Ç.Ü. Sosyal Bilimler Enstitüsü Dergisi, Cilt 29, Sayı 3, 2020, Sayfa 49-69

INVESTIGATION OF APPLICATIONS OF FIBONACCI SEQUENCE AND GOLDEN RATIO IN MUSIC*

Sümeyye BAKIM[1]
Seyit YÖRE[2]

ABSTRACT

In studies presented in the literature, relationships between music and mathematics can sometimes be observed. Leonardo Fibonacci (1170-1250) is well known in mathematics with the Fibonacci Sequence and this sequence used to identify numbers in various music elements, too. In related studies, these numbers have been used to demonstrate the existence of the 'Golden Ratio' using methods and theories borrowed from the components of music. Nevertheless, this relationship has subsequently been seen inaccurate. The studies that previously based some works of Chopin, Mozart, Beethoven, Bach and Bartók on Fibonacci Sequence and Golden Ratio are critically examined in the context of musical and mathematical theories in this study. Qualitative and quantitative research methods were used together in this interdisciplinary research in the field of mathematical sciences and critical musicology. It was examined basically the measure or rhythms (sound duration) within the musical works that allegedly used the Fibonacci Sequence and the Golden Ratio, and it was found these studies yielded values close to the terms of the Fibonacci Sequence and the determined values of the Golden Ratio were 0.618, 1.618, and 0.382. It is determined that mathematical, historical and music theoretical data and findings could not provide enough to support the claims of the related studies. Thus, it was determined that the accuracy of the Fibonacci Sequence and Golden Ratio expressed in the works of the related composers are controversial within the framework of the relevant studies.
Keywords: Fibonacci Sequence, Golden Ratio, Maths, Music, Analysis

FIBONACCI DİZİSİ VE ALTIN ORAN'IN MÜZİKTEKİ UYGULAMALARININ İNCELENMESİ

ÖZ

Literatürde sunulan çalışmalarda, bazen müzik ve matematik arasındaki ilişkiler gözlemlenebilir. Leonardo Fibonacci (1170–1250), Fibonacci Dizisi'yle matematikte iyi bilinir ve bu dizi çeşitli müzik öğelerinde sayıların tanımlanması için de kullanılmıştır. Bu rakamlar, müzik bileşenlerinden ödünç alınan yöntem ve teorileri kullanarak Altın Oran'ın varlığını göstermek için ilgili çalışmalarda yer almıştır. Bununla birlikte, bu ilişkinin daha sonra yanlış olduğu görülmüştür. Daha önce Chopin, Mozart, Beethoven, Bach ve Bartók'tan seçilmiş eseleri Fibonacci Dizisi ve Altın Oran'a dayandıran çalışmalar, bu çalışmada müziksel ve matematiksel teoriler bağlamında eleştirel olarak irdelenmiştir. Matematik bilimleri ve eleştirel müzikoloji alanındaki bu disiplinlerarası araştırmada nitel ve nicel araştırma yöntemleri birlikte kullanılmıştır. Fibonacci Dizisi'ni ve Altın Oran'ı kullandığı iddia edilen müzik eserlerinin ölçüleri veya ritimleri (ses süresi) incelenmiş, bu çalışmaların Fibonacci sekansına yakın değerler verdiği ve Altın Oran'ın belirlenen değerlerinin 0.618, 1.618 ve 0.382 olduğu tespit edilmiştir. İlgili çalışmaların iddialarını destekleyecek matematiksel, tarihsel ve müzik

[1] Lecturer, Department of Mechatronics Engineering, Faculty of Engineering, KTO Karatay University, Konya/Turkey, sumbkm@gmail.com, ORCID: 0000-0002-6957-2328
[2] Prof. Dr., Department of Musicology, State Conservatory, İstanbul University, İstanbul/Turkey, seyityore@gmail.com, ORCID: 0000-0001-5833-4271
* This article was produced from the first author's master thesis (See Bakım, 2014).
Received/Geliş: 27/09/2019 Accepted/Kabul: 02/02/2020, Research Article/Araştırma Makalesi
Cite as/Alıntı: Bakım, S., Yöre, S. (2020), "Investigation of Applications of Fibonacci Sequence and Golden Ratio in Music", Çukurova Üniversitesi Sosyal Bilimler Enstitüsü Dergisi, cilt 29, sayı 3, s.49-69.

teorik verileri yeterince sunamadığı belirlemiştir. Böylece, ilgili bestecilerin eserlerinde ifade edilen Fibonacci Dizisi'nin ve Altın Oran'ın doğruluğunun ilgili çalışmalar çerçevesinde tartışmalı olduğu tespit edilmiştir.
Anahtar Kelimeler: Fibonacci Dizisi, Altın Oran, Matematik, Müzik, Analiz

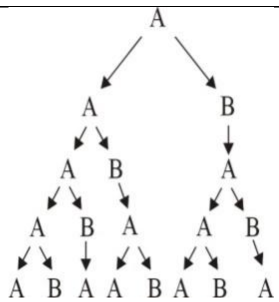
Introduction

Pythagoras (570–495 BC) was an ancient Greek philosopher who is thought to have originated the concept of mathematical and musical relationships. He revealed the arithmetic relationships among sound pitches (Hammond, 2011, p. 1). According to legend, Pythagoras encountered to workers with various types of hammers. The sound from the hammers appeared to be harmonious. When Pythagoras weighed the hammers, he noticed that the weight ratios were 12:9:8:6. Later, Pythagorean followers derived the ratios 2:1, 3:2, 4:3, and 9:8 from 12:9:8:6, creating the accepted standard sound intervals in music. These ratios are integer multiples of one single octave within a frequency (the octave between the starting frequency and twice frequencies), designating the initials of the indicated frequency. Using the "monochord" shown in Fig. 1, frequencies can be easily calculated by changing the length of the wire of the "relative frequencies" (Bora, 2002, p. 54-60).

Leonardo Fibonacci (1170–1250), one of the most famous Italian mathematicians, developed an original theory for identifying numbers, called the Fibonacci Sequence. Depending on the established numbers and the related Golden Ratio that they form, the natural science of these numbers has been used in natural sciences and many musical works around the world. Leonardo Pisano Bigollo or Leonardo Bonacci (whom we now know as Leonardo Fibonacci) in his book *Liberabaci* (Book of Calculations) (1202) provided the key to understanding the mathematical puzzle of the Golden Ratio (Marie, 2012, p. 1). This puzzle is known as the rabbit problem, expressed as follows:

1. On the first day of January, one rabbit couple exists, 2. On the first day of February and after every first day of each month thereafter, they give birth to two rabbits, 3. Each newborn couple becomes adult in one month and from the third month of their lives, they give birth to a new couple every first day of each month. The rabbits do not die. A, B, and T denotes the number of adult couples, baby couples and the total number of rabbits, respectively (Hoggatt, 1969, p. 1-2).

Table 1. Number of rabbit couples in one year (Hoggatt, 1969, p. 2-3).

MONTHS	**RUBBITS**	**A**	**B**	**T**
Begining		**1**	**0**	**1**
January		**1**	**1**	**2**
February		**2**	**1**	**3**
March		**3**	**2**	**5**
April		**5**	**3**	**8**
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

December	**233**	**144**	**377**

We obtain the number of adults, babies and total rabbits from the Table 1.

From the rabbit problem, the following Sequence:
1,2,3,5,8,13,21,34, ... is called a **Fibonacci Sequence,** and the terms are called **Fibonacci numbers.** The *n*th Fibonacci number is denoted by F_n (Hoggatt, 1969, p. 5).

1. Theory: Golden Ratio and Fibonacci Quadratic Equation

When line segment AB is considered and point C separates AB into a piece in the following proportion:

$$\frac{|AB|}{|AC|} = \frac{|AC|}{|CB|}$$

then C is called the "gold section" of AB. This ratio is made up of ratios $|AB|/|AC|$ and $|AB|/|CB|$ are called the **Golden Ratio**.

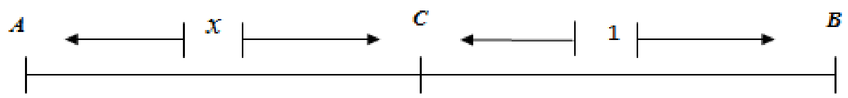


Fig 1. Golden Section

Let $|AC| = x$ and $|CB| = 1$ in line segment $|AB|$ (Fig. 1). Thus, the $\frac{|AB|}{|AC|} = \frac{|AC|}{|CB|}$ ratio can be expressed as

$$\frac{x+1}{x} = \frac{x}{1},$$

which is equivalent to

$$x + 1 = x^2.$$

Then

$x^2 - x - 1 = 0$ forms a quadratic equation. The roots of the quadratic equation are

$$x_1 = \frac{1+\sqrt{5}}{2} = 1.61803..$$

$$x_2 = \frac{1-\sqrt{5}}{2} = 0.61803..$$

$x_1 = 1.61803$ is a five-step value of the Golden Ratio, which was consequently accepted as 1.618. This number is denoted by ϕ(Phi). In this manner, $x_2 = 0.61803$ represents an inverse value of the Golden Ratio accepted as 0.618.

When the Golden Ratio 1.618 is compared with its inverse value 0.618, an interesting feature occurs: 1.618 is a unique number that provides its inverse number 0.618 when the Golden Ratio is subtracted by one.

Therefore,

$$1.618 - 1 = \frac{1}{1.618} = 0.618$$

Thus,
$1.618 \times 1.618 - 1.618 = 1$
$(1.618)^2 = 2.618$

This result means that when one is added to the Golden Ratio, the Golden Ratio yields its own square. No other number possesses this characteristic (5). Meanwhile, any number in the Sequences yields the next two numbers when multiplied by 2.618 (89 × 2.618 = 223) or the previous two numbers when multiplied by 0.382 (89 x 0.382 = 34). The limit of rate F_n/F_{n+2} is 0.382, and that of F_{n+2}/F_n is 2.618.

2. Problem

In the present study, studies on relationship of the Fibonacci Sequence and the Golden Ratio with music were investigated. In related studies, specific discussions have been made regarding whether the Fibonacci Sequence is appropriate for the European-based theories. The principal works of Chopin, Mozart, Beethoven, Bach and Bartók that are claimed to contain the Fibonacci Sequence and Golden Ratio, have been discussed as part of mathematical and musical theories, and the results are provided in Section 5.

3. Method

In this research based on phenomenology, grounded theory research and meta-analysis, qualitative and quantitative research methods were used together. The works of Chopin, Mozart, Beethoven, Bach and Bartók, which are claimed to be based on Fibonacci Sequence and Golden Ratio in the sources selected from many sources were analyzed mathematically and musically by analytical method. These analyzes were compared with critical method according to Fibonacci Sequence and the Golden Ratio theory and the general music theory, and the findings were revealed. The research is interdisciplinary in the framework of mathematical sciences, systematic musicology and critical musicology.

4. Theoretical Discussions on Relationships of the Fibonacci Sequence and the Golden Ratio in Music

The Fibonacci Sequence was associated with music by W. J. Zerger who is the mathematics professor.

1. The English word "music" begins with the 13th letter 'm' and successor letter 'u' is the 21st letter of the English alphabet. The 8th, 13th, and 21st letters can create the word "hum" (song). 8, 13 and 21 are Fibonacci numbers, 2. In the US Library of Congress, the classification of the musical number is M, the 13th letter in the alphabet, 3. The Dewey decimal numbers for the music classification system is 780. The number 780 = 2·2·3·5·13 is a product of a Fibonacci Sequence, 4. Pianos are tuned to the chord standards of 440 cycles per second (440 = 8 × 55). The numbers 8 and 55 are numbers in a Fibonacci Sequence (Koshy, 2001, p. 37).

In these expressions, Fibonacci numbers and music endeavored to associate and demonstrated by the numbers fortuitously obtained from studies. In particular, in the first article based on the English word "music," then derived the English word "hum" (song), we can see a non-universal approach to relationship between the Fibonacci Sequence and music. These and the similar phrases made by authors that tried to establish the relationship between music and the Fibonacci Sequence appeared in several websites and resources. However, this attempt does not imply any meaningful relationship.

Piano keys also make a stunning visual statement of the connection between music and the Fibonacci Sequence. An octave on the keyboard defines musical interval between two notes one with higher than the other. Frequency of higher note is twice of the lower one. Octaves on the keyboard are divided into five black and eight white keys for a total of 13 keys (Fig. 2). The five black keys are divided into two groups as one binary and one one triple. Each of the numbers 2, 3, 5, 8, and 13 are numbers in the Fibonacci Sequence (Koshy, 2001, p. 38).

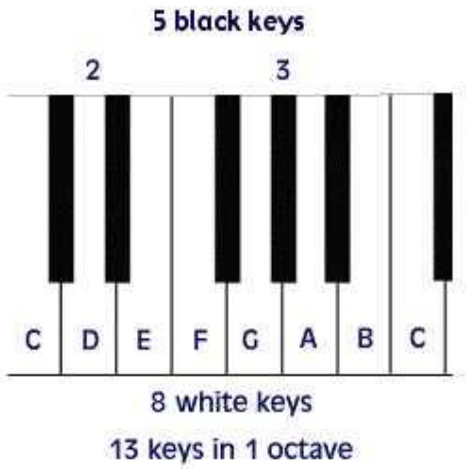


Fig. 2. Fibonacci numbers in one octave of piano keys according to Zerger

As shown in this example, in proving the existence of Fibonacci numbers, expressions on the eight-notes sequences completely wrong. Diatonic sequences have seven notes and not eight. Because the eighth note is the repetition of the first note and moves numerically to the next note. In this case we need to write 2, 3, 5, 7, and 12; however, this is not Fibonacci Sequence. In fact, no relationship with the Fibonacci Sequence exists among the black keys on a piano when divided into two groups as double or one-third. It is impossible to determine which of the two or three numbers comes first (Posamentier&Lehmann, 2007, p. 272).

Musical instruments are often constructed based on the number ϕ. Similar to the design of a violin, in the design of high quality of the vocal cord, (its strings too) the numbers in the Fibonacci Sequence and ϕ are being used. The Golden Ratio in the violin is shown in Fig. 3 (Posamentier&Lehmann, 2007, p. 291).

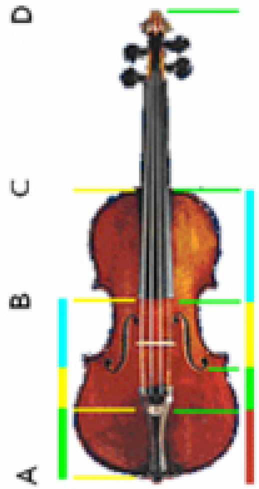


Fig 3. Golden Ratio in a violin.

$\frac{|AB|}{|BC|} = \phi$ and $\frac{|AC|}{|CD|} = \phi$.In the same manner: $\frac{|AD|}{|AC|} = \frac{|AC|}{|AB|} = \frac{|CD|}{|BC|} = \phi$.

5. Findings

In literature, including those reviewed below, famous compositions for example composed by Mozart, Beethoven, Bach, Chopin, Béla Bartók, and others have been used to demonstrate the assumed application of the Fibonacci Sequence and Golden Ratio.

When composers compose their works, they sometimes use direct materials from subconscious emotions/feelings. Sometimes they rely on conscious thinking. Thus, composers sometimes create their work in a method exceptional in the musical structures of their period. This process can reveal incidental or consequential mathematical structured works in which the analytical structures of music are indeed based on physics and mathematics. Hence, the incidental mathematical theories within works, except for those consciously created by composers, do not always mean that they are important assets. The holistic value of a work is the most important factor from the musical point of view. Thus, a certain form of music used by the same composer may consist of equal sections or not, in which many examples in the musical history can be found. In addition, by 20th century in contemporary period, compositions created by mathematical features consciously like 12 sound system in music and has esthetics at the same time may achieve importance (Posamentier&Lehmann, 2007, p. 274-275).

Examples related to the usage of the Fibonacci Sequence and Golden Ratio are provided and discussed in the next subsection.

5.1. Fibonacci numbers and the Golden Ratio in Chopin's *Preludes*

One of the piano works of Frédéric Chopin (1810–1849) from the 19^{th} century is Op. No. 28 *Preludes*. This collection includes 24 of the most exceptional musical miniatures. The first one of these works was based on a game that Chopin played by himself. Fig. 4 shows a basic simplified drawing of the melodic movements in the right hand. Each measure (except the last six) is accompanied by the left hand (the whole note), and the other measures (the black note) that have notes that contain two notes that are one step away from the other, have no accompaniment. This work, lasting approximately 30 seconds, is established by two distinct sections with different measures. The climax of this work is at the 21^{st} measure, corresponding to the Golden Ratio of the 34^{th} measure. Numbers 21 and 34 are both numbers of the Fibonacci Sequence and $34 \times 0.618 \approx 21$. The climax is also located in the same manner in the Golden Ratio in the *Prelude* Op. 28 No. 9. The length of this work is 12 measures and contains 48 beats. The climax point is exactly at beat 29 ($48 \times 0.618 \approx 29$) and appears at the beginning of the eighth measure. The climax point does not always have to agree with the mathematical formula. In some cases, it can reach a value near the Golden Ratio. The preludes are not mostly similar to the Golden Ratio. Perhaps Chopin thought that the use of Phi (ϕ) was not necessary to ensure musical success (Posamentier&Lehmann, 2007, p. 272-273).

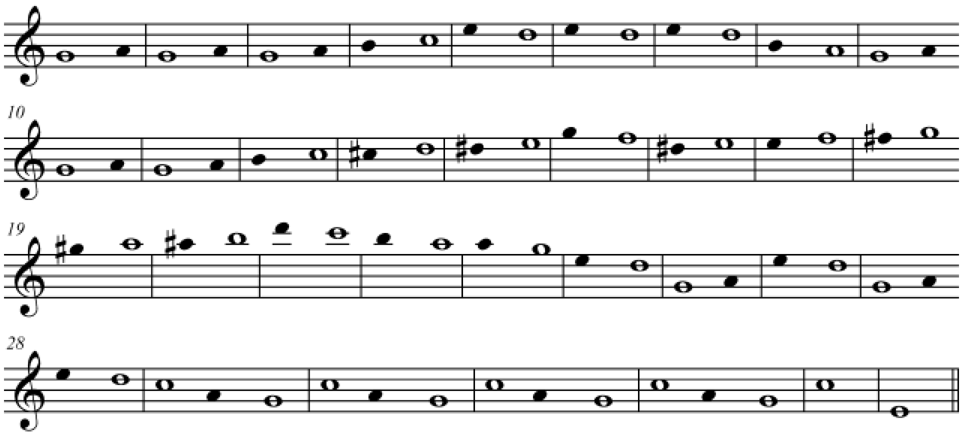


Fig. 4. *Prelude*, No. 1 in C Major (Posamentier&Lehmann, 2007, p. 273).

Fig. 5 is a graphical equivalent of Fig. 4 and shows the Golden Ratio of the climax using a pitch string.

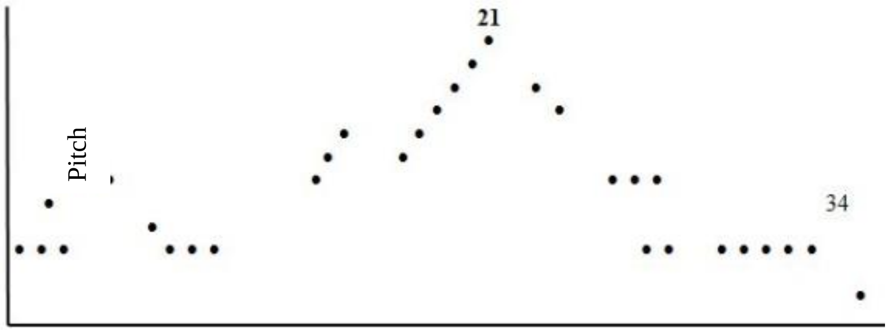


Fig. 5. *Prelude*, No. 1 in C Major (Posamentier&Lehmann, 2007, p. 274).

5.2. Fibonacci numbers and the Golden Ratio in Mozart's piano sonatas
One of the games played to determinate the form of works by some famous composers is related to the use of the Golden Ratio. Because Wolfgang Amadeus Mozart (1756–1791) loved numbers and all types of games, he was known for his addiction to this practice. Mozart composed eighteen sonatas for piano. Except for one, he used the sonata-allegro form in all of them. In the remaining one, the form of "theme and variations" was used. As listed in Table 2, six of the 17 (35%) works can be exactly divided into the Golden Ratio. These are indicated by the word "golden" in the accuracy column. Eight of these works (47%) is very close to the Golden Ratio, and these are shown as error ratios that range between –3 and +4 in the accuracy column. These numbers indicate that no Golden Ratio is present. Because the values 6, 8, and 12 are very high, the remaining three works (18%) are not considered to be close to the Golden Ratio. Statistically, on the basis of these examples, we can appreciate the importance of the Golden Ratio to Mozart (Posamentier&Lehmann, 2007, p. 277).

Table 2. Mozart's piano sonatas (Posamentier&Lehmann, 2007, p. 278)

Mozart Sonata	Key	Length	Exposition	Proportion	Accuracy (%)
No. 1. K. 279	C major	100	38	0.38	golden
No. 2. K. 280	F major	144	56	0.389	golden
No. 3. K. 281	Bb major	109	40	0.367	-2
No. 4. K. 282	Eb major	36	15	0.417	1
No. 5. K. 283	G major	120	53	0.442	8
No. 6. K. 284	D major	127	51	0.402	3
No. 7. K. 309	C major	156	59	0.378	golden

No. 8. K. 310	A minor	133	49	0.368	-1
No. 9. K. 311	D major	112	9	0.348	-3
No. 10 K. 330	C major	149	57	0.383	golden
No. 11. K. 331	A major	135	55	Theme & Variations	
No. 12. K. 332	F major	229	93	0.406	6
No. 13. K. 333	Bb major	170	63	0.371	-1
No. 14. K. 457	C minor	185	74	0.4	4
No. 15. K. 545	C Major	73	28	0.384	golden
No. 16. K. 570	Bb major	209	79	0.378	golden
No. 17. K. 576	D major	160	58	0.363	-2
No. 18. K. 533	F major	240	103	0.429	12

Exhibition part of *Sonata No. 1 (K279)*, one of the six compositions listed in Table 2 marked as word "golden," consists of 100 measures and finishes at the 38th measure. This composition is the closest to the Golden Ratio, as shown in Fig. 6.

Fig. 6. Mozart's *Piano Sonata*, No.1 (K279) (Posamentier&Lehmann, 2007, p. 278).

5.3. Fibonacci Sequence and the Golden Ratio in Haydn's piano sonata

In evaluating whether Franz Joseph Haydn (1732–1803) used the Golden Ratio in his piano sonata or not, we can conclude that Haydn's use of ϕ cannot be compared with that of Mozart. When the same number of randomly-selected piano sonatas was studied, only 18% (3/17) of the sonata-allegro forms have an exact Golden Ratio, and 53% (9/17) are closer to the Golden Ratio. The remaining 29% (5/17) are not taken into consideration, as they are not related to the Golden Ratio. The average calculated number in the Golden Ratio in Mozart's work is 0.388, and it is 0.364 in Haydn's works. In Mozart's works, location of measures in accordance with the ϕ values are found between –3 and +12, whereas they are –6 and +2 in Haydn's works. The quality of music or the aesthetic value is irrelevant in these numbers. In order to test this, when 34 works taken one from Mozart and one from Haydn, it is not possible to conclude which work is closest to the Golden Ratio, but a lot of beautiful musical works are heard (Table 3) (Posamentier&Lehmann, 2007, p. 278-279).

Table 3. Haydn's piano sonatas (Posamentier&Lehmann, 2007, p. 279)

Haydn Sonata	**Key**	**Length**	**Exposition**	**Proportion**	**Accuracy(%)**
No. 14. 1767	E major	100	84	30	-2
No. 15. 1767	D major	144	110	36	-6
No. 16. 1767	Bb major	109	116	38	-6
No. 17. 1767	D major	36	103	42	2
No. 19. 1773	C major	120	150	57	golden
No. 21. 1773	F major	127	127	46	-3
No. 25. 1776	G major	156	143	57	2
No. 26. 1776	Eb major	133	141	52	-2
No. 27. 1776	F major	112	90	31	-4
No. 31. 1778	D major	149	195	69	-6
No. 32. 1778	E minor	135	127	45	-4
No. 33. 1780	C major	229	172	68	2
No. 34. 1780	C # minor	170	100	33	-5
No. 35. 1780	D major	185	103	40	golden
No. 42. 1786	G minor	73	77	30	golden
No. 43. 1786	Ab major	209	112	38	-5
No.49. 1793	Eb major	160	116	43	-1

5.4. Fibonacci numbers and the Golden Ratio in Beethoven's *Symphony No. 5*

In the tempo *Allegro con brio* (cheerful fanfare), the first five measures start the first section of the symphony (Op. 67), probably the most universally-known musical expression in classical music. The form of this work is different from that of Mozart's and Haydn's sonatas and symphonies. In Ludwig van Beethoven's (1770–1827) fifth symphony exposition, the development and recapitulation subsections have approximately the same length. In this manner, the Golden Ratio is not present in these three subsections (Posamentier&Lehmann, 2007, p. 280).

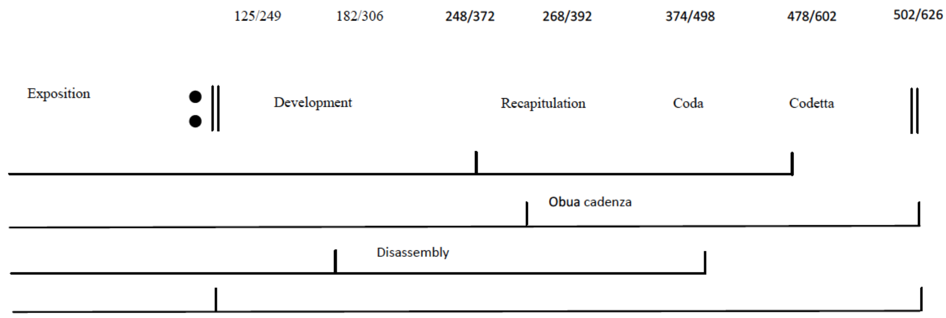


Fig. 7. Beethoven's *Symphony No. 5*: first movement (Posamentier&Lehmann, 2007, p. 280).

Instead of leading the first movement to a conclusion, it is not seen in earlier musical works, the recapitulation turns into a coda and so new (second) development section is formed. When codetta added to coda, a significant and unique fifth subsection appears. Therefore, with five subsections, not four, ends this section between measures 124 and 128. This form is a new type of sonata-allegro form. Fig. 7 shows that this new improved sonata-allegro form contains three Golden Ratio subsections. Firstly, repetition of the exposition occurs at measure 372, and this is the Golden Ratio of the entire section. The length of this section without the codetta is 602 measure, and the Golden Ratio is: $602 \times 0.618 \approx 372$. Measure of the last section of the repetition in this exposition subsection ($124 \times 2 = 248$) is, nearly a Golden Ratio of the whole part ($248 \div 626 \approx 0.396$). If the last two measures of the exposition are removed, the number will be closer to the Golden Ratio ($244 \div 626 \approx 0.389$). In the current work, two stylistic events occur upon the ratio. First occurs in the development subdivision. A four-note motive is separated into two and finally into a single part. Distribution of these four notes in the 306 measure and it is the Golden Ratio of the part to the end of the repetition in measure ($306 \div 498 \approx 0.614$). One of the most important moments of the entire symphony occurs at the 392nd measure of the repetition, the entire orchestra stops except the oboe. No example is available for the short caesura of the oboe, and this is unusual for those who know the sonata-allegro form. This solo takes place only six measure away from the Golden Ratio of the entire part ($626 \times 0.618 \approx 386$). We will never know if Beethoven planned this special moment according to the Golden Ratio or not. However, the result is close to the Golden Ratio (Posamentier&Lehmann, 2007, p. 281-282).

5.5. Fibonacci numbers and the Golden Ratio in Bach's *Chromatic Fantasy*

We can see another example of the Golden Ratio when Johann Sebastian Bach's (1685–1750) *The Chromatic Fantasy* (BWV 903/1) is divided into two unequal parts. In this example, to avoid fractions, quarter notes are used as measures. The same results can be realized by counting the measures. The length of the *Chromatic Fantasy* can be described as 316 quarter notes and it is divided into two formally unequal parts. The length of the first part is 195, second part is 121 quarter notes. The first section is thought to be written in a "toccata" form. For the 195th notes, many sources used the expression "recitativo." At this point, an abrupt change begins, and the remaining part is written in a half-recitative form. The Chromatic Fantasy is division shown in Fig. 8 (Power, 2001, p. 85).

Form: Prelude Recitative

Quarter notes: <----------195-----------------><------------121----------->

<------------------------------316---------------------------->

Fig. 8. Division of *Chromatic Fantasy (BWV 903/1)* (Power, 2001, p. 85).

From the division of all the quarter notes in whole part (316) to the quarter notes in the Prelude (195) can be seen to coincide with the Golden Ratio. Mathematically: $316 \div 195 = 1.621 \approx \phi$. If the larger part of the Prelude is divided into smaller part recitative, it will result in $195 \div 121 = 1.61$. Again this result approaches the ϕ value with small deviation. These two calculations show relation of division of the composition with the Golden Ratio (Power, 2001, p. 87).

The earliest extant da capo arias of Bach in 1713 is seen in the first permanent cantata known as *Was mir behagt, ist nur die muntre Jagd* (Sprightly hunting is the only thing that makes me happy) (BWV 208). In this work, in addition to the two choirs, three arias were present, and two of them holds kind of shape and measure that demonstrates the Golden Ratio. Fig. 9 shows the beginning form. The measurements in this example were from Bach's beginning aria extant from the mentioned cantata. At a first glance, the aria ratio revealed the Golden Ratio. When we looked at the number of measures, we can see that section A had 21 measures and section B had 13, meaning that segments A and B together yielded 34, it gives 55 measures when we calculate$(A + B + A2 = 21 + 13 + 21 = 55)$. Considering the measures of A and B and the combination of segments A and B, these numbers appear to be numbers in the Fibonacci Sequence 1, 2, 3, 5, 8, 13, **21**, **34**, and **55**. This Sequence of numbers, which probably Bach knew, revealed that this part was consciously and not coincidentally composed (Power, 2001, p. 105).

Form: A B A2

Quarter notes: <----------21----------><-----13-----><----------21---------->

<-------------------34------------------>

<-----------------34------------------->

Fig. 9. Bach’s first extand da capo aria (BWV 208/4) (Power, 2001, p. 106).

The following calculations can support the ratios. When the combination of sections A and B and all the aria measures are considered, (the number of measures for the entire aria: $(A + B + A2) \div (A + B) = 55 \div 34 = 1.618 \approx \phi$ can be found. When the combination of sections A and B (totally 34 measure) is divided into section A (21 measure), a similar proximity of ϕ can be observed: $34 \div 21 = 1.619 \approx \phi$. When entire section divided into combination of sections B and A2 another example of Golden Ratio can be observed: $(A + B + A2) \div (B + A2) = 55 \div 34 = 1.618 \approx \phi$ (Power, 2001, p. 106).

5.6. Fibonacci numbers and the Golden Ratio in Bartók’s *Two Sonatas for Piano and Percussion*

The work of Béla Bartók (1881–1945) composed in 1937, namely, *Sonata for Two Pianos and Percussion*, consisted of three sections. Each section was divided into subsections, and the relationships of the ratio of these subsections with one another in terms of the Golden Ratio were examined.

5.6.1. First Movement

The formal division of the first movement, both the long and short sections, agree with the Golden Ratio principles. In this section, in order to examine Golden Ratio, formal divisions lengths were used and calculations made by multiplying lengths by inverse value of the Golden Ratio of 0.618 or the value 0.382 (Simons, 2000, p. 62).

Table 4. Formal divisions of first movement (Simons, 2000, p. 33).

1st measure	**Introduction**	**Foreshadows the Allegro**
m. 32	**Allegro/Exposition** First thematic group	Beginning of the movement proper, in C. Theme 1 (Principle theme)
m.43	First thematic group	Theme 2
m. 84-101	Secondary theme	
m. 105	Codetta	
m. 161	Conclusion	Reference to secondary theme
m. 175	**Transition**	Layers of fourths

m. 195	**Development**	a) in E. Uses second theme of main theme group as ostinato beneath principal theme in imitation
m. 217		b) Short interlude
m. 232		c) in G-sharp. Inverted ostinato
m. 274	**Recapitulation**	
m. 332	**Coda**	Fugato beginning; built on closing theme
m. 443	End	

Length of combination from the first measure of introduction till the last measure of the exposition and thematic materials 171st measure is totally 170 measures. By including previous 171st measure, combination of "transition" section and "development" subsection until 274th measure is **104** measure long. First movements (total 443 measure) Golden Section is seen at 274th measure at the beginning of "recapitulation" $(443x0.618 \approx 274)$. From the 274th measure, remaining part of the section is 169 measures. Golden Ratio of 169 is: $169 \times 0.618 = \mathbf{104.44}$, which shows that Bartók's whole plan for the first section was to create a symmetrical design that adhered to the Golden Ratio (Table 4). By including the previous 171st measure, the combined length of the transition section and "development" subsection until 274th measure is not 104 measure but **103**. The remaining part of the section from the 274th measure is not 169 but 170 (Simons, 2000, p. 62). Therefore, the calculation should be as follows: 70 × 0.618 = **105.06**, which has nothing to do with 103. In this case, the assumption raised about the Golden Ratio is wrong

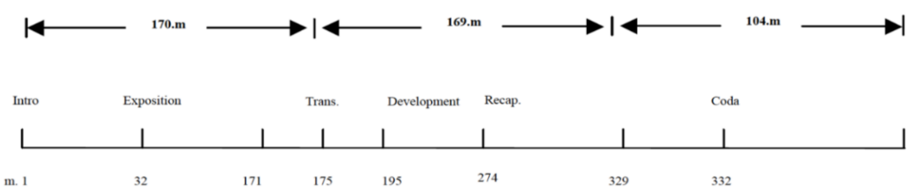


Fig. 10. "Formal divisions of first movement."

Source: Simons, 2000, "Top, total number of measures in large sections: bottom, total number of measures in small sections", (p. 63)

Smaller-scaled Golden Ratios can be observed within this formal outline. For example, the structural differences in the introduction subsection with the separate canonical sections show structural similarities to the Golden Ratio (Fig. 11). The Golden Ratio of 55 beat in the first canonical section (introduction) coincide with fortissimo (very strong) rise at 6th measure or 34th beat (55 × 0.618 = 33.99). The silence at 8th measure is at 52nd of the 55 beats, and it is equivalent to the Golden Ratio of two canonical subsection combinations: The total is 84 beats × 0.618 =54.912. Meanwhile, first four canonical parts consists of two equal sections. After 55-beat subsection, 26-beat thematic material sections appear in each of them. The Golden Ratio of the first and third subsections 55 beats is 26.01, and this is equal to 26 beats of the second and fourth sections of the thematic materials (Fig. 12) (Simons, 2000, p. 63).

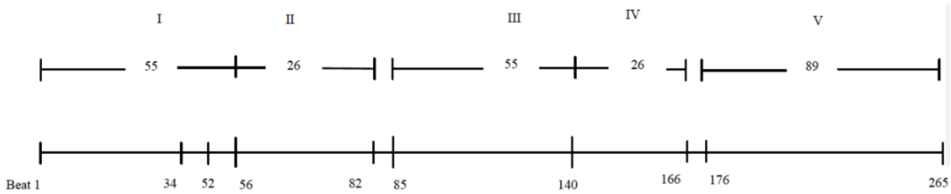


Fig. 11. "Golden Section proportions of movement I, introduction to m.32."
Source: Simons, 2000, Top. Canonic sections by number; middle, total number of beats per section; bottom, beats" (p. 64).

Here, the Golden Ratio of number 55 is incorrectly evaluated. The Golden Ratio of the 55 beats in the first and third subdivisions is not 26.01 but 55 × 0.38 = 21.01. This does not correspond to 26 beats of the thematic material in the second and fourth divisions. Therefore, the alleged Golden Ratio is incorrect. From the beginning of the exposition at the 32nd measure until the end of the secondary theme in the 101st measure, 70 measures exist. The Golden Ratio of 70 is (70 × 0.618) 43.2. This value coincides with the 2nd theme of the first thematic group at the 43rd measure (Fig. 12).

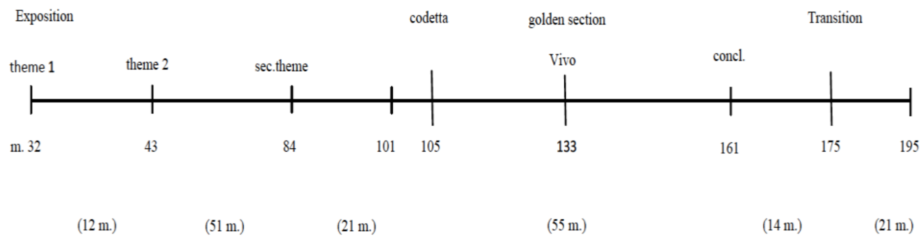


Fig. 12. "Golden Section proportions of movement I"
Source: "Exposition through transition: bottom, total measures per section" (Simons, 2000, p. 65).

In the above-mentioned division, we can also observe small-sized Golden Ratios. For example, the Golden Ratio of the first thematic group of the exposition subsection (84-32=52) is: 52 × 0.618 = 32.13, which corresponds to the first measure of this subsection. Simultaneously, the Golden Ratio of the part from the codetta to vivo (live) (a total of 28 measures) is 28 × 0.618 = 17.304, is the length of the secondary theme. The Golden Ratio of the length from codetta to the concluding subsection is: 55 × 0.618 = 33.99, which is relevant to number 34 obtained from the combination of the conclusion subsection and transition. Thus, the Golden Ratio of the section from codetta to the end of the transition: 89 x 0.618 = 55.02, is equal to the length from the codetta to the conclusion at the 161st measure. These numbers are closely related to the Golden Ratio calculations of the introduction subsection. We can see that Golden Ratio of total measure from the conclusion to transition (34 × 0.618 = 21.01) is closely related to the length of the transition (20 measures). After all, the Golden Ratio is seen in all parts of the entire movement (Fig. 13). Combination from the transition to the end of the development subsection is 99 measure. Its Golden Ratio: $99 \times 0.618 = 67.362$, equal to the total measures of the part from the interlude to the end of the development subsection (Simons, 2000, p. 63-64).

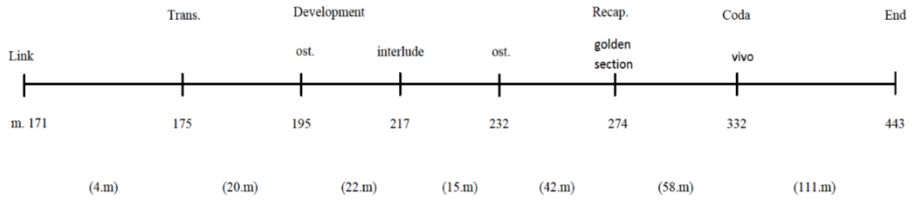


Fig. 13. Golden Ratio of the first movement from transition to conclusion (Simons, 2000, p. 66).

According to the above-mentioned findings, the length from codetta to conclusion is not 55 but 56. Thus, the Golden Ratio of this value is $56 \times 0.618 = 34.60835 \approx 35$. From conclusion section to the transition, the measure is not 34 but 35. From codetta to the end of transition is not 89 but 90. The Golden Ratio of this value is $90 \times 0.618 = 55.62 \approx 56$. This number indicates the measurement of the section from the codetta to the conclusion. From the transition section to the end of development (99 measures), the Golden Ratio: $99 \times 0.618 = 61.18261$. This is not the measure of the length from the interlude to the end of development as mentioned. This length is not 67 but 57. Therefore, it has nothing to do with the number 61 found by Golden Ratio calculations. Here, not only the Golden Ratio of 99 is miscalculated but also the number of measures is miscounted. That is to say, the alleged Golden Ratio findings are wrong (Simons, 2000, p. 65-66).

5.6.2. Second Movement

The whole second movement is based on the Golden Ratio principle. The formal division is shown in Fig. 14.

Fig. 14. "Divisions and subsections of second movement" (Simons, 2000, p. 66).

On a large scale, the total measure (92) when applied to the Golden Ratio results in 56.85. At 56th measure, a significant change occurs in the musical materials. In this section, depending on the frequent fluctuation of the time signals, calculation of the Golden Ratio of measures numbers may be difficult. For more accurate results of the Golden Ratio, various sections should be calculated using the quarter-note beat. Some of the Fibonacci Numbers are as follows: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610.

When the second section formal scheme is considered, the importance of these numbers on the calculation of the quarter notes among formal subsections arise (Fig. 15). Each subsection in Fig. 16 represents a musical event and beat numbers between each event is associated with certain integers of the Fibonacci Sequence (Simons, 2000, p. 72-73).

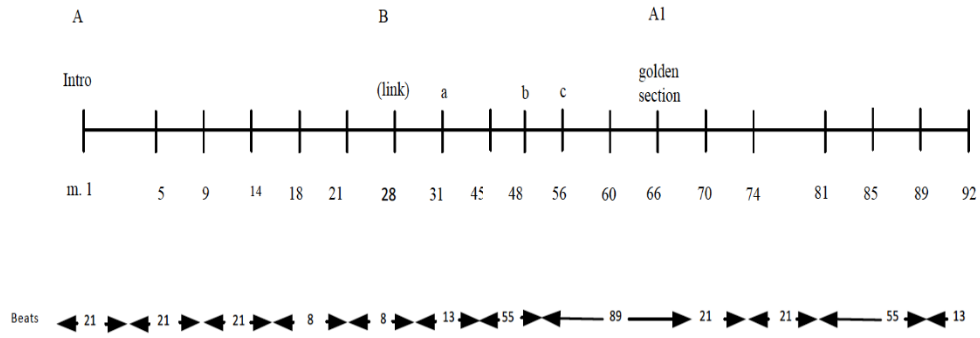


Fig. 15. "Golden Ratios of the second movement"
Source: Bottom, beat numbers of the section" (Simons, 2000, p. 73).

In this section, a total of 377 beats is present, yielding 233 when multiplied by the Golden Ratio. Probably, the most important musical phenomena occur at the 233rd beat, 66th measure on the turn to subsection A. This determines Golden Ratio of this movement both in mathematical and musical meaning. Considering the three main sections of this movement, a close relationship with the Golden Ratio can be clearly observed. Subsection A has a length of 126 quarter-note beats. Subsection B has 114 beats, and subsection A1 has 107 beats. The total beats in both sections A and A1 is 233, and this is Golden Ratio for a total of 377 beats When the beats from the first measure in this section are calculated, we obtain 113 in section A, 157 in section B, and 110 in section C, where the total number of beats is not 377, as previously estimated, but 380. When the Golden Ratio of this value is calculated, no meaning is obtained: 380x0.618=234,84≈ 235. The total beats of the A subsequences (A+A1) is not 233 but 223. This number is not a Golden Ratio of 380. The beats here are incorrectly counted, and an error has been made in the calculation of the Golden Ratio (Simons, 2000, p. 73).

5.6.3. Third Movement

In this section, the principles of the Golden Ratio are included. In the context of the formal outline, the linear expression of the division in the third section reveals the connection between the total length of the subsections and the Fibonacci numbers (Table 5).

Table 5. "Formal outline of third movement (left formal sections; right, sonata-rondo divisions)" (Simons, 2000, p. 77).

A	m. 1-Principal Theme (1)	(C)	A
A1	m. 18-Theme 1 (variation) m.28-Transition	(C) (C)	
B	m. 44-Secondary theme (II)	(E)	B
B1	m. 56-Theme II (variation) m. 91-Transition	(F-sharp) (C-sharp)	
C	m. 103-First Episode (I)	(B)	
A	m. 134-Refrain, Theme I	(G)	A
D	m. 140 Development	(E-flat)	C
A	m. 229-Refrain, Theme I	(B-flat)	A
A	m. 248-Recapitulation m. 260-Retransition	(C) (G)	
B	m. 269-Theme II. m. 287- Retransition	(E-flat) (B-flat)	B
C	m. 301-Second Episode	(E)	
A	m. 351-Refrain, Theme I (variation)	(E-flat)	A
A1	m. 365- Refrain, Theme I (variation) M. 379-Coda (m. 420-End)	(F) (C) (C)	 (Coda)

This section contains 843 quarter-note beats. Division of Fig. 16 into three major parts and calculation of the quarter notes is shown by the below line. The Fibonacci numbers given in parenthesis correspond to the total quarter-note beats, and the total number of measures in smaller section of the formal outline. The large scaled subdivisions of the sonata-rondo plan; A, B, and C corresponds to the Golden Ratio. The total amount of measured numbers among in all A parts is 161, almost the Golden Ratio of the total number in sections B and C (261 × 0.618 – 161.298). Therefore, A is a Golden Ratio of B + C. (Simons, 2000, p. 90).

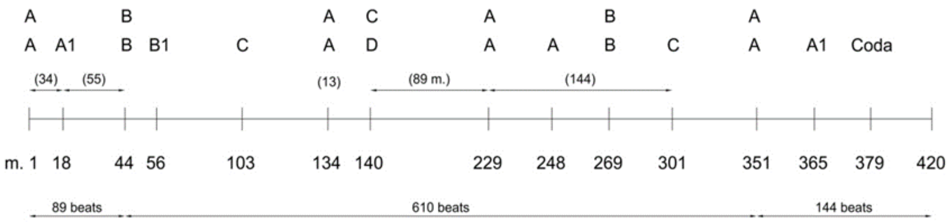


Fig. 16. "Linear expression of the sections in the third movement"
Source: "Top, sonata-rondo parts. (Second line, thematic parts. Numbers in parenthesis show the beat or measurement in the sections" (Simons, 2000, p. 90).

In this section, since measure number of subsection A is not 161 but 158. Besides this, Golden Ratio of the total amount of the measured numbers in section B and C (261 × 0.618 = 161.298) has no meaning. Thus, subsection A is not a Golden Ratio of sections B + C. A Golden Ratio is also observed in the smaller parts of the formal outline. In the first 44 measures of the section, evidence is present on the application of Golden Ratio. If the section articulated as A, A1 and T, each of them relates to the other in Golden Section sense (Fig. 17). Golden Ratio of the total measures: 43.5 (43.5 × 0.168) is 26.88. In parallel, the total measures of sections A and A1 is 27, which almost approaches this number. The length of A1 (10.5 measures) is a full Golden Ratio of the length of A (17 × 0.618 =10.506). The transition section or T immediately starts after the 27th measure of the Golden Ratio division. Meanwhile, the length of T is a slightly less than 17 measures, related to the Golden Ratio equality: A + A1 :27, 5 × 0,168 = 16,995 (Simons, 2000, p. 90-91).

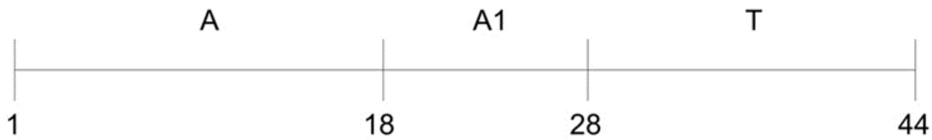


Fig. 17. Third movement measurements 1–44 (Simons, 2000, p. 91).

The Golden Ratio is mostly seen in all of the three main movements in this work. Although some ratios do not cover the whole work, they are also obvious and significant. As a result, a strong commitment to certain ratios shows Bartók's great reliance to formal planning. Moreover, Bartók's musical creativity shines through the use of formulaic rules, which he only used in important cases to emphasize the great musical needs (Simons, 2000, p. 93).

In the above expressions, important composer works from the 17th to the 20th century, the use of the Golden Ratio and Fibonacci Sequence was assumed without verifying whether they consciously did them or not. In particular, in Bartók's years as a student, his familiarity with mathematics and writing numbers over the notes lead to the belief that he used the Fibonacci Sequence numbers and the Golden Ratio in his compositions consciously. Many studies have been conducted to prove this (see 10). As a consequence of both the criticism of these studies made on Bartók's works (see Somfai, 1996) and the conversation of Yöre with Peter Bartók (Béla Bartók's son and publisher), no document is available that provided evidence of Bartók's use of the Fibonacci Sequence and Golden Ratio (personal communication, 23 March 2015). Thus, no similar evidence is available for the other composers.

Conclusion, Discussion and Recommendations

In the present study, according to the related problem, mathematics in relation to music were first discussed. Some mathematical representations of the musical expressions were shown. The mathematical and musical relationship was theoretically studied. Specifically, the use of the Fibonacci Sequence and Golden Ratio in the construction of instruments was discussed, as well as the European-based musical theory. Essentially, studies were performed relative to certain polyphonic music composer works whether they considered the Fibonacci Sequence and Golden Ratio during their composition process.

From the theoretical perspective, according to one octave consisting of eight notes, the assumption that the black and white keys in a piano create numbers in the Fibonacci Sequence is wrong. The octave consists of seven different sounds and therefore, a conclusion that this number is not a number in the Fibonacci Sequence was made.

Investigation of the works in which the alleged use of the Fibonacci Sequence and the Golden Ratio were made was conducted, where only a few exceptions in the conclusion that they are not related to the Golden Ratio were observed. In these particular investigations, with respect to the number of measures, we can see that the ratios in the sections or subsections are each close to the Golden Ratio. However, when the entire studies are considered in the calculation of some musical works, the Golden Ratio was considered to be 1.618, in some works, it was 0.168, whereas in other works, it was 0.382. This concludes that related studies are not aimed at investigating existence of the Golden Ratio in the works, but aimed at confirmation of the assumption.

The individually-considered works were about proximity to the Golden Ratio. However, no error margin determined in calculations. For example, in one work, 1.618 was thought to be the basis of the Golden Ratio; however, the values of 1.571 and 1.616 were also accepted as Golden Ratios. Even between the numbers 1.616 and 1.618, many infinite real numbers exist. This condition shows that no acceptable rule is available that determines whether a value can be considered "golden." Therefore, in the investigated works there exist methodological errors.

In some compositions where we encountered Fibonacci numbers, no evidence was available to prove that it is not coincidental, although it did not also mean that the composer deliberately used the Golden Ratio in his works. For such an assertion to be true, clear data must be available that prove that the composer deliberately used it. Therefore, from these examples selected from polyphonic/contemporary composers, the compositions were not purposely made but were based on artistic intuitions and the mathematical specifications in these works determined by the examinations were made later.

Instead of using the Golden Ratio's three different values (1.618, 0.618 and 0.382) if only one value were considered it would increase the accuracy of the examination results. To determine whether a composer used mathematical features in his works, we recommend examination of the complete works of the composers. In addition, more realistic and scientific results can be obtained if works of contemporary composers (e.g., Iannis Xenakis, Luigi Nono, Ernst Krenek and Karlheinz Stockhausen), who are known to use the Fibonacci Sequence in their works.

References

Antolontez, E.& Susanni, P. (2011). *Béla Bartók: A research and information guide* (3rd ed.). New York and London: Routledge.

Bakım, S. (2014). *Fibonacci dizisi ve Altın Oran'ın müzikte kullanımın incelenmesi* (Yayınlanmamış yüksek lisans tezi). Selçuk Üniversitesi Fen Bilimleri Enstitüsü, Konya.

Bergil, M. S. (1993). *Doğada/bilimde/sanatta Altın Oran*. İstanbul: Arkeoloji ve Sanat Yayınları.

Bora, U. (2002). Bilim ve sanatın kesiştiği temel bir nokta: Matematik ve müzik ilişkisi. *Uludağ University Faculty of Education Journal*, *XV*(1), 53-68. Retrieved from http://dergipark.gov.tr/download/article-file/153191

Hammond, J. K. (2011). Mathematics of music. *UW-L Journal of Undergraduate Research*, *XIV*, 1-11. Retrieved from https://www.uwlax.edu/urc/jur-online/PDF/2011/hammond.MTH.pdf

Hoggatt, V. E. (1969). *Fibonacci and Lucas numbers*. Boston: Houghton Mifflin Company.

Koshy, T. (2001). *Fibonacci and Lucas numbers with applications*. New Jersey: John Wiley & Sons, Inc.

Marie, J. S. (2012). *Theology as the basis for Golden Section analysis: A model of construction for Johann Sebastian Bach's St. John Passion* (Unpublished doctoral dissertation). University of Southern California, California.

Posamentier, A.S.&Lehmann, I. (2007). *The (Fabulous) Fibonacci numbers*. NY: Prometheus Books.

Power, T. (2001). *J. S. Bach and the Divine Proportion* (Unpublished doctoral dissertation). Duke University, Durham.

Simons, H. A. (2000). *Béla Bartók's sonata for two pianos and percussion* (Unpublished doctoral dissertation). University of Alberta, Alberta, Canada.

Somfai, L. (1996). *Béla Bartók: Composition, concepts, and autograph sources*. Oakland, CA: University of California Press.