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LATTICE STRUCTURES OF AUTOMATA

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Abstract. Structures and the number of subautomata of a finite automaton are investigated. It is shown that the set of all subautomata of a finite automaton $\mathcal{A}$ is upper semilattice. We give conditions which allow us to determine whether for a finite upper semilattice $(L, \leq)$ there exists an automaton $\mathcal{A}$ such that the set of all subautomata of $\mathcal{A}$ under set inclusion is isomorphic to $(L, \leq)$. Examples illustrating the results are presented.

1. Introduction

With the advent of electronic computers in the 1950's, the study of simple formal models of computers such as automata was given a lot of attention. The aims were multiple: to understand the limitations of machines, to determine to what extent they might come to replace humans, and later to obtain efficient schemes to organize computations. One of the simplest models that quickly emerged is the finite automaton which, in algebraic terms, is basically the action of a finitely generated free semigroup on a finite set of states and thus leads to a finite semigroup of transformations of the states. From its very beginning, the theory of automata, especially the algebraic one, was based on numerous algebraic ideas and methods. The fact that automata without outputs, and hence the automata without outputs belonging to arbitrary automata, can be treated as algebras whose all fundamental operations are unary, that is as unary algebras. This makes possible to investigate automata from the aspect of Universal algebra and to use its ideas, methods and results [1, 3, 4, 5, 6, 8, 9, 10].

Here we deal with some important concepts of lattice theory and automata theory that will be used in the paper. Let $(P, \leq)$ be a poset and let $a, b \in P$ with $a \neq b$. Then a is called a predecessor of b, and b is called a successor of a if $a \leq c \leq b$ and

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$c \in P$ imply $c = a$ or $c = b$. We denote this relation by $< a, b >$. By $|D|$ we denote the cardinality of D. For every element a of P we set $|\{b \in P :< b, a >\}| = o(a)$, $o(P)$ denotes $\max\{o(a) : a \in P\}$ and $B(P) = \{a \in P : o(a) \leq 1\}$. Two posets $(P_1, \leq_1)$ and $(P_2, \leq_2)$ are said to be isomorphic, denoted by $(P_1, \leq_1) \cong (P_2, \leq_2)$, if there exists a bijection f of P_1 onto P_2 such that for any $a, b \in P_1$, $a \leq_1 b$ if and if only $f(a) \leq_2 f(b)$. A poset $(L, \leq)$ is said to be an upper semilattice if for any $x, y \in L$ there exists the least upper bound of x and y [1, 7, 8].

Let Σ be a nonempty finite set. Denote by Σ^* the free monoid over Σ and ε the empty string of Σ. A finite automaton is a triple $\mathcal{A} = (X, \Sigma, \lambda)$ where X and Σ are nonempty finite sets called a state set and alphabet, respectively and $\lambda : X \times \Sigma^* \to X$ is the transition function satisfying $\forall x \in X$, $\forall a, b \in \Sigma^*$, $\lambda(x, ab) = \lambda(\lambda(x, a), b)$ and $\lambda(x, \varepsilon) = x$. Define a relation $\sim$ on X by $\forall p, q \in X$, $p \sim q$ if and only $\lambda(p, u) = q$, $\lambda(q, v) = p$ for some $u, v \in \Sigma^*$. The relation $\sim$ is an equivalence relation. Let $p \in X$. We denote the equivalence class $\{q \in X : p \sim q\}$ by T_p. This subset T_p is called a layer of X. If $\rho(\mathcal{A}) = \{T_p : p \in X\}$, we define a partial order $\preccurlyeq$ on $\rho(\mathcal{A})$ as follows: For $p, q \in X$, $T_p \preccurlyeq T_q$ if and only if there exists $v \in \Sigma^*$ such that $\lambda(q, v) = p$. An automaton $\mathcal{B} = (X', \Sigma, \theta)$ is called a subautomaton of automaton $\mathcal{A} = (X, \Sigma, \lambda)$ if and only if $X' \subseteq X$ and $\theta = \lambda|_{X' \times \Sigma^*}$, i.e., θ is the restriction of λ to $X' \times \Sigma^*$. We denote the set of all of subautomata of $\mathcal{A}$ by $\sigma(\mathcal{A})$. Let $\mathcal{B}$, $\mathcal{C} \in \sigma(\mathcal{A})$. By $\mathcal{B} \sqsubseteq \mathcal{C}$, we mean that $\mathcal{B}$ is a subautomaton of $\mathcal{C}$. Then $\sqsubseteq$ is a partial order on $\sigma(\mathcal{A})$; hence $(\sigma(\mathcal{A}), \sqsubseteq)$ is a poset, see [7]. Moreover, $(\sigma(\mathcal{A}), \sqsubseteq)$ is a finite upper semilattice by [7, Proposition 2]. Throughout this paper, we shall assume unless otherwise stated, that posets, upper semilattices, lattices and automata are finite.

A directed graph is a graph that is a set of vertices connected by edges, where the edges have a direction associated with them. We define a directed graph on finite poset $(A, \leq)$, $G(A)$, with vertices as elements of A and for two distinct vertices a and b, we have edge (a, b) if and only if $< a, b >$ (for a vertex a, the in-degree of a, $deg^-(a)$, is the number of edges going to a).

2. Subautomaton

Our starting point is the following lemma.

Lemma 1. *Let $(L, \leq)$ be a finite poset. Then there exists an automaton $\mathcal{A} = (X, \Sigma, \lambda)$ such that $(\rho(\mathcal{A}), \preccurlyeq) \cong (L, \leq)$.*

Proof. We construct an automaton $\mathcal{A} = (X, \Sigma, \lambda)$ in the following way. Let $X = L$ and $\Sigma = \{a_1, a_2, \cdots, a_n\}$, where $n = o(L) + 1$ (so $n \geq 1$). Now by the same technique as in [7,Theorem 1], for each $l \in L$, consider $l_1, l_2, \cdots, l_{o(l)}$, all the predecessors of l. Define λ as follows: $\lambda(l, a_i) = l_i$ for $i = 1, 2, \cdots, o(l)$ and $\lambda(l, a_i) = l$ for $i = o(l) + 1, \cdots, n$. This gives $\lambda(l, w) \leq l$ for any $l \in L$ and $w \in \Sigma^*$. Thus $T_l = \{l\}$.

Define $f : L \to \rho(\mathcal{A}) = \{T_l : l \in L\}$ by $f(l) = T_l$. Clearly, f is a bijective mapping.

It remains to prove $l \leq l'$ and $l \neq l'$ if and only if $T_l \preceq T_{l'}$ and $T_l \neq T_{l'}$. Suppose that $l \leq l'$ and $l \neq l'$. Since L is finite set, there exist $x_1 = l, x_2, \cdots, x_m = l'$ of L such that $< x_{i-1}, x_i >$ $(i = 2, \cdots, m)$. Therefor there is an element $a_{i-1} \in \Sigma$ such that $\lambda(x_i, a_{i-1}) = x_{i-1}$. Thus we have $T_{x_{i-1}} \preceq T_{x_i}$ and $T_{x_{i-1}} \neq T_{x_i}$. It follows $T_l \preceq T_{l'}$ and $T_l \neq T_{l'}$. The other implication is clear. So $(\rho(\mathcal{A}), \preccurlyeq) \cong (L, \leq)$. □

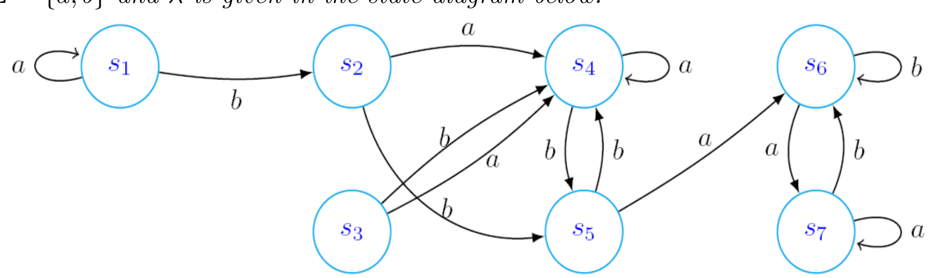
At this point we investigate structures and the number of subautomata of a finite automaton. The following proposition is a reformulation of [7, Theorem 2] and it gives a more explicit description of subautomaton of an automaton.

Proposition 2. *Let $\mathcal{A} = (X, \Sigma, \lambda)$ be an automaton. Then $\mathcal{B} = (X', \Sigma, \theta)$ is a subautomaton of $\mathcal{A}$ if and only if the following conditions are satisfied:*

(i) *The set $X'$is an union of layers of X.*
(ii) *If T_p and T_q are two layers of X with $T_p \subseteq X'$ and $T_q \preccurlyeq T_p$, then $T_q \subseteq X'$.*
(iii) $\theta = \lambda|_{X' \times \Sigma^*}$.

Proof. The sufficiency follows by (i), (ii) and (iii). Conversely, suppose that $\mathcal{B}$ is a subautomaton of $\mathcal{A}$. To see that (i), let $p \in X'$. Then $p \in T_p \subseteq \cup_{q \in X'} T_q$; so $X' \subseteq \cup_{q \in X'} T_q$. For the reverse inclusion, assume that $t \in \cup_{q \in X'} T_q$. Then $t \in T_p$ for some $p \in X'$; hence there exists $\omega \in \Sigma^*$ such that $\lambda(p, \omega) = t$. Now $\lambda(p, \omega) = \theta(p, \omega) = t$ gives $t \in X'$. Thus $X' = \cup_{q \in X'} T_q$. To prove that (ii), from $T_q \preccurlyeq T_p$ we conclude that there exists $\omega \in \Sigma^*$ such that $\lambda(p, \omega) = q$. Therefore $p \in T_p \subseteq X'$ gives $\lambda(p, \omega) = \theta(p, \omega) = q \in X'$. By an argument like that (i), we get $T_q \subseteq X'$. (iii) is clear. □

Example 3. *Consider automaton $\mathcal{D} = (X, \Sigma, \lambda)$, where $X = \{s_1, s_2, \cdots, s_7\}$, $\Sigma = \{a, b\}$ and λ is given in the state diagram below:*



By the definition of layer, we have layers: $T_1 = \{s_1\}$, $T_2 = \{s_2\}$, $T_3 = \{s_3\}$, $T_4 = \{s_4, s_5\}$, $T_5 = \{s_6, s_7\}$ are all of layers of $\mathcal{D}$. The following Figure describes relationship between $T_i, (1 \leq i \leq 7)$. Set $X'_1 = T_5$, $X'_2 = T_4 \cup X'_1$, $X'_3 = T_2 \cup X'_2$, $X'_4 = T_3 \cup X'_2$, $X'_5 = X'_4 \cup X'_3$, $X'_6 = T_1 \cup X'_3$ and $X'_7 = X'_5 \cup T_1$. By Proposition 2, any subautomaton of $\mathcal{D}$ is of the form: $\mathcal{B}_i = (X'_i, \Sigma, \theta_i)$ where $\theta_i = \lambda|_{X'_i \times \Sigma^*}$ for $i = 1, \ldots, 7$.

Definition 4. *A non-empty subset L of a poset $(P, \leq)$ is called a lower set, if for $a \in P$, $b \in L$ and $a \leq b$ implies $a \in L$. In particular, for any $a \in P$ one obtains the*

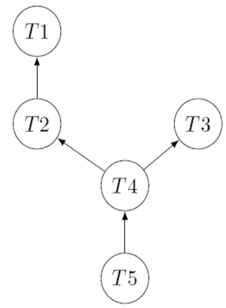


FIGURE 1. $G(\rho(D))$

principle lower set $< a >= \{t \in P : t \leq a\}$.

The set of all lower sets of a poset P is denoted by $LS(P)$. The following theorem shows that relationship between $(\sigma(\mathcal{A}), \sqsubseteq)$ and $(LS(\rho(\mathcal{A})), \subseteq)$.

Theorem 5. *Let $\mathcal{A}$ be any automaton. Then $(\sigma(\mathcal{A}), \sqsubseteq) \cong (LS(\rho(\mathcal{A})), \subseteq)$.*

Proof. By Proposition 2, $\mathcal{B}$ is a subautomaton of $\mathcal{A}$ if and only if $\rho(\mathcal{B}) \in LS(\rho(\mathcal{A}))$. We define the mapping $f : \sigma(\mathcal{A}) \to LS(\rho(\mathcal{A}))$ as follows: $f(\mathcal{B}) = \rho(\mathcal{B})$ for each subautomaton $\mathcal{B}$ of $\mathcal{A}$. An inspection will show that f is a poset isomorphism. □

Corollary 6. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two automata such that*

$$(\rho(\mathcal{A}_1), \preceq) \cong (\rho(\mathcal{A}_2), \preceq).$$

Then

$$(\sigma(\mathcal{A}_1), \sqsubseteq) \cong (\sigma(\mathcal{A}_2), \sqsubseteq).$$

Proof. Apply Theorem 5. □

Proposition 7. *Let L_1, L_2 and L be Lower sets of a poset $(P, \leq)$.*

(i) *$< L_1, L_2 >$ if and only if $L_2 = L_1 \cup \{t\}$, where t is a maximal element in L_2.*

(ii) *If $deg^-(L) \neq 0$ and $D = \{t_i : t_i$ is a maximal element in $L\}$, then $|D| = deg^-(L)$. Moreover, if $deg^-(L) = 0$, then L has an unique maximal element.*

(iii) *If either L_1, L_2 are minimal elements in $LS(P)$ or $< L, L_1 >$ and $< L, L_2 >$, then there exists $L' \in LS(P)$ with $< L_1, L' >$ and $< L_2, L' >$.*

Proof. (i) Assume that $< L_1, L_2 >$. Then $L_2 = L_1 \cup \{t_1, t_2, \ldots, t_n\}$ for some $t_i \notin L_1, (1 \leq i \leq n)$. If $n \geq 2$, then the set $H = \{t_1, t_2, \ldots, t_n\}$ has a minimal element, say t'. Set $L' = L_1 \cup \{t'\}$. By the definition of lower set, $L' \in LS(P)$ such that $L_1 \subsetneqq L' \subsetneqq L_2$ which is a contradiction. Thus

$L_2 = L_1 \cup \{t_1\}$. Now we show that t_1 is a maximal element in L_2. Otherwise, there exists an element t in L_2 such that $t_1 \leq t$ and $t_1 \neq t$. Therefore $t \in L_1$ (since $L_2 = L_1 \cup \{t_1\}$). So $t_1 \in L_1$ which is a contradiction . The other implication is clear.

(ii) Let $t_1, \cdots, t_k$ be the all of distinct maximal elements in L. If L_1 is a lower set of P with $< L_1, L >$, then there exists an unique element t_i (for some $1 \leq i \leq k$) such that $L = L_1 \cup \{t_i\}$ and $t_i \notin L_1$) by (i). Also, each lower set L' with $L' \subseteq L$ such that $|\{t_i : t_i \notin L'\}| = 1$ implies that $< L', L >$; hence $|D| = deg^-(L)$. Finally, If $deg^-(L) = 0$, then L has not a proper subset in $LS(P)$. Therefore $L = \{t\}$ for some element t in P, and the proof is complete.

(iii) Suppose that L_1, L_2 are minimal elements in $LS(P)$. Then $L_1 = \{t_1\}$ and $L_2 = \{t_2\}$. Set $L' = \{t_1, t_2\}$. Clearly, $< L_1, L' >$ and $< L_2, L' >$ by (i). Similarly, if $< L, L_1 >$ and $< L, L_2 >$, then we set $L' = L_1 \cup L_2$; thus $L' \in LS(P)$ gives $< L_1, L' >$ and $< L_2, L' >$ by (i).

□

Lemma 8. *Let P be a poset. Then P is a chain with $|P| = n$ if and only if $LS(P)$ is a chain with $|LS(P)| = n$.*

Proof. Let $LS(P) = \{L_1, \cdots, L_n\}$ with $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n$. Then we can take $L_1 = \{t_1\}$, $L_2 = \{t_1, t_2\}, \ldots,$ and $L_n = L = \{t_1, t_2, \ldots, t_n\}$ by Proposition 7 (i). We claim that $t_1 \leq t_2 \leq \ldots \leq t_n$. Assume to the contrary, let $t_k \nleq t_{k+1}$ for some k $(1 \leq k < n)$. Since t_{k+1} is a maximal element in L_{k+1} by Proposition 7 (i), we get $t_{k+1} \nleq t_k$. Set $L' = \{t_i : t_i \leq t_{k+1}\}$. Then $L' \in LS(P)$ with $L' \nsubseteq L_k$ and $L_k \nsubseteq L'$ that is a contradiction. Conversely, suppose $t_1 \leq t_2 \leq \ldots \leq t_n$. For each j $(1 \leq j \leq n)$, we set $L_j = \{t_1, t_2, \ldots, t_j\}$. Then by the definition of lower set and Proposition 7 (i), L_j $(1 \leq j \leq n)$ is an element of $LS(P)$ with $L_1 \subseteq L_2 \subseteq \ldots \subseteq L_n$. □

In view of the proof of the Proposition 7 and Lemma 8, we have the following corollary for automata.

Corollary 9. •

(a) *Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{C}$ be subautomata of $\mathcal{A} = (X, \Sigma, \lambda)$.*

(i) *$< \mathcal{A}_1, \mathcal{A}_2 >$ if and only if $\rho(\mathcal{A}_1) \subseteq \rho(\mathcal{A}_2)$ and $\rho(\mathcal{A}_2) = \rho(\mathcal{A}_1) \cup \{T\}$, where T is a maximal element in $\rho(\ \mathcal{A}_2)$.*

(ii) *If $deg^-(\mathcal{C}) \neq 0$ and $D = \{T_i : T_i$ is a maximal element in $\rho(\mathcal{C})\}$, then $|D| = deg^-(\mathcal{C})$. Moreover, if $deg^-(\mathcal{C}) = 0$, then C has an unique maximal layer.*

(iii) *If either $\mathcal{A}_1$, $\mathcal{A}_2$ are minimal elements in $\sigma(\mathcal{A})$ or $< \mathcal{C}, \mathcal{A}_1 >$ and $< \mathcal{C}, \mathcal{A}_2 >$, then there exists $\mathcal{A}' \in \sigma(\mathcal{A})$ with $< \mathcal{A}_1, \mathcal{A}' >$ and $< \mathcal{A}_2, \mathcal{A}' >$.*

(b) *Let $\mathcal{A} = (X, \Sigma, \lambda)$ be an automaton. Then $\rho(A)$ is a chain with $|\rho(A)| = n$ if and only if $\sigma(A)$ is a chain with $|\sigma(A)| = n$*

Definition 10. *A poset $(P, \leq)$ is called decomposable, if there exist proper subpoests $(P_1, \leq), \cdots, (P_n, \leq)$ of P such that $P = \cup_{i=1}^{n} P_i$, $P_i \cap P_j = \emptyset$ for $i \neq j$ $(1 \leq i, j \leq n)$ and for every couple $a_i \in P_i$, $b_j \in P_j$ be incomparable where $i \neq j$ $(1 \leq i, j \leq n)$. In this case, we say that P_i is a decomposition component of P $(1 \leq i \leq n)$.*

Proposition 11. *Let $(P_1, \leq)$ and $(P_2, \leq)$ be decomposition components of a poset $(P, \leq)$. Then L is a lower set of P if and only if it satisfies one of the following conditions:*

(i) *Either L is a lower set of P_1 or is a lower set of P_2.*
(ii) *There exist a lower set L_1 of P_1 and a lower set L_2 of P_2 such that $L = L_1 \cup L_2$.*

Proof. Let $L = \{t_1, \ldots, t_n\}$ be a lower set of P. If $L \in LS(P_1)$ or $L \in LS(P_2)$, then we are done. Otherwise, without loss of generality, we can assume that $L_1 = \{t_1, \cdots, t_k\} \subseteq P_1$ and $L_2 = \{t_{k+1}, \cdots, t_n\} \subseteq P_2$ $(1 \leq k < n)$. If $t \in L_i$, $t' \in L$ and $t' \leq t$ for $i = 1, 2$, then $t' \in L_i$ by Definition 10. Now $L_i \in LS(P_i)$, as required. □

Theorem 12. *Let $(P_1, \leq)$ and $(P_2, \leq)$ be decomposition components of a poset $(P, \leq)$. Then*

$$|LS(P)| = (|LS(P_1)| + 1)(|LS(P_2)| + 1) - 1$$

.

Proof. Assume that $|LS(P_1)| = m$ and $|LS(P_2)| = n$. Then the number of lower set that satisfies conditions (i) and (ii) in Proposition 11 are $n+m$ and nm, respectively, as required. □

Corollary 13. (i) *Let $(P_1, \leq), \cdots, (P_n, \leq)$ be decomposition components of a poset $(P, \leq)$. Then $|LS(P)| = \Pi_{i=1}^{n}(|LS(P_i)| + 1) - 1$.*
(ii) *Assume that $(P, \leq)$ is any poset and let p be a prime number such that $|Ls(P)| = p - 1$. Then $(P, \leq)$ is indecomposable.*

Proof. •

(i) The proof is straightforward by induction on n and Theorem 12.
(ii) Assume to the contrary, let there exist decomposition components $(P_1, \leq)$ and $(P_2, \leq)$ of a poset P. By assumption and (i), $(|LS(P_1)|+1)(|LS(P_2)|+1) = p$ that is a contradiction (because $|LS(P_i)| \geq 1$).

□

Definition 14. *Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be two finite posets with $P_1 \cap P_2 = \emptyset$. Then we can define the poset $(P_1 \cup P_2, \leq)$ as follows:*

(i) *For any $i = 1, 2$, $a, b \in P_i$, $a \leq b$ if $a \leq_i b$.*
(ii) *For any $a \in P_1$ and $b \in P_2$, $a \leq b$.*

Lemma 15. *Let $(P_1, \leq_{p_1}) \cong (P'_1, \leq_{p'_1})$ and $(P_2, \leq_{p_2}) \cong (P'_2, \leq_{p'_2})$ with $P_1 \cap P'_1 = P_2 \cap P'_2 = \emptyset$. Then $(P_1 \cup P_2, \leq) \cong (P'_1 \cup P'_2, \leq)$.*

Proof. The proof is straightforward by Definition 14. □

Proposition 16. *Let* $(P_1, \leq_1)$ *and* $(P_2, \leq_2)$ *be two posets with* $P_1 \cap P_2 = \emptyset$ *and let* $(P_1 \cup P_2, \leq)$*. Then the following hold:*

(i) $(LS(P_1 \cup P_2), \subseteq)$ *is poset isomorphic to* $(LS(P_1) \cup LS(P_2), \subseteq)$*. Moreover,* $|LS(P_1 \cup P_2)| = |LS(P_1)| + |LS(P_2)|$

(ii) *If* $(P_1, \leq_1)$ *is an upper semilattice, then* $(B(P_1) \cup B(P_2), \subseteq) = (B(P_1 \cup P_2), \subseteq)$.

Proof. •

(i) Define the mapping $f : LS(P_1) \cup LS(P_2) \to LS(P_1 \cup P_2)$ as follows: If $L \in LS(P_1)$, then $L \in LS(P_1 \cup P_2)$; so we set $f(L) = L$. If $L \in LS(P_2)$, then we set $f(L) = P_1 \cup L$. It is easy to see that f is a poset isomorphism.

(ii) It suffices to show that $B(P_1 \cup P_2) = B(P_1) \cup B(P_2)$. Clearly, $B(P_1 \cup P_2) \subseteq B(P_1) \cup B(P_2)$. For the reverse inclusion, let $l \in P_1$ and $deg^-(l) \leqslant 1$. Then indegree l in $P_1 \cup P_2$ is equal to 1 or 0. Moreover, if $l \in P_2$ and $deg^-(l) = 1$, then indegree l in $P_1 \cup P_2$ is equal to 1. Also, if $l \in P_2$ and $deg^-(l) = 0$, then we have only $< 1, l >$ in $P_1 \cup P_2$ (1 is the greatest element in P_1); hence indegree l in $P_1 \cup P_2$ is equal to 1. So l in $B(P_1 \cup P_2)$, and we have equality.

□

Corollary 17. (i) *Let* $(P, \leq)$ *be a poset and* $t \in P$ *be a maximum element of* P*. Then* $|LS(P)| = |LS(P_1)| + 1$ *where* $P_1 = P \setminus \{t\}$ *is subposet of* P*.*

(ii) *Let* $(P, \leq)$ *be a poset and* $t \in P$ *be a minimum element of* P*. Then* $|LS(P)| = |LS(P_1)| + 1$ *where* $P_1 = P \setminus \{t\}$ *is subposet of* P*.*

Proof. Apply Proposition 16 (i). □

According to the above results, computation $|\sigma(A)|$ become easier.

Example 18. *In the Example 3, consider subposets* $P_1 = (\{T_4, T_5\}, \preceq)$ *and* $P_2 = (\{T_1, T_2, T_3\}, \preceq)$*. Then* $(\rho(\mathcal{D}), \preceq) \cong (P_1 \cup P_2, \preceq)$*. We have* $|LS(P_1)| = 2$ *and* $|LS(P_2)| = 5$ *by Lemma 8 and Theorem 12 , then* $|\sigma(D)| = 7$ *by Proposition 16 (i) .*

The following theorem gives estimate for the number of lower set of a poset.

Theorem 19. *Let* $(P, \leq)$ *is a poset with* $|P| = n$ *and* $t_1, \cdots, t_m$ *be the all of minimal element of* P *. Then the following inequality is valid:*

$$2^m - 1 \leq |LS(P)| \leq 2^{n-1} + 2^{m-1} - 1$$

Proof. Clearly, every nonempty subset of $A = \{t_1, \cdots, t_m\}$ is a lower set of P. we know the number of the all non-empty subsets of A is equal to $2^m - 1$. So $|LS(P)| \geq 2^m - 1$. It remains to prove the other side unequal. It easily seen that every lower set is equal to union the number of principle lower sets. Assume that a, b be two maximal elements of P . Then $< a > \cup < b >$ is a lower set of P which is distinct from $< a >$ and $< b >$. Therefor, if P has more maximal elements, then there exist more lower sets. Now, if each element of P is maximal element, then $n = m$. It follows $|LS(P)| = 2^m - 1$; Hence we are done. otherwise, we consider poset $P = \{t_1, \cdots, t_m, l_1, \cdots, l_{n-m}\}$ where $t_1 \leq l_i$ and each couple t_j, l_i are incomparable for $i = 1, \cdots, n-m$, $j = 2, \cdots, m$. In this case P has $n-1$ maximal elements. Also $P_1 = \{t_1, l_1, \cdots, l_{n-m}\}$ and $P_2 = \{t_2, \cdots, t_m\}$ are decomposition components of P . Thus $|LS(P_1)| = 2^{n-m}$ by Corollary 16(ii) and Corollary 13 (i) and $|LS(P_2)| = 2^{m-1} - 1$ by Corollary 13(i). Now the assertion follows from Theorem 12 . □

Definition 20. *An automaton $\mathcal{A} = (X, \Sigma, \lambda)$ is called decomposable, if there exist proper subautomata $\mathcal{A}_1 = (X_1, \Sigma, \lambda_1), \cdots, \mathcal{A}_n = (X_n, \Sigma, \lambda_n)$ of $\mathcal{A}$ such that $X = \cup_{i=1}^n X_i$ and $X_i \cap X_j = \emptyset$ for $i \neq j$ $(1 \leq i, j \leq n)$. In this case, we say that $\mathcal{A}_i$ is a decomposition component of $\mathcal{A}$ $(1 \leq i \leq n)$.*

The next Theorem follows from Proposition 11, Corollary 13, Proposition 16 and Theorem 19.

Theorem 21. (i) *Let $\mathcal{B} = (X_{\mathcal{B}}, \Sigma, \lambda_{\mathcal{B}})$ and $\mathcal{C} = (X_{\mathcal{C}}, \Sigma, \lambda_{\mathcal{C}})$ be decomposition components of an automaton $\mathcal{A} = (X, \Sigma, \lambda)$. Then $\mathcal{A}' = (X', \Sigma, \lambda')$ is a subautomaton of $\mathcal{A}$ if and only if it satisfies one of the following conditions:*
  1. *Either $\mathcal{A}'$ is a subautomaton of $\mathcal{B}$ or is a subautomaton of $\mathcal{C}$.*
  2. *There exist a subautomaton $\mathcal{B}' = (X_1', \Sigma, \lambda_1)$ of $\mathcal{B}$ and a subautomaton $\mathcal{C}' = (X_2', \Sigma, \lambda_2)$ of $\mathcal{C}$ such that $X' = X_1' \cup X_2'$.*

(ii) (1) *Let $\mathcal{A}_1 = (X_1, \Sigma, \lambda_1), \cdots, \mathcal{A}_n = (X_n, \Sigma, \lambda_n)$ be decomposition components of an automaton $\mathcal{A} = (X, \Sigma, \lambda)$. Then $|\sigma(A)| = \Pi_{i=1}^n (|\sigma(\ \mathcal{A}_i)| + 1) - 1$.*
  (2) *Assume that $\mathcal{A}$ is any automaton and let p be a prime number such that $|\sigma(\mathcal{A}))| = p - 1$. Then $\mathcal{A}$ is indecomposable.*

(iii) *Let $\mathcal{A}$ be an automaton and $(P_1, \preceq)$ and $(P_2, \preceq)$ be subposets of $(\rho(\mathcal{A}), \preceq)$ such that $P_1 \cap P_2 = \emptyset$ and $(\rho(\mathcal{A}), \preceq) \cong (P_1 \cup P_2, \preceq)$. Then $|\sigma(A)| = |LS(P_1)| + |LS(P_2)|$*

(iv) *Let $\mathcal{A}$ be an automaton , $|(\rho(\mathcal{A})| = n$ and $|\{T \in (\rho(\mathcal{A}) : T$ is a minimal layer$\}| = m$. Then Then the following inequality is valid:*

$$2^m - 1 \leq |\sigma(\mathcal{A})| \leq 2^{n-1} + 2^{m-1} - 1$$

The remaining of this paper is dedicated to the following question: For which posets L does there exist an automaton $\mathcal{A}$ such that $\sigma(\mathcal{A})$ is isomorphic to L?.

Theorem 22. *If $(P, \leq)$ is a poset, then $(P, \leq) \cong (B(LS(P)), \subseteq)$.*

Proof. We define the mapping $f : P \to B(LS(P))$ as follows: If $t \in P$, then we set $f(t) = \{t' : t' \in P, t' \leq t\}$. It is clear that $f(t) \in LS(P)$. Since t is an unique maximum element in $f(t)$, Proposition 7 (ii) gives $deg^{-}(f(t)) \leq 1$; so $f(t) \in B(LS(P))$. An inspection will show that f is a poset isomorphism. $\square$

Theorem 23. *If $\mathcal{A} = (X, \Sigma, \lambda)$ is an automaton, then*

$$(B(\sigma(\mathcal{A})), \sqsubseteq) \cong (\rho(\mathcal{A}), \preceq).$$

Proof. By Theorem 5, we have $(B(\sigma(\mathcal{A})), \sqsubseteq) \cong (B(LS(\rho(\mathcal{A})), \subseteq))$; hence $(B(\sigma(\mathcal{A})), \sqsubseteq) \cong (\rho(\mathcal{A}), \preceq)$ by Theorem 22. $\square$

Theorem 24. *An upper semilattice $(L, \leq)$ is isomorphic to an upper semilattice of subautomata of an automaton if and only if $(LS(B(L)), \subseteq) \cong (L, \leq)$.*

Proof. If $(L, \leq)$ is an upper semilattice, then $(B(L), \leq)$ is a poset; hence there exists an automaton $\mathcal{A}$ such that $(B(L), \leq) \cong (\rho(\mathcal{A}), \preceq)$ by Lemma 1 which implies that $(LS(B(L)), \subseteq) \cong (LS(\rho(\mathcal{A})), \subseteq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$ by Theorem 5. If $(L, \leq) \cong (LS(B(L)), \subseteq)$, then we are done. If $(L, \leq) \ncong (LS(B(L)), \subseteq)$, then we show that there is not any automaton $\mathcal{C}$ such that $(L, \leq) \cong (\sigma(\mathcal{C}), \sqsubseteq)$. Assume to the contrary, let $(L, \leq) \cong (\sigma(\mathcal{C}), \sqsubseteq)$ for some automaton $\mathcal{C}$. Since $(B(\sigma(\mathcal{C})), \sqsubseteq) \cong (B(L), \leq)$, it follows that $(B(L), \leq) \cong (\rho(\mathcal{C}), \preceq)$ by Theorem 23, so $(\rho(\mathcal{C}), \preceq) \cong (\rho(\mathcal{A}), \preceq)$; hence $(\sigma(\mathcal{C}), \sqsubseteq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$ by Corollary 6. Thus $(L, \leq) \cong (\sigma(\mathcal{A}), \sqsubseteq) \cong (LS(B(L)), \subseteq)$ that is a contradiction. $\square$

Example 25. (i) *Let L be an upper semilattice as described in Figure 2. Then there is not any automaton $\mathcal{A}$ such that $(L, \leq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$. If it is, then we conclude that the graph G in Figure 3 corresponds to $G(B(L))$, so $|LS(B(L))| = (2+1)(2+1) - 1 = 8$ by Lemma 8 and Theorem 12, but $|L| = 6$. Thus $(LS(B(L)), \subseteq) \ncong (L, \leq)$.*

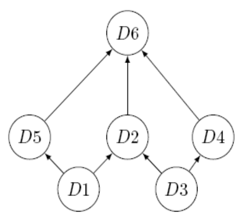


FIGURE 2. L

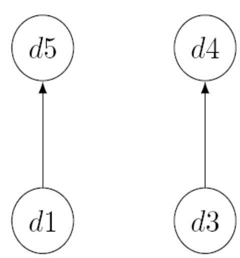


FIGURE 3. G

(ii) *Let L_1 be an upper semilattice as described in Figure 4. Then there exists an automaton $\mathcal{A}$ such that $(L, \leq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$ (The graph G_1 in Figure 5 corresponds to $G(B(L))$. An inspection will show that $(LS(B(L_1)), \subseteq) \cong (L_1, \leq)$).*

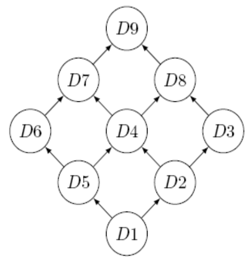


FIGURE 4. L_1

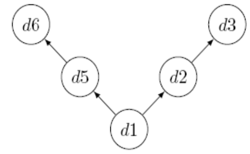


FIGURE 5. G_1

Theorem 26. *Let* $(L_1, \leq_1), (L_2, \leq_2)$ *and* $(L, \leq)$ *be upper semilattices such that* $L_1 \cap L_2 = \emptyset$ *and* $(L, \leq) \cong (L_1 \cup L_2, \leq)$*. Then there exists an automaton* $\mathcal{A}$ *such that* $(\sigma(\mathcal{A}), \sqsubseteq) \cong (L, \leq)$ *if and only if there exist automata* $\mathcal{B}$ *and* $\mathcal{C}$ *such that* $(\sigma(\mathcal{B}), \sqsubseteq) \cong (L_1, \leq_1)$*and* $(\sigma(\mathcal{C}), \sqsubseteq) \cong (L_2, \leq_2)$*.*

Proof. Apply Proposition 16 and Theorem 24 . □

Example 27. *Let* L *be an upper semilattice as described in Figure 6. Then* $L_1 = \{B_1, B_2, B_3, B_4, B_5, B_6\}$ *and* $L_2 = \{B_7, B_8, B_9, B_{10}, B_{11}, B_{12}\}$ *are subupper semilattices satisfy conditions of Theorem 26. Also there is not any automaton* $\mathcal{B}$ *such that* $(L_1, \leq) \cong (\sigma(\mathcal{B}), \sqsubseteq)$ *(see Example 25 (i)). Thus there is not any automaton* $\mathcal{A}$ *such that* $(L, \leq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$*.*

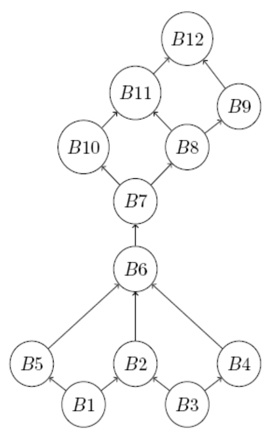


FIGURE 6. L

Example 28. *let* L *be an upper semilattice as described in Figure 7. Then* $L_1 = \{B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9, B_{10}, B_{11}\}$ *and*
$L_2 = \{D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8, D_9\}$ *are subupper semilattices satisfy conditions of Theorem 26. There exists an automaton* $\mathcal{C}$ *such that* $(L_2, \leq) \cong (\sigma(\mathcal{C}), \sqsubseteq)$ *(see Example 25 (ii)). Similarly, there exists an automaton* $\mathcal{B}$ *such that* $(L_1, \leq) \cong$

$(\sigma(\mathcal{B})\ \sqsubseteq)$. Hence there exists an automaton $\mathcal{A}$ such that $(L, \leq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$ By Theorem 26.

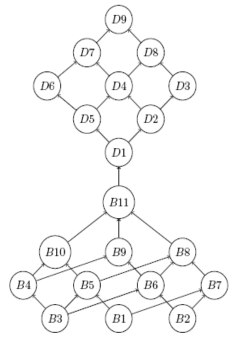


FIGURE 7. L

With the help of results proved by Saliǐ, we find another way to detect whether there exists some automaton $\mathcal{A}$ such that $(\sigma(\mathcal{A}), \sqsubseteq)$ isomorphic to a given finite upper semilattice $(L, \leq)$. Saliǐ consider $= (\emptyset, \Sigma, \lambda)$ as a subautomaton of an automaton $\mathcal{A} = (X, \Sigma, \lambda)$; so $(\sigma(\mathcal{A}) \cup \{\}, \sqsubseteq)$ is a lattice.

Theorem 29. *[3,Theorem 8.4] A finite lattice L isomorphic to the lattice $(\sigma(\mathcal{A}) \cup \{\}, \sqsubseteq)$ for some automaton $\mathcal{A}$ if and only if it is distributive.*

Lemma 30. *Assume that $(L, \leq)$ is an upper semilattice and let 0^* be an element such that $0^* \notin L$. Then $(\bar{L} = \{0^*\} \cup L, \leq)$ is a lattice.*

Proof. This follows from the Definition 14. □

Theorem 31. *An upper semilattice $(L, \leq)$ is isomorphic to the upper semilattice of subautomata of an automaton if and only if $\bar{L}$ is a distributive lattice.*

Proof. Assume that $\bar{L} = \{0^*\} \cup L$ is distributive. Then there exists automaton $\mathcal{A}$ such that $\bar{L}$ is poset isomorphic to $\sigma(\mathcal{A}) \cup \{\}$ by Theorem 29. Note that 0^* and are minimum elements in $\bar{L}$ and $\sigma(\mathcal{A}) \cup \{\}$, respectively. Thus $(L, \leq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$. Conversely, suppose that $(L, \leq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$ for some automaton $\mathcal{A}$. Then $(\bar{L}, \leq) \cong (\sigma(\mathcal{A}) \cup \{\}, \sqsubseteq)$ by Lemma 15. Now the assertion follows from Theorem 29. □

Example 32. *Consider upper semilattices L and L_1 as described in Example 25 (i),(ii), respectively. $\bar{L}$ and $\bar{L_1}$ are defined in Figure 8 and Figure 9. By [2,Theorem 1.7] $\bar{L}$ is not distributive and $\bar{L_1}$ is distributive. Then $(L_1, \leq) \cong (\sigma(\mathcal{A}), \sqsubseteq)$ for some automaton $\mathcal{A}$ and $(L, \leq) \ncong (\sigma(\mathcal{A}), \sqsubseteq)$ for every automaton $\mathcal{A}$ by Theorem 31.*

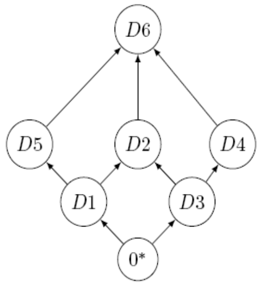


FIGURE 8. $\bar{L}$

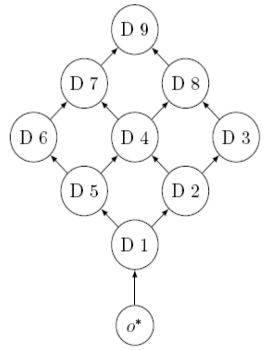


FIGURE 9. $\bar{L_1}$

Definition 33. (a) *[11, Definition 2.1]An algebra* $L = (L, \leq, \wedge, \vee, \bullet, 0, 1)$ *is called a lattice-ordered monoid if*

(1) $L = (L, \leq, \wedge, \vee, 0, 1)$ *is a lattice with the least element 0 and the greatest element 1.*
(2) $(L, \bullet, 1)$ *is a monoid with identity* $1 \in L$ *such that for all* $a, b, c \in L$.
(3) $a \bullet 0 = 0 \bullet a = 0$.
(4) $a \leq b \Rightarrow \forall x \in L,\ a \bullet x \leq b \bullet x$ *and* $x \bullet a \leq x \bullet b$.
(5) $a \bullet (b \vee c) = (a \bullet b) \vee (a \bullet c)$ *and* $(b \vee c) \bullet a = (b \bullet a) \vee (c \bullet a)$.

(b) *[11,Definition 2.2] Let L be a lattice-ordered monoid. Then a 5-tuple* $F = (X, \Sigma, \Gamma, \lambda, \theta)$ *is a called crisp deterministic fuzzy automaton if*

(1) $\mathcal{A} = (X, \Sigma, \lambda)$ *is a finite automaton and* Γ *is a nonempty finite set.*
(2) $\theta : X \times \Sigma^* \times \Gamma^* \longrightarrow L$ *is a map called the output function such that* $\theta(p, \varepsilon, \varepsilon) = 0$, $\theta(p, w, \varepsilon) = \theta(p, \varepsilon, u) = 1$ *for* $\varepsilon \neq w \in \Sigma^*$ *and* $\varepsilon \neq u \in \Gamma^*$ *and* $\theta(p, w_1w_2, u_1u_2) = \theta(p, w_1, u_1) \bullet \theta(\lambda(p, w_1), w_2, u_2)$, *for all* $p \in X$, $w_1, w_2 \in \Sigma^*$ *and* $u_1, u_2 \in \Gamma^*$.

(c) *[11,Definition 2.4] A crisp deterministic fuzzy automaton* $F_1 = (X_1, \Sigma, \Gamma, \lambda_1, \theta_1)$ *is called a subautomaton of a crisp deterministic fuzzy automaton* $F = (X, \Sigma, \Gamma, \lambda, \theta)$ *if* $X_1 \subseteq X$, $\lambda_1 = \lambda|_{X_1 \times \Sigma^*}$ *and* $\theta_1 = \theta|_{X_1 \times \Sigma^* \times \Gamma^*}$

Remark 34. (i) *It is not to hard to see that : If* $F_1 = (X_1, \Sigma, \Gamma, \lambda_1, \theta_1)$ *is a subautomaton of a crisp deterministic fuzzy automaton* $F = (X, \Sigma, \Gamma, \lambda, \theta)$, *then* $\mathcal{A}_1 = (X_1, \Sigma, \lambda_1)$ *is a subautomaton of an automaton* $\mathcal{A} = (X, \Sigma, \lambda)$. *Moreover, if* $\mathcal{A}_1 = (X_1, \Sigma, \lambda_1)$ *is a subautomaton of an automaton* $\mathcal{A}$ *, then* $F_1 = (X_1, \Sigma, \Gamma, \lambda_1, \theta_1)$ *where* $\theta_1 = \theta|_{X_1 \times \Sigma^* \times \Gamma^*}$ *is a subautomaton of a crisp deterministic fuzzy automaton* $F = (X, \Sigma, \Gamma, \lambda, \theta)$. *This gives the results obtained in Theorem 5, Corollary 9 , Theorem 21 is correct similarly for crisp deterministic fuzzy automata.*

(ii) *Let* $\mathcal{A} = (X, \Sigma, \lambda)$ *be an automaton and* $L = \{0, 1\}$ *be a lattice-ordered monoid where* $0 \bullet 0 = 1 \bullet 0 = 0 \bullet 1 = 0$ *and* $1 \bullet 1 = 1$ *. Then* $F = (X, \Sigma, \Gamma, \lambda, \theta)$ *is a crisp deterministic fuzzy automaton where* θ *is defined as follow: if*

$w = u = \varepsilon$, then $\theta(p, w, u) = 0$. Otherwise $\theta(p, w, u) = 1$. From this, we conclude that there exists a crisp deterministic fuzzy automaton for each automaton. Which gives An upper semilattice $(L, \leq)$ is isomorphic to an upper semilattice of subautomata of a crisp deterministic fuzzy automaton if and only if $(LS(B(L)), \subseteq) \cong (L, \leq)$.

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