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On Strongly π-regular Modules

Suat KOÇ*[1]

Abstract

In this article, we introduce the notion of strongly π-regular module which is a generalization of von Neumann regular module in the sense [13]. Let A be a commutative ring with $1 \neq 0$ and X a multiplication A-module. X is called a strongly π-regular module if for each $x \in X$, $(Ax)^m = cX = c^2X$ for some $c \in A$ and $m \in \mathbb{N}$. In addition to give many properties and examples of strongly π-regular modules, we also characterize certain class of modules such as von Neumann regular modules and second modules in terms of this new class of modules. Also, we determine when the localization of any family of submodules at a prime ideal commutes with the intersection of this family.

Keywords: von Neumann regular module, (m, n)-closed ideal, strongly π-regular module, Krull dimension, $(*)$-property, localization.

1. INTRODUCTION

In this article, all rings under consideration will be assumed to be commutative with $1 \neq 0$ and all modules will be nonzero unital. Let A will always denote such a ring and X will denote such an A-module. The set of all prime ideals and maximal ideals will be denoted by $Spec(A)$ and $Max(A)$, respectively. A ring A is said to be a *von Neumann regular (briefly, vn-regular) ring* if for each $c \in A$, there exists $d \in A$ such that $c = c^2d$. In that case the principal ideal $(c) = (a)$ for some idempotent element $a \in A$ [22]. It is clear that A is a vn-regular ring iff $(c) = (c)^2$ for each $c \in A$. The concept of vn-regular ring and its generalizations have been widely studied in many papers. See, for instance, [2], [8], [9], [10], [12], [24]. One of the important generalizations of vn-regular ring is that strongly π-regular ring. A ring A is called a *strongly π-regular ring* if for each $c \in A$, $c^m = c^{m+1}d$ for some $d \in A$ and $m \in \mathbb{N}$, or equivalently, the descending chain $(c) \supseteq (c)^2 \supseteq \cdots \supseteq (c)^m \supseteq \cdots$ of principal powers terminates at some step [14]. It is clear that a ring A is a strongly π-regular iff for each $c \in A$, there exist $m \in \mathbb{N}$ and $d \in A$ such that $(c)^m = (d) = (d)^2$. In that case $(c)^m$ is generated by an idempotent element $a \in A$. It was shown, in [8], that a ring A is strongly π-regular iff its Krull dimension $\dim(A) = 0$ iff the radical commutes

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with intersection of any family of ideals (See, [8, Theorem 3] and [8, Theorem 4]).

Recently, Jayaram and Tekir, in their paper [13], extended the notion of vn-regular rings to modules by defining X-vn-regular elements and weak idempotent elements. An element $c \in A$ is said to be an *X-vn-regular element* if $cX = c^2X$ and $d \in A$ is said to be a weak idempotent if $dx = d^2x$ for each $x \in X$, namely, $d - d^2 \in ann(X)$, where $ann(X) = \{c \in A: cX = 0\}$ is the annihilator of an A-module X. They defined an A-module X as a *vn-regular module* if for each $x \in X$, the cyclic submodule $Ax = cX$ for some X-vn-regular element $c \in A$. An A-module X is said to be a *multiplication module* if its each submodule Y of X is of the form $Y = JX$ for some ideal J of A [5], [11]. Note that X is a multiplication module iff $Y = (Y:X)X$ for each submodule Y of X, where $(Y:X) = ann(X/Y)$. Also it is clear that each vn-regular module is multiplication.

This article aims to study strongly π-regular modules and use them to characterize certain class of modules such as vn-regular modules, second modules, zero dimensional modules and modules satisfying $(*)$-condition. For the sake of completeness, we shall give some notions which will be frequently used throughout the paper. Let X be an A-module. A proper submodule Y of X is said to be prime if for each $c \in A$ and $x \in X$, $cx \in Y$ implies either $c \in (Y:X)$ or $x \in Y$ [16]. In this case, $q = (Y:X) \in Spec(A)$ and Y is said to be q-prime. Let Y be a submodule of X. The radical of Y, denoted by $rad(Y)$, is the intersection of all prime submodules Q of X containing Y. If there is no such a prime submodule, we say $rad(Y) = X$. Note that in a finitely generated (briefly, f.g.) or multiplication modules, there always exists a prime submodule Q of X containing a given proper submodule Y of X. In a multiplication module X, Ameri in his paper [1], defined the product of two submodules and determined the radical of any given proper submodule Y of X. Let $Y = IX$ and $W = JX$ be two submodules of a multiplication module X. Then the product of Y and W is defined by $YW = (IJ)X$. He proved in [1], that the product is well defined and the radical of a proper submodule Y of X is characterized as follows: $rad(Y) = \{x \in X: \exists m \in \mathbb{N}$ such that $(Ax)^m \subseteq Y\}$. If X is a multiplication module, it is clear that the radical commutes with finite intersection of submodules, that is, $rad(\bigcap_{i=1}^{n} Y_i) = \bigcap_{i=1}^{n} rad(Y_i)$. So it is a natural question to ask whether the radical operation commutes (or not) with the infinite intersection. This question was first studied in [6] and continued in [7]. In [6], the authors introduced the $(*)$-property as follows: a multiplication module X is said to satisfy $(*)$-condition if for each family of submodules $\{Y_i\}_{i\in\Delta}$, $x \in rad(Y_i)$ for each $i \in \Delta$ implies $(Ax)^m \subseteq \bigcap_{i\in\Delta} Y_i$ for some $m \in \mathbb{N}$. They also showed that X satisfies $(*)$-condition iff the radical operation commutes with the infinite intersection of a given family of submodules. Also, recall from [7] that a multiplication module X is said to satisfy descending chain condition on principal powers if the descending chain $Ax \supseteq (Ax)^2 \supseteq \cdots \supseteq (Ax)^m \supseteq \cdots$ stops. The authors in [7] showed that a f.g. multiplication module X satisfies $(*)$-property iff it satisfies descending chain condition on principal powers iff its Krull dimension $\dim(X) = 0$ (See [7, Lemma 3], [7, Corollary 3] and [7, Lemma 5]).

We say that an A-module X as a *strongly π-regular module* if for each $x \in X$, there exist $m \in \mathbb{N}$ and $c \in A$ such that $(Ax)^m = cX = c^2X$. Also, we define an A-module X as a *weak π-regular module* if for each $c \in A$, the descending chain $cX \supseteq c^2X \supseteq \cdots \supseteq c^mX \supseteq \cdots$ stops.

Among other things in this paper, in section 2, we study the relations between strongly π-regular modules (weak π-regular modules) and other certain classes of modules such as vn-regular modules, second modules and divisible modules. In particular, we investigate the behavior of strongly π-regular modules under homomorphism, in factor modules, in localization, in cartesian product of modules (See Proposition 2.16, Corollary 2.17, Proposition 2.20, Proposition 2.21 and Theorem 2.23). Also, we give a characterization of strongly π-regular modules (weak π-regular modules) in terms of strongly π-regular rings (See Proposition 2.5, Proposition 2.10, Proposition 2.14 and Proposition 2.15). Finally, we characterize second modules and vn-regular modules in terms of strongly π-regular modules (See, Corollary 2.19 and Theorem 2.24).

In section 3, we deal with commutativity of localization at a prime ideal and arbitrary intersection of any family of submodules. Let Y be a q-prime submodule of X. Then we know that localization at q commutes with the finite intersection of a given family of submodules. Thus it is a natural problem to ask whether this property of localization is true when we replace finite intersection by infinite one. This question has a negative answer (See Example 3.1). We show that q-prime submodule Y of a f.g. multiplication module X is a strongly prime submodule iff the localization commutes with the infinite intersection of a given family of submodules (See, Theorem 3.2). Finally, we give the relations between strongly π-regular modules and other certain class of modules such as weak π-regular modules, zero dimensional modules, modules satisfying $(*)$-condition (See, Theorem 3.6).

2. CHARACTERIZATION OF STRONGLY π-REGULAR MODULES

Definition 2.1. A multiplication A-module X is called a strongly π-regular module if for all $x \in X$, $(Ax)^m = cX = c^2X$ for some $c \in A$ and $m \in \mathbb{N}$.

Example 2.2. A ring A is a strongly π-regular ring iff it is a strongly π-regular A-module.

Recall from [20] that an A-module X is said to be simple if zero and X are the only submodules of X.

Example 2.3. All simple modules are strongly π-regular. To see this take a simple A-module X and $x \in X$. Then either $Ax = 0 = 0X$ or $Ax = X = 1X$. Hence, X is a strongly π-regular module.

Example 2.4. All vn-regular modules are clearly strongly π-regular module. But the converse need not be true. For instance, consider the $\mathbb{Z}$-module $\mathbb{Z}_{q^k}$, where q is a prime number and $k \geq 1$ is an integer. Let $\bar{x} \in \mathbb{Z}_{q^k}$. If $\gcd(x, q) = 1$, then we conclude $\mathbb{Z}\bar{x} = \mathbb{Z}_{q^m} = 1\mathbb{Z}_{q^m}$. If $\gcd(x, q) \neq 1$, then we get $(\mathbb{Z}\bar{x})^m = (\bar{0}) = 0\mathbb{Z}_{q^m}$. But $\mathbb{Z}$-module $\mathbb{Z}_{q^m}$ is not a vn-regular module.

Proposition 2.5. Suppose that X is a f.g. strongly π-regular module. Then $A/ann(X)$ is a strongly π-regular ring.

Proof. Take $q \in Spec(A)$ with $ann(X) \subseteq q$. Then there exist $m \in Max(A)$ with $ann(X) \subseteq q \subseteq m$. Suppose that $q \subsetneq m$. Then $qX \subsetneq mX$ and so there exists $x \in mX - qX$. Since X is strongly π-regular module, $(Ax)^m = cX = c^2X$ for some $c \in A$ and $m \in \mathbb{N}$. Since X is f.g. module, cX is f.g. so by [4, Corollary 2.5], we have $(1 + dc)cX = 0$ for some $d \in A$. This gives that $(1 + dc)c \in ann(X)$. Also by [11, Corollary 2.11], qX is prime and it is clear that $(Ax)^m \nsubseteq qX$ so we get $c \notin (qX{:}X) = q$. As q is a prime ideal and $c(1 + dc) \in ann(X) \subseteq q$, we conclude that $1 + dc \in q \subseteq m$. Also note that $cX \subseteq mX$. This yields that $c \in m$ and so $1 \in m$, a contradiction. Thus $q = m \in Max(A)$. Therefore $A/ann(X)$ is a strongly π-regular ring by [8, Theorem 3].

Definition 2.6. A-module X is called a weak π-regular module if for each $c \in A$, $c^mX = c^{m+1}X$ for some $m \in \mathbb{N}$.

Example 2.7. A ring A is a strongly π-regular ring iff it is a weak π-regular as an A-module.

Example 2.8. Let X be a f.g. vn-regular module. Then by [13, Theorem 1], every $c \in A$ is an X-vn-regular, namely, $cX = c^2X$. Therefore, every f.g. vn-regular module is weak π-regular.

A module X over an integral domain A is called a divisible module if $cX = X$ for each $0 \neq c \in A$ [17]. Also a module X over a commutative ring A (not necessarily a domain) is said to be a second module if for each $c \in X$, either $cX = 0$ or $cX = X$ [23].

Example 2.9.

(i) Every divisible module over an integral domain is a weak π-regular module.

(ii) Every second module is a weak π-regular module.

Proposition 2.10. If X is a f.g. strongly π-regular module, then X is a weak π-regular module.

Proof. Take $c \in A$. Then by Proposition 2.5, $A/ann(X)$ is a strongly π-regular ring and so $\bar{c}^m = \bar{c}^{m+1}\bar{d}$ for some $d \in A$ and $m \in \mathbb{N}$, where $\bar{c} = c + ann(X)$. This implies that $c^m - c^{m+1}d \in ann(X)$ and so $c^m X = c^{m+1}dX \subseteq c^{m+1}X$. Since the other inclusion always holds, we have $c^m X = c^{m+1}X$. Therefore, X is a weak π-regular module.

The converse of Proposition 2.10 need not be true.

Example 2.11. Consider the $\mathbb{Z}$-module $X = \mathbb{Z}_2 \times \mathbb{Z}_2$ and take any integer c. If c is even, then $cX = 0 = c^2X$. Also if c is odd, we have $cX = X = c^2X$. Thus X is a weak π-regular module. Since X is not a multiplication module, X can not be a strongly π-regular module.

The following Lemma is well known in [13] and for the sake of completeness, we remind it here.

Lemma 2.12. Let X be an A-module.

(i) If $c, d \in A$ are weak idempotents, then $c + d(1-c)$ is a weak idempotent.

(ii) $cX + dX = \big(c + d(1-c)\big)X$.

(iii) Suppose that X is a f.g. A-module. Then $c \in A$ is X-vn-regular iff $cX = dX$ for some weak idempotent $d \in A$.

Remark 2.13. Let X be a f.g. A-module and $c^m X = c^{m+1}X$ for some $c \in A$ and $m \in \mathbb{N}$. Then clearly we have $c^m X = (c^m)^2 X$ and so c^m is an X-vn-regular element. By Lemma 2.12 (iii), $c^m X = dX$ for some weak idempotent element $d \in A$.

Proposition 2.14. Suppose that X is a f.g. multiplication A-module. If $A/ann(X)$ is a strongly π-regular ring, then X is a strongly π-regular module.

Proof. Take $c \in A$. Since $A/ann(X)$ is a strongly π-regular ring, $c^m X = c^{m+1}X$ for some $m \in \mathbb{N}$ as in the Proof of Proposition 2.10. By Remark 2.13, we have $c^m X = dX$ for some weak idempotent element $d \in A$. Let $x \in X$. Since X is f.g. multiplication, $Ax = JX$ for some f.g. ideal J of A, where $J = (c_1, c_2, \dots, c_n)$ for some $c_1, c_2, \dots, c_n \in A$. Then we have $Ax = c_1X + c_2X + \cdots + c_nX$. Also note that for each $c_i \in A$, there exist $t_i \geq 1$ and weak idempotent element $d_i \in A$ such that ${c_i}^{t_i}X = d_iX$. Now, put $t = t_1 + t_2 + \cdots + t_n$. Then we get $(Ax)^t = \{(c_1) + (c_2) + \cdots + (c_n)\}^t X = {c_1}^{t_1}X + {c_2}^{t_2}X + \cdots + {c_n}^{t_n}X = d_1X + d_2X + \cdots + d_nX = dX = d^2X$ for some weak idempotent element $d \in A$ by Lemma 2.12 (iii). Therefore, X is a strongly π-regular module.

Proposition 2.15. Suppose that X is a f.g. weak π-regular module. Then $A/ann(X)$ is a strongly π-regular ring.

Proof. Let $\bar{c} = c + ann(X) \in A/ann(X)$ for some $c \in A$. Now, we will show that $(\bar{c})^m = (\bar{c})^{m+1}$ for some $m \in \mathbb{N}$. As X is a weak π-regular module, $c^m X = c^{m+1}X$ for some $m \in \mathbb{N}$. Since X is f.g., by [4, Corollary 2.5], we conclude that $(1 - dc)c^m X = 0$ for some $d \in A$. This implies that $c^m - dc^{m+1} \in ann(X)$ and so $(c^m) + ann(X) = (c^{m+1}) + ann(X)$. Then we deduce $(\bar{c})^m = (\bar{c})^{m+1}$ and hence $A/ann(X)$ is a strongly π-regular ring.

Proposition 2.16. Let $g: X \to X'$ be an epimorphism of A-modules and X be a strongly π-regular module. Then X' is a strongly π-regular module.

Proof. Let $x' \in X'$. Since g is surjective, $g(x) = x'$ for some $x \in X$. Since X is a multiplication module, $Ax = JX$ for some ideal J of A. As X is a strongly π-regular module, $(Ax)^m = J^m X = cX = c^2X$ for some $c \in A, m \in \mathbb{N}$. This implies that $g(Ax) = Ag(x) = Ax' = JX'$ and also note that $(Ax')^m = (JX')^m = J^m X' = J^m g(X) = g(J^m X) = g(cX) = g(c^2X)$. This gives $(Ax')^m = cX' = c^2X'$ which completes the proof.

Corollary 2.17. Let X be a strongly π-regular module and Y a submodule of X. Then X/Y is a strongly π-regular module.

An element $c \in A$ is said to be a zero divisor on X if $cx = 0$ for some $0 \neq x \in X$. Also the set of all zero divisor elements on X is denoted by $z(X)$.

Lemma 2.18. Let X be a f.g. strongly π-regular module. Then $cX = X$ for each $c \in A - z(X)$.

Proof. Let $c \in A - z(X)$. Note that for each $m \in \mathbb{N}$, $c^m \notin z(X)$. Since X is a f.g. strongly π-regular module, by Proposition 2.10, $c^k X = c^{k+1} X$ for some $k \in \mathbb{N}$. Let $x' \in X$. As $c^k X = c^{k+1} X$, $c^k x' = c^{k+1} x''$ for some $x'' \in X$. This yields that $c^k(x' - cx'') = 0$ and so $x' = cx''$. Therefore, we have $cX = X$.

Corollary 2.19.

(i) Suppose that X is a f.g. strongly π-regular module in which $z(X) = ann(X)$. Then X is a second module.

(ii) Suppose that X is a f.g. multiplication module in which $z(X) = ann(X)$. Then X is a second module iff X is a strongly π-regular module.

(iii) Suppose that X is a f.g. faithful strongly π-regular module over an integral domain A. Then X is a divisible module.

(iv) Suppose that X is a torsion-free module, namely, $z(X) = 0$. If X is a f.g. strongly π-regular module, then X is a divisible module.

Proof. (i) Assume that X is a f.g. strongly π-regular module in which $z(X) = ann(X)$. Let $c \in A$. If $c \in z(X) = ann(X)$, then $cX = 0$. So assume that $c \notin z(X)$. Thus by Lemma 2.18, $cX = X$ and so X is a second module.

(ii): Directly from (i), Example 2.9, Proposition 2.14 and Proposition 2.15.

(iii): It is similar to (i).

(iv): It can be obtained from (iii).

Proposition 2.20. Suppose that X is an A-module and $T \subseteq A$ is a multiplicatively closed subset of A. If X is a strongly π-regular module, then $T^{-1}X$ is a strongly π-regular $T^{-1}A$-module.

Proof. Let $\frac{x}{t} \in T^{-1}X$ for some $t \in T, x \in X$. Since X is a strongly π-regular module, $(Ax)^m = cX = c^2X$ for some $c \in A, m \in \mathbb{N}$. As X is a multiplication module, we can write $Ax = JX$. Note that $T^{-1}A\left(\frac{x}{t}\right) = T^{-1}(Ax) = T^{-1}(JX) = T^{-1}(J)T^{-1}(X)$. This implies that $\left[T^{-1}A\left(\frac{x}{t}\right)\right]^m = \left(T^{-1}(J)\right)^m T^{-1}X = T^{-1}(J^m)T^{-1}X = T^{-1}(J^m X) = T^{-1}(cX) = T^{-1}(c^2X)$. Then we have $\left[T^{-1}A\left(\frac{x}{t}\right)\right]^m = \frac{c}{1}T^{-1}X = (\frac{c}{1})^2 T^{-1}X$ which completes the proof.

Recall that a commutative ring A is said to be a *quasi-semi-local* if the number of its maximal ideals is finite.

Proposition 2.21. Let X be a f.g. module over a quasi-semi-local ring. The followings are equivalent.

(i) X is a strongly π-regular module.

(ii) X_q is a strongly π-regular module for each $q \in Spec(A)$.

(iii) X_q is a strongly π-regular module for each $q \in Max(A)$.

Proof. $(i) \Rightarrow (ii)$: It can be obtained from Proposition 2.20.

$(ii) \Rightarrow (iii)$: Directly from $Max(A) \subseteq Spec(A)$.

$(iii) \Rightarrow (i)$: First note that, by [5, Lemma 2], X is a multiplication module. As X is a f.g. multiplication module, it is sufficient to show that X is a weak π-regular module. Let $c \in A$. Now, we will show that $c^m X = c^{m+1} X$ for some $m \in \mathbb{N}$. Since A is quasi semi-local, A has finitely many maximal ideals $q_1, q_2, \ldots, q_t$. As X_{q_i} is a strongly π-regular module, $\left(\frac{c}{1}\right)^{m_i} X_{q_i} = \left(\frac{c}{1}\right)^{m_i+1} X_{q_i}$ for some $m_i \in \mathbb{N}$. Now, put $m = \max\{m_1, m_2, \ldots, m_t\}$. Then we have $\left(\frac{c}{1}\right)^m X_{q_i} = \left(\frac{c}{1}\right)^{m+1} X_{q_i}$ and so $(c^m X)_{q_i} = (c^{m+1} X)_{q_i}$. This implies that $c^m X = c^{m+1} X$ which completes the proof.

Recall from [19] that a proper submodule Y of X is said to be a *pure submodule* if $cX \cap Y = cY$ for each $c \in A$.

Lemma 2.22. Let X be a strongly π-regular module and Y a pure submodule of X. Then Y is a strongly π-regular module.

Proof. First, we will show that Y is a multiplication module. Take a submodule V of Y. Since X is multiplication, $V = (V{:}X)X$. Let $x \in V$. Then we can write $x = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ for some $c_i \in (V{:}X)$ and $x_i \in X$. As Y is pure and $c_ix_i \in c_iX \cap Y = c_iY$, we have $c_ix_i = c_ix_i{}'$ for some $x_i{}' \in Y$. This implies that $x = c_1x_1{}' + c_2x_2{}' + \cdots + c_nx_n{}' \in (V{:}X)Y \subseteq (V{:}Y)Y$. Then we have $V = (V{:}X)X \subseteq (V{:}X)Y \subseteq (V{:}Y)Y \subseteq V$. Thus $V = (V{:}Y)Y$. Now, we will show that Y is a strongly π-regular module. Let $x \in Y$. As X is a strongly π-regular module, we conclude that $(Ax)^m = dX = d^2X$ for some $d \in A$ and $m \in \mathbb{N}$. Since Y is a pure submodule of X, we have $dY = dX \cap Y = d^2X \cap Y = d^2Y$. Thus we conclude that $(Ax)^m = (Ax)^m \cap Y = dX \cap Y = dY = d^2Y$. Therefore, Y is a strongly π-regular module.

Theorem 2.23. Let X_i be a multiplication A_i-module for all $i = 1,2,\dots,n$. Then $A = \prod_{i=1}^{n} A_i$-module $X = \prod_{i=1}^{n} X_i$ is a strongly π-regular module iff X_i is a strongly π-regular module for all $i = 1,2,\dots,n$.

Proof. $\Rightarrow$: Suppose X is a strongly π-regular A-module and choose $t \in \{1,2,\dots,n\}$. Now, we shall show that X_t is a strongly π-regular module. To see this, take $x_t \in X_t$. Put $x = (0,0,\dots,x_t,0,0,\dots,0) \in X$. Since X is a strongly π-regular module, we get $(Ax)^m = cX = c^2X$, where $c = (c_1,c_2,c_3,\dots,c_n) \in A$. As X_t is multiplication module, we have $A_tx_t = J_tX_t$ for some ideal J_t of A_t. This implies that $Ax = JX$, where $J = \{0\} \times \{0\} \times \dots \times J_t \times \{0\} \times \dots \times \{0\}$. Then we conclude that $(Ax)^m = (\{0\} \times \{0\} \times \dots \times J_t{}^m \times \{0\} \times \dots \times \{0\})X = cX = c^2X$. Then we get $(A_tx_t)^m = J_t{}^mX_t = c_tX_t = c_t{}^2X_t$. Therefore, X_t is a strongly π-regular module.

$\Leftarrow$: Suppose X_i is a strongly π-regular module for every $i = 1,2,\dots,n$. Let $x = (x_1,x_2,\dots,x_n) \in X$. Note that $Ax = \prod_{i=1}^{n}(A_ix_i)$. Since X_i is a multiplication A_i-module, for all $i = 1,2,\dots,n$, $A_ix_i = J_iX_i$. Note that $Ax = \prod_{i=1}^{n}(J_iX_i) = (\prod_{i=1}^{n}J_i)X$ and also $(Ax)^t = (\prod_{i=1}^{n}J_i)^tX = \prod_{i=1}^{n}(J_i{}^tX_i)$ for all $t \in \mathbb{N}$. Since X_i is a strongly π-regular module, $(A_ix_i)^{t_i} = J_i{}^{t_i}X_i = c_iX_i = c_i{}^2X_i$ for some $t_i \geq 1$. Let $m = \max\{t_1,t_2,\dots,t_n\}$. Then it is easy to see that $(A_ix_i)^m = J_i{}^mX_i = c_iX_i = c_i{}^2X_i$. Thus we conclude that $(Ax)^m = \prod_{i=1}^{n}(J_i{}^mX_i) = \prod_{i=1}^{n}(c_iX_i) = \prod_{i=1}^{n}(c_i{}^2X_i)$. This yields that $(Ax)^m = cX = c^2X$, where $c = (c_1,c_2,\dots,c_n) \in A$.

Recall that an A-module X is said to be a *reduced module* if for each $c \in A, x \in X, c^2x = 0$ implies $cx = 0$ [15]. It was shown, in [13, Lemma 10], that a f.g. vn-regular module is reduced. Also, it is well known that a ring A is a vn-regular ring iff A is reduced strongly π-regular ring. Now, we prove this fact for vn-regular modules.

Theorem 2.24. The followings are equivalent for any f.g A-module X.

(i) X is a vn-regular module.

(ii) X is a reduced strongly π-regular module.

Proof. $(ii) \Rightarrow (i)$: We will show that $cX = c^2X$ for each $c \in A$. Since X is a f.g. strongly π-regular module, by Proposition 2.5, $A/ann(X)$ is a strongly π-regular ring. Let $c \in A$. Then we have $c^m - dc^{m+1} \in ann(X)$ for some $d \in A$ and $m \in \mathbb{N}$. This implies that $c^m(1-dc) \in ann(X)$ and so $(c(1-dc))^mX = 0$. Since X is a reduced module, we conclude that $c(1-dc)X = 0$ and so $cX = c^2X$. Therefore, by [13, Theorem 1], X is a vn-regular module.

$(i) \Rightarrow (ii)$: Directly from [13, Lemma 10] and Example 2.4.

Recall from [3] that a proper ideal J of A is said to be an (m,n)-closed ideal if $c^m \in J$, then $c^n \in J$. Now, we will characterize vn-regular modules in terms of (m,n)-closed ideals.

Theorem 2.25. Let X be a f.g. reduced multiplication module. Then X is a vn-regular module iff every proper ideal of $A/ann(X)$ is a (3,2)-closed ideal.

Proof. ⇒: Let J' be a proper ideal of $A/ann(X)$. Then there exists a proper ideal J of A containing $ann(X)$ such that $J' = J/ann(X)$. Now, we will show that J is a (3,2)-closed ideal of A. To see this take $c^3 \in J$. Then we have $c^3X \subseteq JX$. Since X is a vn-regular module, we conclude that $c^2X = c^3X \subseteq JX$. By [21, Corollary to Theorem 9], we have $c^2 \in J + ann(X) = J$. Thus J is a (3,2)-closed ideal of A. By [3, Corollary 2.11], J' is a (3,2)-closed ideal of $A/ann(X)$.

⇐: Let $c \in A$. Now, we will show that $c^2X = cX$. If $c^3X = X$, then it is easily seen that $cX = X = c^2X$. So suppose that c^3X is a proper submodule of X. Then by assumption $(c^3X:X)/ann(X)$ is a (3,2)-closed ideal of $A/ann(X)$. Then by [3, Corollary 2.11], $(c^3X:X)$ is a (3,2)-closed ideal of A. Since $c^3 \in (c^3X:X)$, we get $c^2 \in (c^3X:X)$ and so $c^2X = c^3X$. Since X is f.g., by [4, Corollary 2.5], we have $(1-dc)c^2X = 0$ for some $d \in A$ and so $[(1-dc)c]^2X = 0$. As X is reduced module, we obtain that $(1-dc)cX = 0$ and this yields that $cX = c^2X$. By [13, Theorem 1], X is a vn-regular module.

3. WHEN DOES LOCALIZATION AT A PRIME IDEAL COMMUTE WITH INFINITE INTERSECTION?

Let $q \in Spec(A)$ and X an A-module. Then we know that $(\bigcap_{i=1}^{n} Y_i)_q = \bigcap_{i=1}^{n} (Y_i)_q$. One can naturally ask whether this property is true when we replace the finite intersection by infinite one. This question has a negative answer.

Example 3.1. Consider the $\mathbb{Z}$-module $\mathbb{Z}$ and $q = 2\mathbb{Z}$. Let $Y_i = 3^i\mathbb{Z}$ for each $i \in \mathbb{N}$. Then it is clear that $(\bigcap_{i=1}^{\infty} Y_i)_q = 0_q \neq \mathbb{Z}_{(2)} = \bigcap_{i=1}^{\infty} (Y_i)_q$.

Recall from [18] that a prime submodule Q of X is said to be a *strongly prime submodule* if $\bigcap_{i\in\Delta} Y_i \subseteq Q$, then for some $j \in \Delta$, $Y_j \subseteq Q$. In particular, a multiplication module M is said to be a *strongly zero dimensional module* if its each prime submodule is strongly prime. In [7], the authors showed that a f.g. multiplication module X is a strongly zero dimensional module iff its Krull dimension $\dim(X) = 0$ and X is quasi-semi-local. Now, we determine when the localization commutes with infinite intersection in terms of strongly prime submodules.

Theorem 3.2. Let X be a f.g. multiplication module and Q a q-prime submodule of X. Then Q is a strongly prime submodule iff the localization at q commutes with the intersection of any family of submodules of X.

Proof. ⇒: Let Q be a strongly prime submodule and $q = (Q:X)$. Now we will show that $(\bigcap_{i\in\Delta} Y_i)_q = \bigcap_{i\in\Delta} (Y_i)_q$ for every family of submodules $\{Y_i\}_{i\in\Delta}$ of X. We know that the inclusion $(\bigcap_{i\in\Delta} Y_i)_q \subseteq \bigcap_{i\in\Delta} (Y_i)_q$ always holds. Let $\frac{x}{t} \in \bigcap_{i\in\Delta} (Y_i)_q$. Then for each $i \in \Delta$, $t_i x \in Y_i$ for some $t_i \notin q$. This implies that $(Y_i : Ax) \nsubseteq q$. As X is a f.g. multiplication module, we have $(Y_i : Ax)X \nsubseteq Q$. As Q is a strongly prime submodule, we conclude that $(\bigcap_{i\in\Delta} (Y_i : Ax)X) = (\bigcap_{i\in\Delta} (Y_i : Ax))X \nsubseteq Q$. Thus we have $\bigcap_{i\in\Delta} (Y_i : Ax) \nsubseteq q$ and so there exists $c \in \bigcap_{i\in\Delta} (Y_i : Ax) - q$. This implies that $\frac{x}{t} \in (\bigcap_{i\in\Delta} Y_i)_q$.

⇐: Assume that the localization at q commutes with the infinite intersection of any family of submodules. Now, we will show that Q is a strongly prime submodules. As X is a f.g. multiplication module, X_q is a quasi-local module over a quasi-local ring A_q. Let $\bigcap_{i\in\Delta} Y_i \subseteq Q$. Then we get $(\bigcap_{i\in\Delta} Y_i)_q = \bigcap_{i\in\Delta} (Y_i)_q \subseteq Q_q$. Since Q_q is the unique maximal submodule of X_q, we have $(Y_i)_q \subseteq Q_q$ for some $i \in \Delta$. As Q is a q-prime submodule, we deduce $Y_i \subseteq Q$ for some $i \in \Delta$. Therefore, Q is a strongly prime submodule of X.

Now, we will end this section by studying $(*)$-condition on modules and improving some results in [6]. The following proposition gives the relations between $(*)$-condition and strongly π-regular module.

Proposition 3.3.

(i) Let X be a f.g. strongly π-regular module. Then X satisfies $(*)$-condition.

(ii) Let X be a f.g. multiplication module satisfying $(*)$-condition. Then X is a strongly π-regular module.

Proof. (i): Suppose that X is a f.g. strongly π-regular module. Then it is clear that X satisfies descending chain condition on principal powers. The rest follows from [7, Lemma 3].

(ii): Let X be a f.g. multiplication module satisfying $(*)$-condition. Then by [7, Lemma 3] and [7, Corollary 3], $\dim(X) = 0$. Then by [7, Lemma 4], it can be seen that $\dim(A/ann(X)) = 0$. Again by [8, Theorem 3], $A/ann(X)$ is a strongly π-regular ring. The rest follows from Proposition 2.14.

Now, we will characterize strongly zero dimensional modules in terms of localization at a prime ideal.

Theorem 3.4. The followings are equivalent for every f.g. multiplication module X.

(i) X is a strongly zero dimensional module.

(ii) X is a zero dimensional quasi-semi-local module.

(iii) X satisfies $(*)$-property and no maximal submodule contains the intersection of other maximal submodules.

(iv) X satisfies descending chain condition on principal powers and no maximal submodule contains the intersection of the other maximal submodules.

(v) For every q-prime submodule Q of X, the localization at q commutes with the infinite intersection of any family of submodules of X.

Proof. $(i) \Leftrightarrow (ii)$: It can be obtained from [7, Corollary 4].

$(i) \Leftrightarrow (iii) \Leftrightarrow (iv)$: It can be obtained from [7, Theorem 4] and [7, Lemma 3].

$(i) \Leftrightarrow (v)$: It can be obtained from Theorem 3.2.

Lemma 3.5. Let X be a f.g. multiplication A-module and $x \in X$ such that $Ax = JX$ for some ideal J of A. The followings are equivalent.

(i) The descending chain $Ax \supseteq (Ax)^2 \supseteq \cdots \supseteq (Ax)^m \supseteq \cdots$ stops.

(ii) $JX + \bigcup_{m=1}^{\infty}(0:_X J^m) = X$.

Proof. $(ii) \Rightarrow (i)$: Suppose that $JX + \bigcup_{m=1}^{\infty}(0:_X J^m) = X$. Since X is a multiplication module, $(0:_X J^m) = \big((0:_X J^m):X\big)X = (0:J^m X)X$. This gives $(J + \bigcup_{m=1}^{\infty}(0:J^m X))X = X$. Since X is f.g. module, we obtain that $J + \bigcup_{m=1}^{\infty}(0:J^m X) = A$. Then we have $c + d = 1$ for some $c \in J$, $d \in (0:J^m X)$ and $m \in \mathbb{N}$. This implies that $J^m X = cJ^m X \subseteq J^{m+1}X \subseteq J^m X$. Thus the descending chain $Ax \supseteq (Ax)^2 \supseteq \cdots \supseteq (Ax)^m \supseteq \cdots$ stops at the m^{th} step.

$(i) \Rightarrow (ii)$: Suppose that (i) holds. Then $(Ax)^m = (Ax)^{m+1}$ for some $m \in \mathbb{N}$. As X is a f.g. multiplication module, $Ax = JX$ for some f.g. ideal J of A. Then we have $J^m X = J^{m+1}X$. Since J and X are f.g., so is $J^m X$. Then by [4, Corollary 2.5], $(1-c)J^m X = 0$ for some $c \in J$. This implies that $(1-c)X \subseteq (0:_X J^m)$. Since $cX \subseteq JX$, we obtain that $X = cX + (1-c)X \subseteq JX + \bigcup_{m=1}^{\infty}(0:_X J^m) \subseteq X$ which completes the proof.

Theorem 3.6. The followings are equivalent for every f.g. multiplication module X.

(i) X is a strongly π-regular module.

(ii) X is a weak π-regular module.

(iii) $A/ann(X)$ is a strongly π-regular module.

(iv) X satisfies $(*)$-condition.

(v) X satisfies descending chain condition on principal powers.

(vi) $\dim(X) = 0$.

Proof. $(i) \Rightarrow (ii)$: Directly from Proposition 2.10.

$(ii) \Rightarrow (iii)$: It can be obtained from Proposition 2.15.

$(iii) \Rightarrow (i)$: It can be obtained from Proposition 2.14.

$(i) \Leftrightarrow (iv)$: It can be obtained from Proposition 3.3.

$(iv) \Leftrightarrow (v) \Leftrightarrow (vi)$: It can be obtained from [7, Lemma 3], [7, Corollary 3] and [7, Lemma 5].

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