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Some New Inequalities for (α, m_1, m_2)-GA Convex Functions

Mahir KADAKAL[*1]

Abstract

In this manuscript, firstly we introduce and study the concept of (α, m_1, m_2)-Geometric-Arithmetically (GA) convex functions and some algebraic properties of such type functions. Then, we obtain Hermite-Hadamard type integral inequalities for the newly introduced class of functions by using an identity together with Hölder integral inequality, power-mean integral inequality and Hölder-İşcan integral inequality giving a better approach than Hölder integral inequality. Inequalities have been obtained with the help of Gamma function. In addition, results were obtained according to the special cases of α, m_1 and m_2.

Keywords: (α, m_1, m_2)-GA convex function, Hölder integral inequality, power-mean inequality, Hölder-İşcan inequality, Hermite-Hadamard integral inequality.

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1. INTRODUCTION

Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

hold. Both inequalities hold in the reversed direction if the function f is concave [4, 6]. The above inequalities were firstly discovered by the famous scientist Charles Hermite. This double inequality is well-known in the literature as Hermite-Hadamard integral inequality for convex functions. This inequality gives us upper and lower bounds for the integral mean-value of a convex function. Some of the classical inequalities for means can be derived from Hermite-Hadamard inequality for appropriate particular selections of the function f.

Convexity theory plays an important role in mathematics and many other sciences. It provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. Readers can find more information in the recent studies [1, 5, 8, 10, 11, 15, 19, 20, 23, 24, 25] and the references therein for different convex classes and related Hermite-Hadamard integral inequalities.

Definition 1. ([17,18]) *A function $f: I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is said to be GA-convex function on I if the inequality*

$$f\left(x^\lambda y^{1-\lambda}\right) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$, where $x^\lambda y^{1-\lambda}$ and $\lambda f(x) + (1-\lambda)f(y)$ are respectively the weighted geometric mean of two positive numbers x and y and the weighted arithmetic mean of $f(x)$ and $f(y)$.

Definition 2. ([22]) *A function $f: [0,b] \to \mathbb{R}$ is said to be m-convex for $m \in (0,1]$ if the inequality*

$$f(\alpha x + m(1-\alpha)y) \leq \alpha f(x) + m(1-\alpha)f(y)$$

holds for all $x, y \in [0,b]$ and $\alpha \in [0,1]$.

Definition 3. ([12]) *The function $f: [0,b] \to \mathbb{R}$, $b > 0$, is said to be (m_1, m_2)-convex, if the inequality*

$$f(m_1 tx + m_2(1-t)y) \leq m_1 t f(x) + m_2(1-t)f(y)$$

holds for all $x, y \in I$, $t \in [0,1]$ and $(m_1, m_2) \in (0,1]^2$.

Definition 4. ([13]) *$f: [0,b] \to \mathbb{R}$, $b > 0$, is said to be (α, m_1, m_2)-convex function, if the inequality*

$$f(m_1 tx + m_2(1-t)y) \leq m_1 t^\alpha f(x) + m_2(1-t^\alpha)f(y)$$

holds for all $x, y \in I$, $t \in [0,1]$ and $(\alpha, m_1, m_2) \in (0,1]^3$.

Definition 5. ([16]) *For $f: [0,b] \to \mathbb{R}$ and $(\alpha, m) \in (0,1]^2$, if*

$$f(tx + (1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

is valid for all $x, y \in [0,b]$ and $t \in [0,1]$, then we say that $f(x)$ is an (α, m)-convex function on $[0,b]$.

Definition 6. ([17]) *The GG-convex functions (called in what follows multiplicatively convex functions) are those functions $f: I \to J$ (acting on subintervals of $(0,\infty)$) such that*

$$x, y \in I \text{ and } \lambda <\in [0,1] \Longrightarrow f(x^{1-t}y^t) \leq f(x)^{1-\lambda}f(y)^\lambda$$

i.e., it is called log-convexity and it is different from the above.

Definition 7. ([9]) *Let the function $f: [0,b] \to \mathbb{R}$ and $(\alpha, m) \in [0,1]^2$. If*

$$f\left(x^t y^{m(1-t)}\right) \leq t^\alpha f(a) + m(1-t^\alpha)f(b). \quad (1.1)$$

for all $[a,b] \in [0,b]$ and $t \in [0,1]$, then $f(x)$ is said to be (α, m)-geometric arithmetically convex function or, simply speaking, an (α, m)-GA-convex function. If (1.1) reversed, then $f(x)$ is *said to be (α, m)-geometric arithmetically concave function or, simply speaking, an (α, m)-GA-concave function.*

A refinement of Hölder integral inequality better approach than Hölder integral inequality can be given as follows:

Theorem 1. (Hölder-İşcan integral inequality [7]) *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on $[a, b]$ and if $|f|^p$, $|g|^q$ are integrable functions on the interval $[a, b]$ then*

$$\int_a^b |f(x)g(x)|dx$$

$$\leq \frac{1}{b-a}\left\{\left(\int_a^b (b-x)|f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b (b-x)|g(x)|^q dx\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\int_a^b (x-a)|f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b (x-a)|g(x)|^q dx\right)^{\frac{1}{q}}\right\}.$$

Definition 8. (Gamma function) *The classic gamma function is usually defined for $Rez > 0$ by*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

The main purpose of this paper is to introduce the concept of (α, m_1, m_2)-geometric arithmetically (GA) convex functions and establish some results connected with new inequalities similar to the Hermite-Hadamard integral inequality for these classes of functions.

2. MAIN RESULTS FOR (α, m_1, m_2)-GA CONVEX FUNCTIONS

In this section, we introduce a new concept, which is called (α, m_1, m_2)-GA convex functions and we give by setting some algebraic properties for the (α, m_1, m_2)-GA convex functions, as follows:

Definition 9. *Let the function $f: [0, b] \to \mathbb{R}$ and $(\alpha, m_1, m_2) \in (0,1]^3$. If*

$$f\left(a^{m_1 t} b^{m_2(1-t)}\right) \leq m_1 t^\alpha f(a) + m_2(1 - t^\alpha) f(b) \quad (2.1)$$

for all $[a, b] \in [0, b]$ and $t \in [0,1]$, then the function f is said to be (α, m_1, m_2)-geometric arithmetically convex function, if the inequality (2.1) reversed, then the function f is said to be (α, m_1, m_2)-geometric arithmetically concave function.

Example 1. $f(x) = c, c < 0$ is a (α, m_1, m_2)-*geometric arithmetically convex function.*

We discuss some connections between the class of the (α, m_1, m_2)-GA convex functions and other classes of generalized convex functions.

Remark 1. *When $m_1 = m_2 = \alpha = 1$, the (α, m_1, m_2)-geometric arithmetically convex (concave) function becomes a geometric arithmetically convex (concave) function defined in [17, 18].*

Remark 2. *When $m_1 = 1$, $m_2 = m$, the (α, m_1, m_2)-geometric arithmetically convex (concave) function becomes an (α, m)-geometric arithmetically convex (concave) function defined in [9].*

Remark 3. *When $m_1 = m_2 = 1$ and $\alpha = s$, the (α, m_1, m_2)-geometric arithmetically convex (concave) function becomes a geometric arithmetically-s convex (concave) function defined in [14].*

Remark 4. *When $\alpha = 1$, the (α, m_1, m_2)-geometric arithmetically convex (concave) function becomes a (m_1, m_2)-GA convex (concave) function defined in [21].*

Proposition 1. *The function $f: I \subset (0, \infty) \to \mathbb{R}$ is (α, m_1, m_2)-GA convex function on I if and only if $f \circ exp: lnI \to \mathbb{R}$ is (α, m_1, m_2)-convex function on the interval $lnI = \{lnx | x \in I\}$.*

Proof. ($\Rightarrow$) Let $f: I \subset (0, \infty) \to \mathbb{R}$ (α, m_1, m_2)-GA convex function. Then, we write

$$(f \circ exp)(m_1 t lna + m_2(1-t) lnb)$$

$\leq m_1t^\alpha(f\circ exp)(lna)+m_2(1-t^\alpha)(f\circ exp)(lnb).$

From here, we get

$$f\left(a^{m_1t}b^{m_2(1-t)}\right)\leq m_1t^\alpha f(a)+m_2(1-t^\alpha)f(b).$$

Hence, the function $f\circ\exp$ is (α,m_1,m_2)-convex function on the interval lnI.

($\Leftarrow$) Let $f\circ\exp:\ln I\to\mathbb{R}$, (α,m_1,m_2)-convex function on the interval lnI. Then, we obtain

$$f\left(a^{m_1t}b^{m_2(1-t)}\right)=f\left(e^{m_1tlna+\quad_2(1-t)lnb}\right)$$

$$=(f\circ exp)(m_1tlna+m_2(1-t)lnb)$$

$$\leq m_1t^\alpha f(e^{lna})+m_2(1-t^\alpha)f(e^{lnb})$$

$$=m_1t^\alpha f(a)+m_2(1-t^\alpha)f(b),$$

which means that the function $f(x)$ (α,m_1,m_2)-GA convex function on I.

Theorem 2. *Let $f,g:I\subset\mathbb{R}\to\mathbb{R}$. If f and g are (α,m_1,m_2)-GA convex functions, then $f+g$ is an (α,m_1,m_2)-GA convex function and cf is an (α,m_1,m_2)-GA convex function for $c\in\mathbb{R}_+$.*

Proof. Let f,g be (α,m_1,m_2)-GA convex functions, then

$$(f+g)\left(a^{m_1t}b^{m_2(1-t)}\right)$$

$$=f\left(a^{m_1t}b^{m_2(1-t)}\right)+g\left(a^{m_1t}b^{m_2(1-t)}\right)$$

$$\leq m_1t^\alpha f(a)+m_2(1-t^\alpha)f(b)$$

$$+m_1t^\alpha g(a)+m_2(1-t^\alpha)g(b)$$

$$=m_1t^\alpha(f+g)(a)+m_2(1-t^\alpha)(f+g)(b)$$

Let f be (α,m_1,m_2)-GA convex function and $c\in\mathbb{R}$*(*$c\geq 0$*)*, then

$$(cf)\left(a^{m_1t}b^{m_2(1-t)}\right)$$

$$\leq c[m_1t^\alpha f(x)+m_2(1-t^\alpha)f(y)]$$

$$=m_1t^\alpha(cf)(x)+m_2(1-t^\alpha)(cf)(y).$$

This completes the proof of the theorem.

Theorem 3. *If $f:I\to J$ is a (m_1,m_2)-GG convex and $g:J\to\mathbb{R}$ is a (α,m_1,m_2)-GA convex function and nondecreasing, then $g\circ f:I\to\mathbb{R}$ is a (α,m_1,m_2)-GA convex function.*

Proof. For $a,b\in I$ and $t\in[0,1]$, we get

$$(g\circ f)\left(a^{m_1t}b^{m_2(1-t)}\right)$$

$$=g\left(f\left(a^{m_1t}b^{m_2(1-t)}\right)\right)$$

$$\leq g\left([f(a)]^{m_1t}[f(b)]^{m_2(1-t)}\right)$$

$$\leq m_1t^\alpha g(f(x))+m_2(1-t^\alpha)g(f(y)).$$

This completes the proof of the theorem.

Theorem 4. *Let $b>0$ and $f_\beta:[a,b]\to\mathbb{R}$ be an arbitrary family of (α,m_1,m_2)-GA convex functions and let $f(x)=sup_\beta f_\beta(x)$. If $J=\{u\in[a,b]:f(u)<\infty\}$ is nonempty, then J is an interval and f is an (α,m_1,m_2)-GA convex function on J.*

Proof. Let $t\in[0,1]$ and $x,y\in J$ be arbitrary. Then

$$f\left(a^{m_1t}b^{m_2(1-t)}\right)$$

$$=\sup_\beta f_\beta\left(a^{m_1t}b^{m_2(1-t)}\right)$$

$$\leq\sup_\beta\left[m_1t^\alpha f_\alpha(x)+m_2(1-t^\alpha)f_\beta(y)\right]$$

$$\leq m_1t^\alpha\sup_\beta f_\beta(x)+m_2(1-t^\alpha)\sup_\beta f_\beta(y)$$

$$=m_1t^\alpha f(x)+m_2(1-t^\alpha)f(y)<\infty.$$

This shows simultaneously that J is an interval since it contains every point between any two of its points, and that f is an (α,m_1,m_2)-GA convex function on J. This completes the proof of the theorem.

Theorem 5. *If the function $f:[a,b]\to\mathbb{R}$ is an (α,m_1,m_2)-GA convex function then f is bounded on the interval $[a,b]$.*

Proof. Let $K=\max\{m_1f(a),m_2f(b)\}$ and $x\in[a,b]$ is an arbitrary point. Then there exists a $t\in[0,1]$ such that $x=a^{m_1t}b^{m_2(1-t)}$. Thus, since $m_1t^{\alpha}\le 1$ and $m_2(1-t^{\alpha})\le 1$ we have

$$f(x)=f\left(a^{m_1t}b^{m_2(1-t)}\right)$$

$$\le m_1t^{\alpha}f(a)+m_2(1-t^{\alpha})f(b)\le 2K=M.$$

Also, for every $x\in[a^{m_1},b^{m_2}]$ there exists a $\lambda\in\left[\sqrt{\frac{a^{m_1}}{b^{m_2}}},1\right]$ such that $x=\lambda\sqrt{a^{m_1}b^{m_2}}$ and $x=\frac{\sqrt{a^{m_1}b^{m_2}}}{\lambda}$. Without loss of generality we can suppose $x=\lambda\sqrt{a^{m_1}b^{m_2}}$. So, we have

$$f\left(\sqrt{a^{m_1}b^{m_2}}\right)$$

$$=f\left(\sqrt{\left[\lambda\sqrt{a^{m_1}b^{m_2}}\right]\left[\frac{\sqrt{a^{m_1}b^{m_2}}}{\lambda}\right]}\right)$$

$$\le\sqrt{f(x)f\left(\frac{\sqrt{a^{m_1}b^{m_2}}}{\lambda}\right)}.$$

By using M as the upper bound, we obtain

$$f(x)\ge\frac{f^2\left(\sqrt{a^{m_1}b^{m_2}}\right)}{f\left(\frac{\sqrt{a^{m_1}b^{m_2}}}{\lambda}\right)}\ge\frac{f^2\left(\sqrt{a^{m_1}b^{m_2}}\right)}{M}=m.$$

This completes the proof of the theorem.

3. HERMITE-HADAMARD INEQUALITY FOR (α,m_1,m_2)-GA CONVEX FUNCTION

This section aims to establish some inequalities of Hermite-Hadamard type integral inequalities for (α,m_1,m_2)-GA convex functions. In this section, we will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on the interval $[a,b]$.

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be an (α,m_1,m_2)-GA convex function. If $a<b$ and $f\in L[a,b]$, then the following Hermite-Hadamard type integral inequalities hold:*

$$f\left(\sqrt{a^{m_1}b^{m_2}}\right)\le\frac{1}{lnb^{m_2}-lna^{m_1}}\int_{a^{m_1}}^{b^{m_2}}\frac{f(u)}{u}du$$

$$\le\frac{m_1f(a)}{\alpha+1}+\frac{\alpha m_2f(b)}{\alpha+1}. \tag{3.1}$$

Proof. Firstly, from the property of the (α,m_1,m_2)-GA convex function of f, we can write

$$f\left(\sqrt{a^{m_1}b^{m_2}}\right)=f\left(\sqrt{a^{m_1t}b^{m_2(1-t)}a^{m_1(1-t)}b^{m_2t}}\right)$$

$$\le\frac{f\left(a^{m_1t}b^{m_2(1-t)}\right)+f\left(a^{m_1(1-t)}b^{m_2t}\right)}{2}.$$

Now, if we take integral in the last inequality with respect to $t\in[0,1]$, we deduce that

$$f\left(\sqrt{a^{m_1}b^{m_2}}\right)$$

$$\le\frac{1}{2}\int_0^1 f\left(a^{m_1t}b^{m_2(1-t)}\right)dt+\frac{1}{2}\left(a^{m_1(1-t)}b^{m_2t}\right)dt$$

$$=\frac{1}{2}\frac{1}{ln\ ^{m_2}-lna^{m_1}}\int_{a^{m_1}}^{b^{m_2}}\frac{f(u)}{u}du$$

$$+\frac{1}{2}\frac{1}{lnb^{m_2}-lna^{m_1}}\int_{a^{m_1}}^{b^{m_2}}\frac{f(u)}{u}du$$

$$=\frac{1}{lnb^{m_2}-lna^{m_1}}\int_{a^{m_1}}^{b^{m_2}}\frac{f(u)}{u}du.$$

Secondly, by using the property of the (α,m_1,m_2)-GA convex function of f, if the variable is changed as $u=a^{m_1t}b^{m_2(1-t)}$, then

$$\frac{1}{lnb^{m_2}-lna^{m_1}}\int_{a^{m_1}}^{b^{m_2}}\frac{f(u)}{u}du$$

$$=\int_0^1 f\left(a^{m_1t}b^{m_2(1-t)}\right)dt$$

$$\le\int_0^1\left[m_1t^{\alpha}f(a)+m_2(1-t^{\alpha})f(b)\right]dt$$

$$=m_1f(a)\int_0^1 t^{\alpha}dt+m_2f(b)\int_0^1(1-t^{\alpha})dt$$

$$=\frac{m_1f(a)}{\alpha+1}+\frac{\alpha m_2f(b)}{\alpha+1}$$

This completes the proof of the theorem.

Corollary 1. *By considering the conditions of Theorem 6, if we take* $m_1 = m_2 = 1$ *and* $\alpha = 1$ *in the inequality (3.1), then we get*

$$f(\sqrt{ab}) \leq \frac{1}{lnb-lna}\int_a^b \frac{f(u)}{u}du \leq \frac{f(a)+f(b)}{2}.$$

This inequality coincides with the inequality in [2].

Corollary 2. *By considering the conditions of Theorem 6, if we take* $\alpha = 1$ *in the inequality (3.1), then we get*

$$f\left(\sqrt{a^{m_1}b^{m_2}}\right) \leq \frac{1}{lnb^{m_2}-ln\ \ ^{m_1}}\int_{a^{m_1}}^{b^{m_2}} \frac{f(u)}{u}du$$

$$\leq \frac{m_1 f(a)+m_2 f(b)}{2}.$$

This inequality coincides with the inequality in [14].

4. SOME NEW INEQUALITIES FOR (α, m_1, m_2)-GA CONVEX FUNCTIONS

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard integral inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is (α, m_1, m_2)-GA convex function. Ji et al. [9] used the following lemma. Also, we will use this lemma to obtain our results.

Lemma 1. ([3]) *Let* $f: I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ *be differentiable function and* $a, b \in I$ *with* $a < b$. *If* $f' \in L([a,b])$, *then*

$$\frac{b^2 f(a)-a^2 f(b)}{2} - \int_a^b xf(x)dx$$

$$= \frac{lnb-lna}{2}\int_0^1 a^{3(1-t)}b^{3t}f'(a^{1-t}b^t)dt.$$

Theorem 7. *Let the function* $f: \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ *be a differentiable function and* $f' \in L[a,b]$ *for* $0 < a < b < \infty$. *If* $|f'|$ *is* (α, m_1, m_2)*-GA convex on* $\left[0, max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$ *for* $[\alpha, m_1, m_2] \in (0,1]^3$, *then the following integral inequalities hold*

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2} - \int_a^b xf(x)dx\right| \tag{4.1}$$

$$\leq \frac{m_1}{2}\left|f'\left(a^{\frac{1}{m_1}}\right)\right|$$

$$\left[\frac{b^3-a^3}{3} - \frac{a^3\Gamma(\alpha+1,3(lna-lnb))-a^3\Gamma(\alpha+1,0)}{3^{\alpha+1}(lna-lnb)^\alpha}\right]$$

$$+\frac{m_2}{2}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\left[\frac{a^3\Gamma(\alpha+1,3(lna-lnb))-a^3\Gamma(\alpha+1,0)}{3^{\alpha+1}(lna-lnb)^\alpha}\right],$$

where Γ is the Gamma function.

Proof. By using Lemma 1 and the inequality

$$|f'(a^{1-t}b^t)| = \left|f'\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)} f'\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right|$$

$$\leq m_1(1-t^\alpha)\left|f'\left(a^{\frac{1}{m_1}}\right)\right| + m_2 t^\alpha\left|f'\left(b^{\frac{1}{m_2}}\right)\right|,$$

we get

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2} - \int_a^b xf(x)dx\right|$$

$$\leq \frac{ln(b/a)}{2}\int_0^1 \left|a^{3(1-t)}b^{3t}\right||f'(a^{1-t}b^t)|dt$$

$$\leq \frac{ln(b/a)}{2}\int_0^1 a^{3(1-t)}b^{3t}\begin{bmatrix} m_1(1-t^\alpha)\left|f'\left(a^{\frac{1}{m_1}}\right)\right| \\ +m_2 t^\alpha\left|f'\left(b^{\frac{1}{m_2}}\right)\right| \end{bmatrix}dt$$

$$= m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|\frac{ln(b/a)}{2}\int_0^1 (1-t^\alpha)a^{3(1-t)}b^{3t}dt$$

$$+m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\frac{ln(b/a)}{2}\int_0^1 t^\alpha a^{3(1-t)}b^{3t}dt$$

$$= \frac{m_1}{2}\left|f'\left(a^{\frac{1}{m_1}}\right)\right|\left[\frac{b^3-a^3}{3} - \frac{a^3\Gamma(\alpha+1,3(lna-lnb))-a^3\Gamma(\alpha+1,\ 0)}{3^{\alpha+1}(lna-lnb)^\alpha}\right]$$

$$+\frac{m_2}{2}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\left[\frac{a^3\Gamma(\alpha+1,\ 3(lna-l\quad))-a^3\Gamma(\alpha+1,\ 0)}{3^{\alpha+1}(lna-lnb)^\alpha}\right].$$

This completes the proof of the theorem.

Corollary 3. *By considering the conditions of Theorem 7, if we take $m_1 = m_2 = 1$ and $\alpha = 1$ then we get*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right|$$

$$\leq \frac{|f'(a)|}{6}[L(a^3, b^3) - a^3] + \frac{|f'(b)|}{6}[b^3 - L(a^3, b^3)],$$

where L is the logarithmic mean.

Corollary 4. *By considering the conditions of Theorem 7, if we take $\alpha = 1$ in the inequality (4.1), then we get*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right|$$

$$\leq \frac{m_1}{2}\left|f'\left(a^{\frac{1}{m_1}}\right)\right| [L(a^3, b^3) - a^3]$$

$$+ \frac{m_2}{2}\left|f'\left(b^{\frac{1}{m_2}}\right)\right| [b^3 - L(a^3, b^3)].$$

Theorem 8. *Let the function $f: \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be a differentiable function and $f' \in L[a, b]$ for $0 < a < b < \infty$. If $|f'|^q$ is (α, m_1, m_2)-GA convex on $\left[0, max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$ for $[\alpha, m_1, m_2] \in (0,1]^3$ and $q \geq 1$ then*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \tag{4.2}$$

$$\leq \frac{lnb - lna}{2} L^{1-\frac{1}{q}}(a^3, b^3)$$

$$\cdot\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q \left(\frac{b^3 - a^3}{3(lnb - lna)} - \frac{a^3\Gamma(\alpha+1,\ 3(ln\qquad)) - a^3\Gamma(\alpha+1,\ 0)}{3^{\alpha+1}(lnb - l\quad)(lna - lnb)^{\alpha}}\right)\right.$$

$$\left.+ m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q \left(\frac{a^3\Gamma(\alpha+1,\ 3(lna - l\quad)) - a^3\Gamma(\alpha+1,\ 0)}{3^{\alpha+1}(lnb - lna)(lna - lnb)^{\alpha}}\right)\right]^{\frac{1}{q}},$$

where L is the logarithmic mean.

Proof. By using both Lemma 1, power-mean inequality and the (α, m_1, m_2)-GA convexity of $|f'|^q$ on the interval $\left[0, max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$, that is, the inequality

$$|f'(a^{1-t}b^t)| = \left|f'\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)} f'\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right|^q$$

$$\leq m_1(1 - t^{\alpha})\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q + m_2 t^{\alpha}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q,$$

is satisfied and we get

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right|$$

$$\leq \frac{ln\left(\frac{b}{a}\right)}{2}\left[\int_0^1 a^{3(1-t)}b^{3t}dt\right]^{1-\frac{1}{q}}$$

$$\left[\int_0^1 a^{3(1-t)}b^{3t}\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right|^q dt\right]^{\frac{1}{q}}$$

$$\leq \frac{ln\left(\frac{b}{a}\right)}{2}\left[\int_0^1 a^{3(1-t)}b^{3t}dt\right]^{1-\frac{1}{q}}$$

$$\cdot\left(\int_0^1 a^{3(1-t)}b^{3t}\begin{bmatrix} m_1(1 - t^{\alpha})\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q \\ +m_2 t^{\alpha}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q \end{bmatrix}dt\right)^{\frac{1}{q}}$$

$$= \frac{ln\left(\frac{b}{a}\right)}{2}\left[\int_0^1 a^{3(1-t)}b^{3t}dt\right]^{1-\frac{1}{q}}$$

$$\times\begin{bmatrix} m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q \int_0^1 (1 - t^{\alpha})a^{3(1-t)}b^{3t}dt \\ +m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q \int_0^1 t^{\alpha}a^{3(1-t)}b^{3t}dt \end{bmatrix}^{\frac{1}{q}}$$

$$= \frac{lnb - l}{2} L^{1-\frac{1}{q}}(a^3, b^3)$$

$$\cdot\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q \left(\frac{b^3 - a^3}{3(lnb - lna)} - \frac{a^3\Gamma(\alpha+1,\ 3(lna - lnb)) - a^3\Gamma(\alpha+1,\ 0)}{3^{\alpha+1}(lnb - lna)(lna - lnb)^{\alpha}}\right)\right.$$

$$\left.+m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q \left(\frac{a^3\Gamma(\alpha+1,\ 3(lna - lnb)) - a^3\Gamma(\alpha+1,\ 0)}{3^{\alpha+1}(lnb - lna)(lna - lnb)^{\alpha}}\right)\right]^{\frac{1}{q}}.$$

This completes the proof of the theorem.

Corollary 5. *By considering the conditions of Theorem 8, if we take $m_1 = m_2 = 1$ and $\alpha = 1$ in the inequality (4.2), then we get*

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2} - \int_a^b xf(x)dx\right| \le \frac{lnb-lna}{2} L^{1-\frac{1}{q}}(a^3,b^3)$$

$$\times\left[|f'(a)|^q \frac{L(a^3,b^3)-b^3}{3(lnb-lna)} + |f'(b)|^q \frac{b^3-L(a^3,b^3)}{3(lnb-lna)}\right]^{\frac{1}{q}},$$

where L is the logarithmic mean.

Corollary 6. *By considering the conditions of Theorem 8, if we take $q = 1$, then*

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2} - \int_a^b xf(x)dx\right| \le$$

$$\times\left[\frac{m_1}{2}\left|f'\left(a^{\frac{1}{m_1}}\right)\right|\left(\frac{b^3-a^3}{3} - \frac{a^3\Gamma(\alpha+1,3(lna-lnb))-a^3\Gamma(\alpha+1,0)}{3^{\alpha+1}(lna-lnb)^\alpha}\right)\right.$$

$$\left.\frac{m_2}{2}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\left(\frac{a^3\Gamma(\alpha+1,\ 3(lna-lnb))-a^3\Gamma(\alpha+1,\ 0)}{3^{\alpha+1}(lna-lnb)^\alpha}\right)\right].$$

This inequality coincides with the inequality (4.1).

Corollary 7. *By considering the conditions of Theorem 8, if we take $m_1 = m_2 = 1$ and $\alpha = q = 1$ in the inequality (4.2), then we get*

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2} - \int_a^b xf(x)dx\right|$$

$$\le\left[\frac{|f'(a)|}{6}(L(a^3,b^3)-b^3) + \frac{|f'(b)|}{6}(b^3-L(a^3,b^3))\right],$$

where L is the logarithmic mean.

Corollary 8. *By considering the conditions of Theorem 8, if we take $m_1 = m$ and $m_2 = 1$ in the inequality (4.2), then we get*

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2} - \int_a^b xf(x)dx\right| \le \frac{lnb-lna}{2} L^{1-\frac{1}{q}}(a^3,b^3)$$

$$\cdot\left[m\left|f'\left(a^{\frac{1}{m}}\right)\right|^q\left(\frac{b^3-a^3}{3(lnb-lna)} - \frac{a^3\Gamma(\alpha+1,\ 3(lna-lnb))-a^3\Gamma(\alpha+1,\ 0)}{3^{\alpha+1}(lnb-lna)(lna-lnb)^\alpha}\right)\right.$$

$$\left.+|f'(b)|^q\left(\frac{a^3\Gamma(\alpha+1,\ 3(lna-l\quad))-a^3\Gamma(\alpha+1,\ 0)}{3^{\alpha+1}(lnb-lna)(lna-lnb)^\alpha}\right)\right]^{\frac{1}{q}}.$$

This inequality coincides with the inequality in [9].

Theorem 9. *Let the function $f: \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ be a differentiable function and $f' \in L[a,b]$ for $0 < a < b < \infty$. If $|f'|^q$ is (α,m_1,m_2)-GA convex on $\left[0, max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$ for $[\alpha,m_1,m_2] \in (0,1]^3$ and $q > 1$, then,*

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2} - \int_a^b xf(x)dx\right| \le \frac{ln(b/a)}{2}$$

$$.\, L^{\frac{1}{p}}(a^{3p},b^{3p})\left[\frac{\alpha m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q}{\alpha+1} + \frac{m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q}{\alpha+1}\right]^{\frac{1}{q}}, \quad (4.3)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using both Lemma 1, Hölder integral inequality and the (α,m_1,m_2)-GA-convexity of the function $|f'|^q$ on the interval $\left[0, max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$, that is, the inequality

$$|f'(a^{1-t}b^t)| = \left|f'\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)} f'\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right|^q$$

$$\le m_1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q + m_2 t\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q,$$

we get

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2} - \int_a^b xf(x)dx\right|$$

$$\le \frac{ln(b/a)}{2}\left[\int_0^1 \left(a^{3(1-t)}b^{3t}\right)^p dt\right]^{\frac{1}{p}}$$

$$\times\left[\int_0^1 \left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right|^q dt\right]^{\frac{1}{q}}$$

$$\leq \frac{ln(b/a)}{2}\left[\int_0^1 \left(a^{3(1-t)}b^{3t}\right)^p dt\right]^{\frac{1}{p}}$$

$$.\left[\int_0^1 \left[m_1(1-t^\alpha)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q + m_2 t^\alpha \left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right]dt\right]^{\frac{1}{q}}$$

$$= \frac{ln(b/a)}{2}\left[\int_0^1 a^{3p(1-t)}b^{3pt}dt\right]^{\frac{1}{p}}$$

$$\times\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q \int_0^1 (1-t^\alpha)dt + m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q \int_0^1 t^\alpha dt\right]^{\frac{1}{q}}$$

$$= \frac{ln(b/a)}{2}L^{\frac{1}{p}}(a^{3p},b^{3p})\left[\frac{\alpha m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q}{\alpha+1}+\frac{m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q}{\alpha+1}\right]^{\frac{1}{q}}.$$

This completes the proof of the theorem.

Corollary 9. *By considering the conditions of Theorem 9, if we take* $m_1 = m_2 = 1$ *in the inequality (4.3), then we get*

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2} - \int_a^b xf(x)dx\right|$$

$$\leq \frac{\ln(b/a)}{2}L^{\frac{1}{p}}(a^{3p},b^{3p})\left[\frac{\alpha|f'(a)|^q}{\alpha+1}+\frac{|f'(b)|^q}{\alpha+1}\right]^{\frac{1}{q}}.$$

Corollary 10. *By considering the conditions of Theorem 9, if we take* $m_1 = m, m_2 = 1$ *in the inequality (4.3) then we obtain*

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2} - \int_a^b xf(x)dx\right|$$

$$\leq \frac{ln(b/a)}{2}L^{\frac{1}{p}}(a^{3p},b^{3p})\left[\frac{\alpha m\left|f'\left(a^{\frac{1}{m}}\right)\right|^q}{\alpha+1}+\frac{|f'(b)|^q}{\alpha+1}\right]^{\frac{1}{q}}.$$

Corollary 11. *By considering the conditions of Theorem 9, if we take* $m_1 = m_2 = 1$ *in the inequality (4.3) then we obtain*

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2} - \int_a^b xf(x)dx\right|$$

$$\leq \frac{ln\left(\frac{b}{a}\right)}{2}L^{\frac{1}{p}}(a^{3p},b^{3p})A^{\frac{1}{q}}(|f'(a)|^q,|f'(b)|^q).$$

Theorem 10. *Let the function* $f:\mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ *be a differentiable function and* $f' \in L[a,b]$ *for* $0 < a < b < \infty$. *If* $|f'|^q$ *is* (α,m_1,m_2)*-GA convex on* $\left[0, max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$ *for* $[\alpha,m_1,m_2] \in (0,1]^3$ *and* $q > 1$, *then the following integral inequalities hold*

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2} - \int_a^b xf(x)dx\right| \tag{4.4}$$

$$\leq \frac{ln(b/a)}{2}\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|\left(\begin{array}{c}L(a^{3q},b^{3q})\\ -\frac{a^{3q}\Gamma(\alpha+1,3q(lna-lnb))-a^{3q}\Gamma(\alpha+1,0)}{(3q)^{\alpha+1}(lna-lnb)^\alpha(lnb-lna)}\end{array}\right)\right.$$

$$\left.+m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\left(\frac{a^{3q}\Gamma(\alpha+1,3q(lna-lnb))-a^{3q}\Gamma(\alpha+1,0)}{(3q)^{\alpha+1}(lna-lnb)^\alpha(lnb-lna)}\right)\right]^{\frac{1}{q}},$$

where L is the logarithmic mean, Γ is the Gamma function and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From both Lemma 1, Hölder integral inequality and the (α,m_1,m_2)-GA-convexity of the function $|f'|^q$ on the interval $\left[0, max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$, we get

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2} - \int_a^b xf(x)dx\right|$$

$$\leq \frac{ln(b/a)}{2}\left(\int_0^1 1dt\right)^{\frac{1}{p}}$$

$$.\left[\int_0^1 a^{3q(1-t)}b^{3qt}\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right|^q dt\right]^{\frac{1}{q}}$$

$$\leq \frac{ln(b/a)}{2}\left(\int_0^1 a^{3(1-t)q}b^{3tq}\left[\begin{array}{c}m_1(1-t^\alpha)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\\ +m_2 t^\alpha\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\end{array}\right]dt\right)^{\frac{1}{q}}$$

$$= \frac{ln(b/a)}{2}\begin{bmatrix} m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q \int_0^1 (1-t^\alpha)a^{3q(1-t)}b^{3qt}dt \\ +m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q \int_0^1 t^\alpha a^{3q(1-t)}b^{3qt}dt \end{bmatrix}^{\frac{1}{q}}$$

$$= \frac{ln(b/a)}{2}\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|\left(\begin{matrix} L(a^{3q},b^{3q}) \\ -\frac{a^{3q}\Gamma(\alpha+1,3q(lna-lnb))-a^{3q}\Gamma(\alpha+1,0)}{(3q)^{\alpha+1}(lna-lnb)^\alpha(lnb-lna)}\end{matrix}\right)\right.$$

$$\left.+m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\left(\frac{a^{3q}\Gamma(\alpha+1,3q(lna-lnb))-a^{3q}\Gamma(\alpha+1,0)}{(3q)^{\alpha+1}(lna-lnb)^\alpha(lnb-lna)}\right)\right]^{\frac{1}{q}}.$$

This completes the proof of the theorem.

Corollary 12. *By considering the conditions of Theorem 10, if we take $m_1 = m_2 = 1$ in the inequality (4.4), then we get*

$$\left|\frac{b^2f(a)-a^2f(b)}{2} - \int_a^b xf(x)dx\right| \le \frac{ln(b/a)}{2}$$

$$.\left[|f'(a)|\left(L(a^{3q},b^{3q}) - \frac{a^{3q}\Gamma(\alpha+1,3q(lna-lnb))-a^{3q}\Gamma(\alpha+1,0)}{(3q)^{\alpha+1}(lna-lnb)^\alpha(lnb-lna)}\right)\right.$$

$$\left.+|f'(b)|\left(\frac{a^{3q}\Gamma(\alpha+1,3q(lna-lnb))-a^{3q}\Gamma(\alpha+1,0)}{(3q)^{\alpha+1}(lna-lnb)^\alpha(lnb-lna)}\right)\right]^{\frac{1}{q}}.$$

Corollary 13. *By considering the conditions of Theorem 10, if we take $m_1 = m_2 = 1$ and $\alpha = 1$ in the inequality (4.4), then we get*

$$\left|\frac{b^2f(a)-a^2f(b)}{2} - \int_a^b xf(x)dx\right|$$

$$\le \frac{ln(b/a)}{2}\left[|f'(a)|\left(\frac{L(a^{3q},b^{3q})-a^{3q}}{3q(lnb-lna)}\right) + |f'(b)|\left(\frac{b^{3q}-L(a^{3q},b^{3q})}{3q(lnb-lna)}\right)\right]^{\frac{1}{q}}.$$

Theorem 11. *Let the function $f:\mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ be a differentiable function and $f' \in L[a,b]$ for $0 < a < b < \infty$. If $|f'|^q$ is (α,m_1,m_2)-GA convex function on the interval $\left[0, max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$ for $[\alpha,m_1,m_2] \in (0,1]^3$ and $q > 1$, then the following integral inequalities hold*

$$\left|\frac{b^2f(a)-a^2f(b)}{2} - \int_a^b xf(x)dx\right| \tag{4.5}$$

$$\le \frac{lnb-lna}{2}\left[\frac{L(a^{3p},b^{3p})-a^{3p}}{3(lnb-lna)}\right]^{\frac{1}{p}}$$

$$.\left[\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\left(\frac{\alpha(\alpha+3)m_1}{2(\alpha^2+3\alpha+2)}\right) + \left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\left(\frac{m_2}{\alpha^2+3\alpha+2}\right)\right]^{\frac{1}{q}}$$

$$+\frac{lnb-lna}{2}\left[\frac{b^{3p}-L(a^{3p},b^{3p})}{3(lnb-lna)}\right]^{\frac{1}{p}}$$

$$.\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\left(\frac{\alpha}{2(\alpha+2)}\right) + m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\left(\frac{1}{\alpha+2}\right)\right]^{\frac{1}{q}},$$

where *L* is the logarithmic mean and $\frac{1}{p}+\frac{1}{q} = 1$.

Proof. From Lemma 1, Hölder-İşcan integral inequality and the (α,m_1,m_2)-GA convexity of the function $|f'|^q$ on the interval $\left[0, max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$, we obtain

$$\left|\frac{b^2f(a)-a^2f(b)}{2} - \int_a^b xf(x)dx\right|$$

$$\le \frac{lnb-lna}{2}\left[\int_0^1 (1-t)\left(a^{3(1-t)}b^{3t}\right)^p dt\right]^{\frac{1}{p}}$$

$$.\left[\int_0^1 (1-t)\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2t}\right)\right|^q dt\right]^{\frac{1}{q}}$$

$$+\frac{lnb-lna}{2}\left[\int_0^1 t\left(a^{3(1-t)}b^{3t}\right)^p dt\right]^{\frac{1}{p}}$$

$$\times\left[\int_0^1 t\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2t}\right)\right|^q dt\right]^{\frac{1}{q}}$$

$$\le \frac{lnb-lna}{2}\left[\int_0^1 (1-t)a^{3p(1-t)}b^{3pt}dt\right]^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \begin{bmatrix} m_1(1-t)(1-t^\alpha)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q \\ +m_2(1-t)t^\alpha\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q \end{bmatrix} dt\right)^{\frac{1}{q}}$$

$$+\frac{lnb-lna}{2}\left[\int_0^1 ta^{3p(1-t)}b^{3pt}dt\right]^{\frac{1}{p}}$$

$$.\left[\int_0^1\left[m_1 t(1-t^\alpha)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q+ m_2 t t^\alpha\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right]dt\right]^{\frac{1}{q}}$$

$$=\frac{lnb-lna}{2}\left[\frac{L(a^{3p},b^{3p})-a^{3p}}{3(lnb-lna)}\right]^{\frac{1}{p}}$$

$$.\left[\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\left(\frac{\alpha(\alpha+3)m_1}{2(\alpha^2+3\alpha+2)}\right)+\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\left(\frac{m_2}{\alpha^2+3\alpha+2}\right)\right]^{\frac{1}{q}}$$

$$+\frac{lnb-lna}{2}\left[\frac{b^{3p}-L(a^{3p},b^{3p})}{3(lnb-lna)}\right]^{\frac{1}{p}}$$

$$.\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\left(\frac{\alpha}{2(\alpha+2)}\right)+m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\left(\frac{1}{\alpha+2}\right)\right]^{\frac{1}{q}}.$$

This completes the proof of the theorem.

Corollary 14. *By considering the conditions of Theorem 11, if we take $m_1 = m_2 = 1$ in the inequality (4.5), then we get*

$$\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right|$$

$$\leq\frac{lnb-lna}{2}\left[\frac{L(a^{3p},b^{3p})-a^{3p}}{3(lnb-lna)}\right]^{\frac{1}{p}}$$

$$.\left[|f'(a)|^q\left(\frac{\alpha(\alpha+3)}{2(\alpha^2+3\alpha+2)}\right)+|f'(b)|^q\left(\frac{1}{\alpha^2+3\alpha+2}\right)\right]^{\frac{1}{q}}$$

$$+\frac{lnb-lna}{2}\left[\frac{b^{3p}-L(a^{3p},b^{3p})}{3(lnb-lna)}\right]^{\frac{1}{p}}$$

$$\times\left[|f'(a)|^q\left(\frac{\alpha}{2(\alpha+2)}\right)+|f'(b)|^q\left(\frac{1}{\alpha+2}\right)\right]^{\frac{1}{q}},$$

Corollary 15. *By considering the conditions of Theorem 11, if we take $m_1 = m_2 = 1$ and $\alpha = 1$ in the inequality (4.5), then we get*

$$\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right|$$

$$\leq\frac{lnb-lna}{2}\left[\frac{L(a^{3p},b^{3p})-a^{3p}}{3(lnb-lna)}\right]^{\frac{1}{p}}\left[\frac{|f'(a)|^q}{3}+|f'(b)|^q\left(\frac{1}{6}\right)\right]^{\frac{1}{q}}$$

$$+\frac{lnb-lna}{2}\left[\frac{b^{3p}-L(a^{3p},b^{3p})}{3(lnb-lna)}\right]^{\frac{1}{p}}\left[\frac{|f'(a)|^q}{6}+\frac{|f'(b)|^q}{3}\right]^{\frac{1}{q}}.$$

5. CONCLUSION

New Hermite-Hadamard type integral inequalities can be obtained by using (α, m_1, m_2)-GA convexity and different type identities.

Research and Publication Ethics

This paper has been prepared within the scope of international research and publication ethics.

Ethics Committee Approval

This paper does not require any ethics committee permission or special permission.

Conflict of Interests

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this paper.

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