Araştırma Makalesi / Research Article

Lucas Type Statistical Convergence of Order α

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Abstract
The main goal of the article is to establish a new regular matrix and new sequence space with the help of Lucas numbers. Also, we examine statistical convergence of order α and its some properties by using Lucas sequence which is obtained from the terms of Lucas matrix. Also, we give some topological properties and inclusion relations about these two concepts.

Keywords: Lucas sequence, Lucas numbers, Statistical Convergence, Sequence space.

Lucas Tipi α. Dereceden İstatistiksel Yakınsaklık

Öz
Bu çalışmada, Lucas sayıları yardımıyla yeni bir regüler matris ve yeni bir dizi uzayı oluşturuyoruz. Ayrıca, Lucas matrisinin terimleriyle elde edilen Lucas dizisini kullanarak $\alpha.$ dereceden istatistiksel yakınsaklık kavramını inceliyoruz. Bununla birlikte, bu iki kavramla ilgili bazı topolojik özellikler ve kapsama bağıntıları veriyoruz.

Anahtar kelimeler: Lucas dizisi, Lucas sayıları, İstatistiksel yakınsaklık, Dizi uzayı.

1. Introduction

Lucas number sequence which was defined by French mathematician Edward Lucas is obtained such that $L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2}, n \geq 2$ similar to Fibonacci recurrence relation. Researches up to the present have proven that there are interesting and close relations between these two number sequences. Despite this relation, they possess distinct properties. Some basic characteristics of Lucas numbers can be found in Vajda [1], Kalman and Mena [2]. By using Lucas and Fibonacci numbers, some authors have introduced new sequence spaces and examined their topological and geometric properties such as, Candan and Kara [3], Kılınc and Candan [4,5], Kara [6], Kara and Başarır [7], Karakaş [8], Karakaş and Karakaş[9], Karakaş et al. [10]. The following articles can also be viewed to gain different perspectives such as Candan [11-15].

Fast [16] and Steinhaus [17] identified the notion of statistical convergence independently from each other in the same year. Throughout the years, several authors have studied this concept and its generalizations or applications such as Fridy [18], Connor [19], Cinar et al. [20], Et et al. [21,22], Işık and Akbaş [23], Mohiuddine et al. [24], Mursaleen [25], Salat [26], Srivastava and Et [27], Çolak and Bektaş [28] and many others.

During the recent years, some authors have been studying Fibonacci and Lucas numbers associated to summability theory and by viewpoint of sequence spaces. Also, the notion of statistical convergence have been working for many years by a lot of authors under different names and by distinct point of view in several areas. Lately, new results have been obtained and added to the literature. In the light of such information, we give new outcomes by combining the notions of Lucas numbers and statistical convergence with degree in this work.

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2. Material and Method

Let $\mathbb{N}$ be the set of all natural numbers, $M \subseteq \mathbb{N}$ and $M(n) = \{k \le n : k \in M\}$. The natural density of M is defined by $\delta(M) = \lim_n \frac{1}{n}|M(n)|$ if the limit exists. The vertical bars show the number of the elements in inclose set.

The sequence $x = (x_k)$ is called statistically convergent to the number L if for every $\varepsilon > 0$ $M(\varepsilon) = \{k \le n : |x_k - L| \ge \varepsilon\}$ has natural density zero ,i.e. for every $\varepsilon > 0$

$$\lim_n \frac{1}{n}\left|\{k \le n : |x_k - L| \ge \varepsilon\}\right| = 0.$$

The sequence $x = (x_k)$ is called statistically Cauchy sequence if for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$\lim_n \frac{1}{n}\left|\{k \le n : |x_k - x_N| \ge \varepsilon\}\right| = 0.$$

Gadjiev and Orhan [29] introduced the statistical convergence for degree $0 < \beta < 1$. Later, Çolak [30] studied the statistical convergence of order α and strong p-Cesaro summability of order α.

Let $x = (x_k) \in \omega$ and $\alpha \in (0,1]$. The sequence (x_k) is named statistically convergent of order α if there is a complex number ℓ such that $\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \le n : |x_k - \ell| \ge \varepsilon\}| = 0$ i.e. for a.a.k(α) $|x_k - \ell| < \varepsilon$ for all $\varepsilon > 0$, and so we mean that x is statistically convergent of order α, to ℓ. In the present case we write $S^\alpha - lim x_k = \ell$.

Let $\alpha \in (0,1]$ and $p \in \mathbb{R}^+$. A sequence $x = (x_k)$ is called strongly p-Cesaro summable of order α, if there exists a complex number ℓ such that $\lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{k=1}^{n} |x_k - \ell|^p = 0$.

Now, we give two famous theorems that we will use in the continuation of this study.

Theorem 2.1. A matrix $A = (a_{nk})_{n,k=1}^{\infty}$ is regular if and only if the three conditions below hold:

i. There is a $K > 0$ so much so that $\sum_{k=1}^{\infty} |a_{nk}| \le K$ holds for every $n = 1,2,3,\ldots$;

ii. $\lim_{n\to\infty} a_{nk} = 0$ for every $k = 1,2,3,\ldots$;

iii. $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$ [31].

Theorem 2.2. Let X be a $BK-$space. Then, X_T is a $BK-$space such that $\|X\|_T = \|T(x)\|$ for all $x \in X_T$ [32].

3. Main Results

3.1. Lucas matrix and Lucas Sequence Space

Now, we define the following Lucas matrix

$$\hat{H} = (L_{nk}) = \begin{cases} \dfrac{L_k}{L_{n+2}-3}, & 1 \le k \le n \\ 0, & other \end{cases}.$$

If we write the terms of this matrix, we get the following matrix:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{8} & \frac{3}{8} & \frac{4}{8} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{15} & \frac{3}{15} & \frac{4}{15} & \frac{7}{15} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{26} & \frac{3}{26} & \frac{4}{26} & \frac{7}{26} & \frac{11}{26} & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

This matrix is clearly triangular, that is $L_{nn} \neq 0$ and $L_{nk} = 0$ for $k > n$. However, H is regular since the conditions of the Silverman-Toeplitz theorem are hold. On the other hand, the inverse of the matrix H is defined as follows:

$$H^{-1} = \begin{cases} \dfrac{L_{n+2}-3}{L_k}, & k = n \\ -\dfrac{L_{n+1}-3}{L_{k+1}}, & k = n-1 \\ 0, & other \end{cases}$$

Now, let us decsribe $y = (y_k) = H_k(x)$ which is named the $H-$transform of a sequence $x = (x_k)$ by

$$y = (y_k) = H_k(x) = \frac{1}{L_{k+2}-3}\sum_{i=1}^{k} L_i x_i . \tag{1}$$

By using this transformation, we present the Lucas sequence space

$$X(H) = \{x = (x_k) \in w : y = (y_k) \in X\}.$$

Theorem 3.1.1 The space $X(H)$ is $BK-$space normed by

$$\|x\|_{X(H)} = \|H(x)\|_X = \|y\|_X = \begin{cases} \sup_k |y_k|, & X = \ell_\infty, c \text{ or } c_0 \\ \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{1/p}, & X = \ell_p (1 \le p < \infty) \end{cases}. \tag{2}$$

Proof. Since the matrix H is triangular, the space $X(H)$ is $BK-$space by (2) and Theorem 2.2.

The following two theorems can be proved by similar techniques in Karakaş [7]. So, we give them without proof.

Theorem 3.1.2. The space $X(H)$ is isometrically isomorphic to the space X, that is $X(H) \cong X$.

Theorem 3.1.3. The inclusion $c_0(H) \subset c(H) \subset \ell_\infty(H)$ strictly holds.

3.2. Statistical convergence of order α defined by Lucas sequence

In this section, we'll take the sequence $Hx=(Hx)_k$ which is constructed by the terms of Lucas matrix instead of the sequence $x=(x_k)$ and give the notion of statistical convergence of order α.

Definition 3.2.1. Let $Hx=(Hx_k)\in w$ and $0<\alpha\leq 1$. If there exists a complex number ℓ such that $\lim\limits_{n\to\infty}\frac{1}{n^\alpha}|\{k\leq n:|Hx_k-\ell|\geq\varepsilon\}|=0$, then the sequence $Hx=(Hx_k)$ is called Lucas type statistiscally convergent of order α to ℓ, that is to say, if for every $\varepsilon>0$ and $a.a.\mathrm{k}(\alpha)$, $|Hx_k-\ell|<\varepsilon$, we state that Hx is Lucas type statistically convergent of order α to ℓ. We write for this $S^\alpha(H)-limx_k=\ell$

We'll use the notation $S_0^\alpha(H)$ to show the set of all Lucas type statistically convergent null sequences of order α. Lucas type statistical convergence of order α is well-defined for $\alpha\in(0,1]$, but for $\alpha>1$, it isn't well-defined. To see this, let us take the sequence

$$x=(x_k)=\left(0,\frac{4}{3},-1,\frac{15}{7},-\frac{15}{11},\frac{44}{18},-\frac{44}{29},\frac{120}{47},\ldots\right).$$

Hence, we obtain the sequence $Hx=(Hx_k)$ as follows:

$$(Hx_k)=\begin{cases}1, & k=2n\\ 0, & k\neq 2n\end{cases}\quad n=1,2,3,\ldots$$

In this case, both

$$\lim_{n\to\infty}\frac{1}{n^\alpha}|\{k\leq n:|Hx_k-1|\geq\varepsilon\}|\leq\lim_{n\to\infty}\frac{n}{2n^\alpha}=0 \text{ and } \lim_{n\to\infty}\frac{1}{n^\alpha}|\{k\leq n:|Hx_k|\geq\varepsilon\}|\leq\lim_{n\to\infty}\frac{n}{2n^\alpha}=0.$$

Accordingly, $S^\alpha(H)-limx_k=1$ and $S^\alpha(H)-limx_k=0$, but it is impossible.

Theorem 3.2.2. Let $\alpha\in(0,1]$ and $x=(x_k), y=(y_k)$ be sequences of complex numbers. Then,

i. $S^\alpha(H)-\lim cx_k=cx_0$, if $c\in\mathbb{C}$ and $S^\alpha(H)-limx_k=x_0$.
ii. $S^\alpha(H)-lim(x_k+y_k)=x_0+y_0$, if $S^\alpha(H)-limx_k=x_0$ and $S^\alpha(H)-\lim y_k=y_0$.

Proof. It can be demonstrated by using similar method in Çolak [24].

Every convergent sequence is Lucas type statistically convergent of order α but the opposite is not true. For instance, if we consider the sequence $x=(x_k)=\left(1,-\frac{1}{3},0,0,0,0,0,\frac{120}{47},\ldots\right)$, the sequence $Hx=(Hx_k)$ is equal to

$$(Hx_k)=\begin{cases}1, & k=n^3\\ 0, & k\neq n^3\end{cases}. \tag{3}$$

Since $S^\alpha(H)-limx_k=0$ for $\alpha>\frac{1}{3}$, the sequence is Lucas type statistically convergent of order α. But it is not convergent.

Theorem 3.2.3. $S^\alpha(H)\subseteq S^\beta(H)$ for $0<\alpha\leq\beta\leq 1$ and the inclusion is strict for some α and β where $\alpha<\beta$.

Proof. For every $\varepsilon > 0$, the inequality

$$\frac{1}{n^{\beta}} \left| \{ k \le n : | Hx_k - \ell | \ge \varepsilon \} \right| \le \frac{1}{n^{\alpha}} \left| \{ k \le n : | Hx_k - \ell | \ge \varepsilon \} \right|$$

holds and so it is obtained that $S^{\alpha}(H) \subseteq S^{\beta}(H)$ for $0 < \alpha \le \beta \le 1$. Now, let us show the strictness of the inclusion. To do this, take the sequence

$$x = (x_k) = \left(1, -\frac{1}{3}, 0, \frac{15}{7}, -\frac{15}{11}, 0, 0, 0, \ldots \right). \tag{4}$$

Then, we find the sequence $Hx = (Hx_k)$ such that $(Hx_k) = \begin{cases} 1, & k = n^2 \\ 0, & k \ne n^2 \end{cases}$ and this gives that $S^{\beta}(H) - lim x_k = 0$, that is $x \in S^{\beta}(H)$ for $\beta \in \left(\frac{1}{2}, 1 \right]$ but $x \notin S^{\alpha}(H)$ for $\alpha \in \left(0, \frac{1}{2} \right]$.

From Theorem 3.2.3, we can give the result below:

Corollary 3.2.4. $S^{\alpha}(H) = S^{\beta}(H)$ if and only if $\alpha = \beta$.

Definition 3.2.5. Let $p \in \mathbb{R}^{+}$ and $\alpha \in (0,1]$. If there is a complex number ℓ such that $\lim_{n \to \infty} \frac{1}{n^{\alpha}} \sum_{k=1}^{n} \left| Hx_k - \ell \right|^{p} = 0$, a sequence $Hx = (Hx_k)$ is called Lucas type strongly $p-$Cesaro summable of order α. We'll apply the notation $w_p^{\alpha}(H)$ for the set of all Lucas type strongly $p-$Cesaro summable sequences of order α.

Theorem 3.2.6. $w_p^{\alpha}(H) \subseteq w_p^{\beta}(H)$ for $0 < \alpha \le \beta \le 1$ and $p \in \mathbb{R}^{+}$. Also, the inclusion is strict for some α and β where $\alpha < \beta$.

Proof. By using the sequence $x = (x_k)$ defined in (4), the required result is easily obtained by standard method in Çolak [24].

Corollary 3.2.7. Let $0 < \alpha \le \beta \le 1$ and $p \in \mathbb{R}^{+}$. Then, $w_p^{\alpha}(H) = w_p^{\beta}(H)$ if and only if $\alpha = \beta$.

Theorem 3.2.8. Let $0 < p < \infty$ and α, β be fixed real numbers such that $0 < \alpha \le \beta \le 1$. Then, a sequence is Lucas type statistically convergent of order β if it is Lucas type strongly $p-$Cesaro summable of order α.

Proof. We can write $\sum_{k=1}^{n} \left| Hx_k - \ell \right|^{p} \ge \left| \left\{ k \le n : \left| Hx_k - \ell \right|^{p} \ge \varepsilon \right\} \right| . \varepsilon^{p}$ for any sequence $Hx = (Hx_k)$ and $\varepsilon > 0$. Therefore, we get

$$\frac{1}{n^{\alpha}} \sum_{k=1}^{n} \left| Hx_k - \ell \right|^{p} \ge \frac{1}{n^{\alpha}} \left| \left\{ k \le n : \left| Hx_k - \ell \right|^{p} \ge \varepsilon \right\} \right| . \varepsilon^{p} \ge \frac{1}{n^{\beta}} \left| \left\{ k \le n : \left| Hx_k - \ell \right|^{p} \ge \varepsilon \right\} \right| . \varepsilon^{p}.$$

From this inequality, the proof is completed.

Remark 3.2.9. If we look closely, the opposite of the previous theorem is not generally true. It is seen that a sequence is not obliged to Lucas type strongly $p-$Cesaro summable of order α for $\alpha \in (0,1)$ when it is bounded and Lucas type statistically convergent of order α. As an example, we consider the following sequence:

$$(Hx_k) = \begin{cases} \frac{1}{\sqrt{k}}, & k \neq m^3 \\ 1, & k = m^3 \end{cases} .$$

So, it is clear that $x \in S^{\alpha}(H)$. Now, let us remember the inequality $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$ for every positive integer $n \geq 2$ and take $M_n = \{k \leq n : k \neq m^3, m = 1,2,3,...\}$ and choose $p=1$. Due to

$$\begin{aligned} \sum_{k=1}^{n} |Hx_k|^p &= \sum_{k=1}^{n} |Hx_k| = \sum_{k \in M_n, 1 \leq k \leq n} |Hx_k| + \sum_{k \notin M_n, 1 \leq k \leq n} |Hx_k| \\ &= \sum_{k \in M_n, 1 \leq k \leq n} \frac{1}{\sqrt{k}} + \sum_{k \notin M_n, 1 \leq k \leq n} 1 > \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n} \end{aligned}$$

we obtain

$$\frac{1}{n^{\alpha}} \sum_{k=1}^{n} |Hx_k|^p = \frac{1}{n^{\alpha}} \sum_{k=1}^{n} |Hx_k| > \frac{1}{n^{\alpha}} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \frac{1}{n^{\alpha}} \sqrt{n} = \frac{1}{n^{\alpha - \frac{1}{2}}} \to \infty .$$

From here, if $p=1$, $x \notin w_p^{\alpha}(H)$ for $\alpha \in \left(0, \frac{1}{2}\right)$. As a consequence, $x \in S^{\alpha}(H) - w_p^{\alpha}(H)$.

4. Conclusion

Fibonacci and Lucas numbers have become a part of approximation of introducing a sequence space by tha aid of matrix domain of an infinite matrix in the last decade. Accordingly, we also define new matrix and new sequence space by means of Lucas numbers. In addition to this, we investigate the notions of statistical convergence and strongly $p-$Cesaro summability with degree α by using the sequences obtained from the terms of the matrix we define. The results in this article can be extended and studied by researchers for different types of statistical convergence. Also, it can be considered from a different perspective.

Author's Contributions

This study is generated by the master thesis titled "Statistical Convergence of Order α Defined By Lucas Numbers" of Hacer Dönmez and her supervisor Murat Karakaş.

Statement of Conflicts of Interest

No conflicts of interest was reported by authors.

Statement of Research and Publication Ethics

The authors declare that this study complies with Research and Publication Ethics.

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