SOME COMMENTS ON METHODOLOGY OF CUBIC RANK TRANSMUTED DISTRIBUTIONS

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ABSTRACT. In this study, at first a new polynomial rank transmutation is proposed. Then, a new cubic rank transmutation is introduced by simplifying the set of transmutation parameters in order to improve its usefulness in statistical modeling. The purpose of this comment is to clarify some issues that exist in the methodology of obtaining the distribution by the cubic transmutation and the stage of proofing it. In this way, both the parameter space is expanded and the process of establishing the cubic transformed distribution family is given.

1. INTRODUCTION

In this study, we inspire the quadratic rank transmutation map (QRTM) proposed by [16]. The mapping is given as

$$u \rightarrow u + \lambda u (1 - u) \tag{1}$$

where $u \in [0, 1]$ and $\lambda \in [-1, 1]$. Using this transmutation many distributions have been derived and still continue to be derived. Beside this, there are also some studies on the modifications of the QRTM. Some of the pioneering works on proposing modified QRTM can be given as follows: [1] proposed a new Weibull distribution by using exponentiated QRTM. [2] generated a new distribution family by considering exponentiated distribution as the baseline distribution. [11] studied a new distribution by taking the baseline distribution as exponentiated exponential distribution. [5] introduced transmuted exponentiated modified Weibull distribution, and [3] introduced transmuted exponentiated Lomax distribution. The last three

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studies can be seen as a special case of [2]. [9] introduced a new transmutation map by adding extra two parameters to get more flexible distribution. Then, [10] introduced a new Lindley distribution by using this new transmutation map approach. [4] introduced a kind of generalization of QRTM by considering sum of k- dimensional vector of transmutation parameters. There are two similar studies which are the generalized transmuted G family by [12] and generalized transmuted Weibull distribution by [13]. Also, by taking into account recent works, [15] introduced a new distribution named as transmuted generalized Gamma distribution. They use QRTM to generate this distribution family.

In this study, a new polynomial rank transmutation is proposed additionally to [17]. Since the parameter set is still complex, a new cubic rank transmutation is introduced in the light of the idea behind QRTM. In our study, since an extra transmutation parameter is added, the distribution has become more flexible.

2. MOTIVATION

[17] proposed polynomial rank transmutation map to demonstrate Skew-kurtotic transmutations. *Figure 3* of [17] indicates that admissible parameter region. However this region is quite complex structure, the points on *Figure 5* of them show some special cases related to family of order statistics up to 3-sized sample. Under the leadership of this idea, we propose a new polynomial rank transmutation to get simpler structure of parameter region. Let $G(u)$ stand for the polynomial rank transmutation defined on $[0, 1]$. Then, we have

$$G(u) = u + \lambda_1 u (1 - u) + \lambda_2 u^2 (1 - u) \tag{2}$$

with $G(0) = 0$ and $G(1) = 1$. Note that, λ_1 and λ_2 are the transmutation parameters. Parameter region will be defined with following discussion. Since G should be non-decreasing, non-negativity of the first derivative of G with respect to u is examined. Thus, the shape of the parameter region is determined. By calling this derivative with g, we have

$$g(u, \lambda_1, \lambda_2) = -3\lambda_2 u^2 - 2u(\lambda_1 - \lambda_2) + (1 + \lambda_1). \tag{3}$$

Non-negativity of $g(u, \lambda_1, \lambda_2)$ at the end-points, namely the inequalities $g(0, \lambda_1, \lambda_2) = 1 + \lambda_1 \geq 0$ and $g(1, \lambda_1, \lambda_2) = 1 - \lambda_1 - \lambda_2 \geq 0$ both requires that

$$\begin{aligned} \lambda_1 &\geq -1 \\ \lambda_1 + \lambda_2 &\leq 1. \end{aligned} \tag{4}$$

From these two inequalities, it is clear that $\lambda_2 \leq 2$. When the eq. (3) is taken into account, $g(u, \lambda_1, \lambda_2)$ is a concave function for $\lambda_2 \in (0, 2]$. As long as the inequality (4) is valid, $g(u, \lambda_1, \lambda_2)$ will take non-negative values. For $\lambda_2 \leq 0$, we will investigate the sufficient conditions on non-negativity of $g(u, \lambda_1, \lambda_2)$. In this case, $g(u, \lambda_1, \lambda_2)$ has a minimum point since it is a convex function. If this minimum point is within $(0, 1)$, the value at that point of the function $g(u, \lambda_1, \lambda_2)$ must be

positive. Accordingly, the minimum point is obtained by taking the derivative of the eq. (3) and equating them to zero as follows:

$$g'(u, \lambda_1, \lambda_2) = -6\lambda_2 u - 2(\lambda_1 - \lambda_2) = 0 \Rightarrow u^* = \frac{-(\lambda_1 - \lambda_2)}{3\lambda_2} \tag{5}$$

Then, the value of $g(u, \lambda_1, \lambda_2)$ at u^* must satisfy

$$g\left(\frac{-(\lambda_1 - \lambda_2)}{3\lambda_2}, \lambda_1, \lambda_2\right) = \frac{\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2(3 + \lambda_2)}{3\lambda_2} \geq 0 \tag{6}$$

Hence, it is necessary to say that the value of the numerator in (6) is non-positive. If this statement given by the numerator is considered as a second order polynomial of λ_1, the roots are given by

$$\lambda_{1_{1,2}} = \frac{-\lambda_2 \pm \sqrt{-3\lambda_2(\lambda_2 + 4)}}{2} \tag{7}$$

Here, we can say that the condition $-4 \leq \lambda_2$ must also occur in order for the roots to be real valued. Thus, under the condition $-4 \leq \lambda_2 < 0$, we have bounds for λ_1 as follows:

$$\frac{-\lambda_2 - \sqrt{-3\lambda_2(\lambda_2 + 4)}}{2} \leq \lambda_1 \leq \frac{-\lambda_2 + \sqrt{-3\lambda_2(\lambda_2 + 4)}}{2} \tag{8}$$

For these bounds, the numerator in (6) has a negative sign. This leads to the following conclusion: The range of λ_1 is as in (8) for $\lambda_2 \in [-4, 0)$. However, the minimum value of the lower bound in (8) can be -1, while the maximum value of the upper bound can be 3. From this we can say that the range of λ_1 is $[-1, 3]$. Thus, combining this results, the parameter region for (λ_1, λ_2) appears as shown in the Figure 1.

By considering this parameter set of (λ_1, λ_2), many well defined distributions are generated from the eq. (2) with the baseline distribution F. Now, let's get a map of the integer values of the pair (λ_2, λ_1) to see the known distributions tabulated in Table 1.

The distributions specified by the star in Table 1 are described below how they correspond to some known failure distributions.

Let $X_{r:n}$ be the rth order statistic in a sample of size n. By noting that, for $\lambda_1 = -1$, $\lambda_2 = -1$ generated distribution indicates the failure distribution of the lifetime of three-component parallel system, namely, this distribution indicates the distribution of the random variable $X_{3:3} = \max\{X_1, X_2, X_3\}$ where X_1, X_2 and X_3 are independent and identically distributed as F. Similarly, for $\lambda_1 = 2$, $\lambda_2 = -1$ generated distribution indicates the distribution of the random variable $X_{1:3} = \min\{X_1, X_2, X_3\}$. For $\lambda_1 = -1$, $\lambda_2 = 1$ generated distribution indicates the distribution of $\max\{X_1, \min\{X_2, X_3\}\}$. For $\lambda_1 = 0$, $\lambda_2 = 1$ generated distribution indicates the distribution of $\min\{X_1, \max\{X_2, X_3\}\}$. For $\lambda_1 = -1$, $\lambda_2 = 2$ generated distribution indicates the failure distribution of the lifetime of the three-out-of- two system, namely, this distribution indicates the distribution of the random
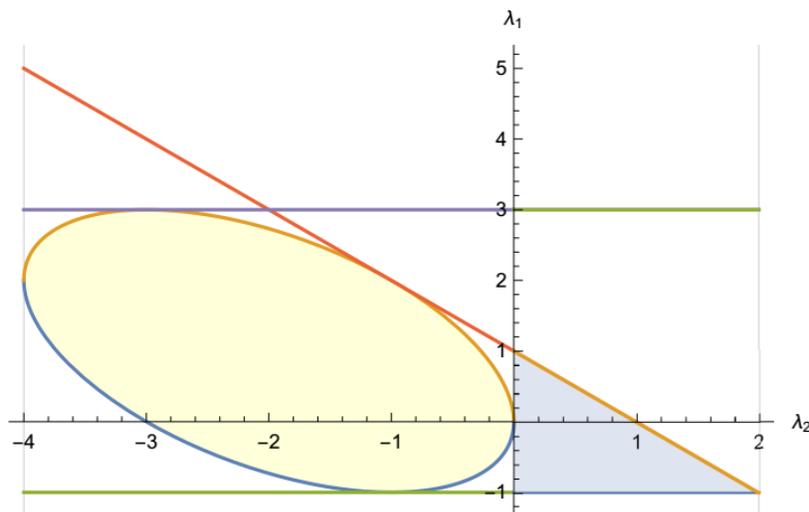
FIGURE 1. Valid parameter set (ellipsoid and triangle on the right-side)

TABLE 1. Some generated distributions according to special cases for the parameter values

λ_2	λ_1	Some Generated Distributions	λ_2	λ_1	Some Generated Distributions
-4	2	$4F^3 - 6F^2 + 3F$	-1	0	$F^3 - F^2 + F$
-3	0	$3F^3 - 3F^2 + F$	-1	1	$F^3 - 2F^2 + 2F$
-3	1	$3F^3 - 4F^2 + 2F$	-1	2	$F^3 - 3F^2 + 3F*$
-3	2	$3F^3 - 5F^2 + 3F$	0	-1	$F^{2}*$
-3	3	$3F^3 - 6F^2 + 4F$	0	0	$F*$
-2	0	$2F^3 - 2F^2 + F$	0	1	$2F - F^{2}*$
-2	1	$2F^3 - 3F^2 + 2F$	1	-1	$2F^2 - F^{3}*$
-2	2	$2F^3 - 4F^2 + 3F$	1	0	$F^2 + F - F^{3}*$
-1	-1	$F^{3}*$	2	-1	$3F^2 - 2F^{3}*$

variable $X_{2:3} = \max\{\min\{X_1, X_2\}, \min\{X_1, X_3\}, \min\{X_2, X_3\}\}$. On the other hand, for $\lambda_1 = -1$, $\lambda_2 = 0$ generated distribution indicates the failure distribution of the lifetime of the two-component parallel system, namely distribution of $X_{2:2} = \max\{X_1, X_2\}$. For $\lambda_1 = 1$, $\lambda_2 = 0$ generated distribution indicates the failure distribution of the lifetime of the two-component series system, namely distribution of $X_{1:2} = \min\{X_1, X_2\}$.

In this case, in addition to the known distributions introduced by the quadratic transmutation, more informative distribution functions occure. However, the set of

the transformation parameters of the proposed cubic transmutation is still complicated.

In order to eliminate of this complexity, by referring to the concept of reliability evaluation of coherent system by using signature (see, [6, 7]), we come up with an idea inspired by both works of [16, 18] as follows:

$$\Pr\left(X_{2:2} \leq t\right) = \Pr\left(\max\left\{X_1, X_2\right\} \leq t\right) = F^2\left(t\right)$$

and

$$\Pr\left(X_{1:2} \leq t\right) = \Pr\left(\min\left\{X_1, X_2\right\} \leq t\right) = 2F\left(t\right) - F^2\left(t\right)$$

where $F\left(t\right)$ indicates the failure distribution of the component lifetime, namely, $\Pr\left(X_1 \leq t\right) = F\left(t\right)$. Hence there exists a stochastic ordering relation such as $X_{1:2} \prec_{st} X \prec_{st} X_{2:2}$. In this case, these three failure distributions can be ordered as $F^2\left(t\right) \leq F\left(t\right) \leq 2F\left(t\right) - F^2\left(t\right)$. From the latter inequality, we can say that $F(t)$ is represented by a convex combination of $2F\left(t\right) - F^2\left(t\right)$ and $F^2\left(t\right)$ where the value of the combination parameter is $\frac{1}{2}$. On the other hand, it is possible to obtain many distributions besides F. Let G stand for the distribution obtained by this convex combination. Then, for $\delta \in [0, 1]$, we have

$$\begin{aligned} G\left(t\right) &= \delta\left(2F\left(t\right) - F^2\left(t\right)\right) + \left(1 - \delta\right)\left(F^2\left(t\right)\right) \\ &= 2\delta F\left(t\right) + \left(1 - 2\delta\right)F^2\left(t\right) \end{aligned} \tag{9}$$

Here, the combination parameter is reparametrized by taking $\delta = \frac{1+\lambda}{2}$ to attain quadratic rank transmutation. Now, the new parameter λ takes the values in $[-1, 1]$. As can be seen immediately, $\lambda = 0$ corresponds to $\delta = \frac{1}{2}$. In eq. (9), substituting δ by λ, we have

$$G\left(t\right) = \left(1 + \lambda\right)F\left(t\right) - \lambda F^2\left(t\right). \tag{10}$$

The above expression is the quadratic rank trasmutation proposed by [16]. Now, we concentrate on 3-component systems with similar thinking. Let X_1, X_2 and X_3 be independent random variables distributed as F. Let $X_{r:3}$ denote rth order statistic of (X_1, X_2, X_3) with corresponding distribution $F_{r:3}$. Then we have

$$F_{3:3}\left(t\right) = \Pr\left(X_{3:3} \leq t\right) = \Pr\left(\max\left\{X_1, X_2, X_3\right\} \leq t\right) = F^3\left(t\right) \tag{11}$$

$$\begin{aligned} F_{2:3}\left(t\right) &= \Pr\left(X_{2:3} \leq t\right) \\ &= \Pr\left(\max\left\{\min\left\{X_1, X_2\right\}, \min\left\{X_1, X_3\right\}, \min\left\{X_2, X_3\right\}\right\} \leq t\right) \\ &= 3F^2\left(t\right) - 2F^3\left(t\right) \end{aligned} \tag{12}$$

$$\begin{aligned} F_{1:3}\left(t\right) &= \Pr\left(X_{1:3} \leq t\right) = \Pr\left(\min\left\{X_1, X_2, X_3\right\} \leq t\right) \\ &= 3F\left(t\right) - 3F^2\left(t\right) + F^3\left(t\right). \end{aligned} \tag{13}$$

According to [18], the properties $F_{3:3} \leq F_{2:3} \leq F_{1:3}$ and $F = \frac{1}{3}F_{3:3} + \frac{1}{3}F_{2:3} + \frac{1}{3}F_{1:3}$. are hold. In other words, F can be represented by a convex combination of $F_{1:3}$, $F_{2:3}$ and $F_{3:3}$. On the other hand, there is also an ordering for F such

that $F_{3:3} \leq F \leq F_{1:3}$. If $F_{2:3}$ is also included in this ordering, we have for $F \geq \frac{1}{2}$, $F_{3:3} \leq F \leq F_{2:3} \leq F_{1:3}$ and for $F < \frac{1}{2}$, $F_{3:3} \leq F_{2:3} \leq F \leq F_{1:3}$. Hence, we can suggest a convex combination to cover both ordering situations. Our aim is to determine exactly where F is. In this case, we can write the following convex combination obtained by $F_{1:3}$ and $F_{2:3}$, called as G^*.

$$G^* = \delta_1 F_{1:3} + (1 - \delta_1) F_{2:3} \tag{14}$$

where $\delta_1 \in [0, 1]$. Now, let's write a convex combination between G^* and $F_{3:3}$. Denoting this convex combination by G, we have

$$G = \delta_2 G^* + (1 - \delta_2) F_{3:3} \tag{15}$$

where $\delta_2 \in [0, 1]$. Combining with the equations (14) and (15), we obtain G as

$$G = \delta_2 \delta_1 F_{1:3} + \delta_2 (1 - \delta_1) F_{2:3} + (1 - \delta_2) F_{3:3}. \tag{16}$$

If the notation F is used for the representation of $F_{r:3}$, and rearranging with respect to polynomial degree of F, the following expression is obtained:

$$\begin{aligned} G &= \delta_2 \delta_1 \left(3F - 3F^2 + F^3 \right) + \delta_2 (1 - \delta_1) \left(3F^2 - 2F^3 \right) + (1 - \delta_2) F^3 \\ &= 3\delta_1 \delta_2 F + 3\delta_2 (1 - 2\delta_1) F^2 + (1 - 3\delta_2 + 3\delta_1 \delta_2) F^3 \end{aligned} \tag{17}$$

Undoubtedly, G is a distribution function. However, reparameterization is made on the model in order to achieve the similar structure of the quadratic rank transmutation. Now, by taking $w_1 = \delta_1 \delta_2$ and $w_2 = \delta_2 - \delta_1 \delta_2$, eq. (17) can be rewritten as follows:

$$G = 3w_1 F + 3 (w_2 - w_1) F^2 + (1 - 3w_2) F^3 \tag{18}$$

where $w_1, w_2 \in [0, 1]$. In eq. (18), by the reparametrizating as $w_1 = \frac{1 + \lambda_1}{3}$ and $w_2 = \frac{1 + \lambda_2}{3}$, we have

$$G = (1 + \lambda_1) F + (\lambda_2 - \lambda_1) F^2 - \lambda_2 F^3 \tag{19}$$

where $\lambda_1, \lambda_2 \in [-1, 2]$. Since $\delta_2 = w_1 + w_2$, the parameter set is also constrained by the condition $\lambda_1 + \lambda_2 \leq 1$. Consequently, the parameter set of λ_1 and λ_2 is presented in a simpler form than the parameter region given in Figure 1. This transmutation defined in eq. (19) is called as cubic rank transmutation and transformed distribution G is named as CRT-F.

As can be seen immediately, CRT-F defines a quadratic rank transmuted distribution at $\lambda_2 = 0$, and $\lambda_1 = \lambda_2 = 0$ gives the baseline distribution F. For this reason, CRT-F can be seen as a generalized form of QRT. The parameter set of λ_1 and λ_2, which is defined as $\{(\lambda_1, \lambda_2) : \lambda_1, \lambda_2 \in [-1, 2], \lambda_1 + \lambda_2 \leq 1\}$ can be figure out in Figure 2. Now, referring to the integer values of λ_1 and λ_2, we can determine the generated distribution functions by the Table 2.

Identifications given in Table 2 show that *Table 1* of [17] is included by CRT-F according to special choices of transmutation parameters.
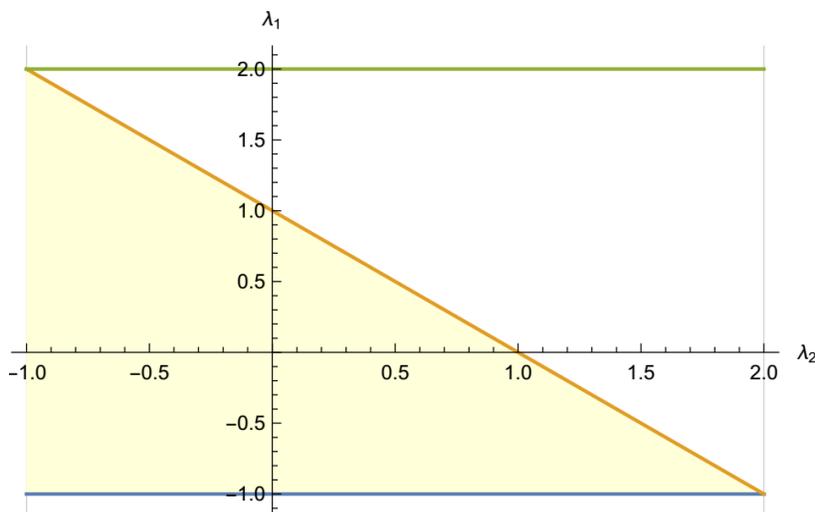
FIGURE 2. Region of the parameter set of CRT-F

TABLE 2. Identifications of CRT-F distribution for special values of transmutation parameters

λ_1	λ_2	CRT-F	Identification
-1	-1	F^3	Distribution of $T_{3:3}$
-1	0	F^2	Distribution of $T_{2:2}$
-1	1	$2F^2 - F^3$	Distribution of $\max\{X_1, \min\{X_2, X_3\}\}$
-1	2	$3F^2 - 2F^3$	Distribution of $T_{2:3}$
0	-1	$F^3 - F^2 + F$	$\frac{1}{3}F_{1:3} + \frac{2}{3}F_{3:3}$
0	0	F	Baseline Distribution
0	1	$F^2 + F - F^3$	Distribution of $\min\{X_1, \max\{X_2, X_3\}\}$
1	-1	$F^3 - 2F^2 + 2F$	$\frac{2}{3}F_{1:3} + \frac{1}{3}F_{3:3}$
1	0	$2F - F^2$	Distribution of $T_{1:2}$
2	-1	$F^3 - 3F^2 + 3F$	Distribution of $T_{1:3}$

Note that, by taking into account the parameter set of (19), the distribution family CRT-F is different as compared with the families proposed by [8, 14].

[8] proposed a cubic rank transmuted distribution family motivated by a study of [17]. The paper contained one theorem (referred to as Theorem 2.1), deriving cubic transmuted distribution. Here, We would like to point out that the result of Theorem 2.1 can be reduced to an explicit and understandable form.

Parameter space of eq. (3) of Theorem 2.1 given by [8] needs to be revised according to mixture probabilities π_i $(i = 1, 2, 3)$. Otherwise, eq. (3) does not provide the distribution function in some cases of λ_1 and λ_2. For instance, suppose that $\lambda_1 = 0$ and $\lambda_2 = -1$ in eq. (3), then we have

$$F(x) = -G^2(x) + 2G^3(x) = G^2(x)(2G(x) - 1).$$

Thus, the function $F(x)$ is positively signed when x is greater than $G^{-1}(1/2)$ which is the median of $G(x)$. Otherwise, $F(x)$ is negatively signed. More generally, according to eq. (3), we have the second-order convex polynomial as $\varphi(u) = \lambda_1 + (\lambda_2 - \lambda_1) u + (1 - \lambda_2) u^2$. After some algebraic manipulation, we say that having positively sign of $\varphi(u)$ depends on $\lambda_1 \geq \frac{(\lambda_1 + \lambda_2)^2}{4}$. Furthermore, in proof of Theorem 2.1, one condition is missing which comes with $0 \leq \pi_1 + \pi_2 \leq 1$. To clarify the above claims, recall the distribution of order statistics associated with sample size of 3:

$$\Pr(X_{1:3} \leq x) = 1 - \Pr(X_{1:3} > x) = 1 - (1 - F(x))^3 = 3F(x) - 3F^2(x) + F^3(x),$$

$$\Pr(X_{2:3} \leq x) = 3F(x)^2 - 2F^3(x),$$

$$\Pr(X_{3:3} \leq x) = F^3(x).$$

Now, a new random variable T is defined by mixing the above order statistics as follows:

$$T = \begin{cases} X_{1:3}, & \text{with probability } \pi_1 \\ X_{2:3}, & \text{with probability } \pi_2 \\ X_{3:3}, & \text{with probability } \pi_3, \end{cases}$$

where $\pi_1 + \pi_2 + \pi_3 = 1$. Then the distribution of T is as follows:

$$\Pr(T \leq t) = \pi_1 \left(3F(t) - 3F^2(t) + F^3(t) \right) + \pi_2 \left(3F^2(t) - 2F^3(t) \right) + \pi_3 \left(F^3(t) \right)$$

$$= (3\pi_1) F(t) + 3(\pi_2 - \pi_1) F^2(t) + (\pi_1 - 2\pi_2 + \pi_3) F^3(t).$$

By letting $\pi_3 = 1 - \pi_1 - \pi_2$ then we have

$$\Pr(T \leq t) = 3\pi_1 F(t) + 3(\pi_2 - \pi_1) F^2(t) + (1 - 3\pi_2) F^3(t).$$

Since $\pi_i \in [0, 1]$, $(i = 1, 2, 3)$, appropriate parameterization for π_1 and π_2, can be taken into account as $3\pi_1 = 1 + \lambda_1$, and $3\pi_2 = 1 + \lambda_2$. New parameters are both in the interval $[-1, 2]$. Recalling the condition $0 \leq \pi_1 + \pi_2 \leq 1$, hence we have $-2 \leq \lambda_1 + \lambda_2 \leq 1$. Accordingly, latter probability is as follows:

$$\Pr(T \leq t) = (1 + \lambda_1) F(t) + (\lambda_2 - \lambda_1) F^2(t) - \lambda_2 F^3(t),$$

with $\lambda_1, \lambda_2 \in [-1, 2]$ and $-2 \leq \lambda_1 + \lambda_2 \leq 1$. It should be noted that this is also obtained by (19).

Therefore, based on the assumptions in proof of Theorem 2.1., eq. (3) cannot be obtained. Also, comparing with the family introduced by [14], they give the similar cubic transmuted distribution family with a narrower parameter space as $\lambda_1, \lambda_2 \in [-1, 1]$ and $-2 \leq \lambda_1 + \lambda_2 \leq 1$.

3. Conclusion

In this article, we propose a new version of polynomial rank transmutation. Since the parameter set is still complex, a new cubic rank transmutation is introduced in the light of the idea behind QRTM technique. Compared to the two techniques in the literature, it is seen that the proposed technique covers them in terms of parameter space.

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