can be used to determine unfavorable alternative. Moreover, some related results were presented. I think that these concepts proposed for better management of decision-making processes for uncertainty problems may be useful in the future.

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