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INVERSE CONTINUOUS WAVELET TRANSFORM IN WEIGHTED VARIABLE EXPONENT AMALGAM SPACES

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Abstract. The wavelet transform is an useful mathematical tool. It is a mapping of a time signal to the time-scale joint representation. The wavelet transform is generated from a wavelet function by dilation and translation. This wavelet function satisfies an admissible condition so that the original signal can be reconstructed by the inverse wavelet transform. In this study, we firstly give some basic properties of the weighted variable exponent amalgam spaces. Then we investigate the convergence of the $\theta-$means of f in these spaces under some conditions. Finally, using these results the convergence of the inverse continuous wavelet transform is considered in these spaces.

1. Introduction

Recently, the variable exponent Lebesgue $L^{p(\cdot)}(\mathbb{R}^d)$ spaces and a class of nonlinear problems with variable exponential growth have been new and interesting topics. The space has several applications, such as electrorheological fluids (see [31]), elastic mechanics (see [43]) and image processing model. Moreover, the spaces $L^{p(\cdot)}(\mathbb{R}^d)$ and $L^{p}(\mathbb{R}^d)$ have many common properties, such as Banach space, reflexivity, separability, uniform convexity, Hölder inequalities and embeddings. One of the most important differences between these spaces is that the space $L^{p(\cdot)}(\mathbb{R}^d)$ is not translation invariant [27]. It is also well known that the maximal operator is bounded in $L^{p(\cdot)}(\mathbb{R}^d)$. For more comprehensive information (see [10], [12], [13] and [14]).

The amalgam of L^p and l^q on the real line is the space (L^p, l^q), which is also larger than the space L^p, consisting of functions which are locally in L^p and have l^q

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behavior at infinity. Many different forms of amalgam spaces have been studied by some authors (see [25], [33], [24], [15] and [18]). Moreover, this space play important roles in recent developments in time frequency analysis and sampling theory, which are modern branches of harmonic analysis. Signal analysis and wireless communication issues are quite popular in amalgam spaces (see [20]).

Variable exponent amalgam spaces $\left(L^{p(\cdot)}, l^q\right)$ and some basic properties, such as Banach function space, Hölder type inequalities, interpolation, bilinear multipliers and the boundedness of maximal operator, have been investigated recently. Some interesting articles have been published on this subject, but not many. So there are many open problems in this function spaces [5], [21], [26], [30], [22], [28], [3], [7], [2], [6].

The so called θ-summation method is investigated by some authors, such as [36], [32], [38], [39], [40], [34], [8]. The θ-summation is defined by

$$\sigma_T^\theta f(x) = \int\limits_{\mathbb{R}^d} f(x-t)\, T^n \theta(Tt)\, dt$$

for an integrable function θ on $\mathbb{R}$. This summability is a generalized form of the well-known summability methods, like Fejér, Riesz, Weierstrass, Abel, etc. by a suitable chosen of θ. Feichtinger and Weisz ([16], [17], [42]) showed that the θ-means $\sigma_T^\theta f$ converges to f almost everywhere and in norm as $T \to \infty$ for $f \in L^p(\mathbb{R}^d)$, (L^p, l^q). Also we characterize the points of the set of a.e. convergence as the Lebesgue points. Moreover, Uribe and Fiorenza [10], Szarvas and Weisz [34] obtained similar results for the space $L^{p(\cdot)}\left(\mathbb{R}^d\right)$.

In this study we will discuss the convergence of the inverse continuous wavelet transform in weighted variable exponent amalgam spaces. Also, we investigate the convergence of the θ-means of f almost everywhere and in norm in these spaces under which conditions. Hence we obtain more general results with respect to [34].

2. Weighted Variable Exponent Lebesgue and Amalgam spaces

In this section we give some required definitions and information about wavelet transform and weighted variable exponent amalgam spaces.

Definition 1. *Let $x \in \mathbb{R}^d$, $s \in \mathbb{R}$ and $s \neq 0$. The continuous wavelet transform is defined by*

$$W_g f(x,s) = |s|^{-\frac{d}{2}} \int\limits_{\mathbb{R}^d} f(t)\, \overline{g\left(s^{-1}(t-x)\right)} dt = \langle f, T_x D_s g \rangle$$

for f and g, where D_s is the dilation operator, and T_x is the translation operator, i.e.,

$$D_s f(t) = |s|^{-\frac{d}{2}} f\left(\frac{t}{s}\right) \text{ and } T_x f(t) = f(t-x) \quad \left(x, t \in \mathbb{R}^d,\ 0 \neq s \in \mathbb{R}\right)$$

[11], [19]. If η is radial, non-increasing as a function on $(0,\infty)$, non-negative, bounded, $|f| \leq \eta$ and $\eta \in L^1\left(\mathbb{R}^d\right)$, then η is a radial majorant of f. If in addition $\eta(.)\ln(|.|+2) \in L^1\left(\mathbb{R}^d\right)$, then η is a radial log-majorant of f.

Definition 2. *A point $x \in \mathbb{R}^d$ is called a Lebesgue point $f \in L^1_{loc}\left(\mathbb{R}^d\right)$ if*

$$\lim_{h\to 0+}\left(\frac{1}{|B(0,h)|}\int\limits_{B(0,h)} |f(x+u)-f(x)|\,du\right)=0,$$

where

$$B(a,\delta)=\left\{x\in\mathbb{R}^d : \|x-a\|<\delta\right\}.$$

Definition 3. *Let $g^*(x)=\overline{g(-x)}$ be involution operator. Then the operators $\rho_S f$ and $\rho_{S,T} f$ are defined by*

$$\rho_S f=\int\limits_S^\infty \int\limits_{\mathbb{R}^d} W_g f(x,s)\,T_x D_s\gamma \frac{dxds}{s^{d+1}}$$

and

$$\rho_{S,T} f=\int\limits_S^T \int\limits_{\mathbb{R}^d} W_g f(x,s)\,T_x D_s\gamma \frac{dxds}{s^{d+1}},$$

where $0<S<T<\infty$. Let define the operator $C'_{g,\gamma}$ with

$$C'_{g,\gamma}=-\int\limits_{\mathbb{R}^d}(g^* * \gamma)(x)\ln(|x|)\,dx.$$

Then $C'_{g,\gamma}$ is finite [29], where g and γ both have radial log-majorants.

Let g and γ be radial, i.e., $\int_{\mathbb{R}^d}(g^* * \gamma)(x)\,dx=0$. Assume that g and γ have a radial log-majorant. Then we get

$$\lim_{S\to 0^+, T\to\infty}\rho_{S,T} f(x)=\lim_{S\to 0^+}\rho_S f(x)=C'_{g,\gamma} f(x)$$

at every Lebesgue point for any $f \in L^p\left(\mathbb{R}^d\right)$ $(1\leq p<\infty)$. The convergence is proved with respect to L^p-norm for $T=\infty$, [29]. Under some similar conditions, Weisz has proved similar results [41].

Definition 4. *Let $p(.)$ be a measurable function from $\mathbb{R}^d$ into $[1,\infty)$ (called a variable exponent on $\mathbb{R}^d$) satisfying the condition $1\leq p^-\leq p(.)\leq p^+<\infty$, where*

$$p^-=\operatorname*{ess\,inf}_{x\in\mathbb{R}^d} p(x), \quad p^+=\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} p(x).$$

The set $P(\mathbb{R}^d)$ denotes variable exponents on $\mathbb{R}^d$. Let $p(.) \in P(\mathbb{R}^d)$. The variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^d)$ consist of all measurable functions f such that $\varrho_{p(.)}(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0 : \varrho_{p(.)}(\frac{f}{\lambda}) \leq 1\right\},$$

where

$$\varrho_{p(.)}(f) = \int\limits_{\mathbb{R}^d} |f(x)|^{p(x)}\, dx.$$

If $p^+ < \infty$, then $f \in L^{p(\cdot)}(\mathbb{R}^d)$ iff $\varrho_{p(.)}(f) < \infty$. The space $\left(L^{p(\cdot)}(\mathbb{R}^d), \|.\|_{p(.)}\right)$ is a Banach space. If $p(.) = p$ is a constant function, then the norm $\|.\|_{p(.)}$ coincides with the usual Lebesgue norm $\|.\|_p$, [27]. A measurable and locally integrable function $\omega : \mathbb{R}^d \longrightarrow (0,\infty)$ is called a weight function. The weighted modular is defined by

$$\varrho_{p(.),\omega}(f) = \int\limits_{\mathbb{R}^d} |f(x)|^{p(x)}\, \omega(x)dx.$$

The space $L^{p(\cdot)}_{\omega}(\mathbb{R}^d)$ is of all measurable functions such that $\|f\|_{L^{p(\cdot)}_{\omega}(\mathbb{R}^d)} = \left\|f\omega^{\frac{1}{p(\cdot)}}\right\|_{p(.)} < \infty$. The dual space of $L^{p(\cdot)}_{\omega}(\mathbb{R}^d)$ is $L^{q(\cdot)}_{\omega^*}(\mathbb{R}^d)$, where $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$ and $\omega^* = \omega^{1-q(.)} = \omega^{-\frac{1}{p(.)-1}}$. Also, $L^{p(\cdot)}_{\omega}(\mathbb{R}^d)$ is a uniformly convex Banach space, thus reflexive for $1 < p^- \leq p(.) \leq p^+ < \infty$, [3], [4].

Definition 5. *The maximal operator M is defined by*

$$M(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|\, dy,$$

for $f \in L^1_{loc}(\mathbb{R})$.

Hästö and Diening [23] defined the class $A_{p(.)}$ consists of those weights ω such that

$$\|\omega\|_{A_{p(.)}} = \sup_{B\in ß} |B|^{-p_B} \|\omega\|_{L^1(B)} \left\|\frac{1}{\omega}\right\|_{L^{\frac{p'(.)}{p(.)}}(B)} < \infty,$$

where ß denotes the set of all balls in $\mathbb{R}^d$, $p_B = \left(\frac{1}{|B|}\int_B \frac{1}{p(x)}dx\right)^{-1}$ and $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$. If $p(.)$ is a constant function, then $A_{p(.)} = A_p$, where A_p is ordinary Muckenhoupt class.

If $p(.)$ satisfies the following inequality

$$|p(x) - p(y)| \leq \frac{C}{\log\left(e + \frac{1}{|x-y|}\right)}$$

for all $x, y \in \mathbb{R}^d$, then $p(.)$ provides the local log-Hölder continuity condition. Moreover, if the inequality

$$|p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)}$$

holds for some $p_\infty > 1$, $C > 0$ and all $x \in \mathbb{R}^d$, then we say that $p(.)$ satisfies the local log-Hölder decay condition. We denote by $P^{\log}(\mathbb{R}^d)$ the class of variable exponents which are log-Hölder continuous, i.e. which satisfy the local log-Hölder continuity condition and local log-Hölder decay condition [4], [37].

Let $p \in P^{\log}(\mathbb{R}^d)$ and $1 < p^- \leq p(.) \leq p^+ < \infty$. Then $M : L^{p(.)}_\omega(\mathbb{R}^d) \hookrightarrow L^{p(.)}_\omega(\mathbb{R}^d)$ if and only if $\omega \in A_{p(.)}$ by Theorem 1.1 in [23].

The space $L^{p(.)}_{loc,\omega}(\mathbb{R}^d)$ is to be space of functions on $\mathbb{R}^d$ such that f restricted to any compact subset K of $\mathbb{R}^d$ belongs to $L^{p(.)}_w(\mathbb{R}^d)$.

In this study we take $d = 1$, and define the weighted variable exponent amalgam spaces on $\mathbb{R}$.

Definition 6. *Let $1 \leq p(.), q < \infty$ and $J_k = [k, k+1)$, $k \in \mathbb{Z}$. The weighted variable exponent amalgam spaces $\left(L^{p(.)}_\omega, l^q\right)$ are defined by*

$$\left(L^{p(.)}_\omega, l^q\right) = \left\{ f \in L^{p(.)}_{loc,\omega}(\mathbb{R}) : \|f\|_{\left(L^{p(.)}_\omega, l^q\right)} < \infty \right\},$$

where $\|f\|_{\left(L^{p(.)}_\omega, l^q\right)} = \left(\sum_{k\in\mathbb{Z}} \left\|f\chi_{J_k}\right\|^q_{L^{p(.)}_\omega(\mathbb{R})}\right)^{\frac{1}{q}}$. If the weight ω is a constant function, then the space $\left(L^{p(.)}_\omega, l^q\right)$ coincides with $\left(L^{p(.)}, l^q\right)$ (see [7], [26]).

In 2014, Meskhi and Zaighum showed that the maximal operator is bounded in weighted variable exponent amalgam spaces under some conditions [30].

Throughout this paper, we assume that $p(.) \in P^{\log}(\mathbb{R})$, $1 < p^- \leq p(.) \leq p^+ < \infty$ and $\omega \in A_{p(.)}$.

3. $\theta-$Summability on the Weighted Variable Exponent Wiener Amalgam Spaces

Lemma 1. *Let $1 \leq p(.), q < \infty$ and $0 < c \leq \omega$. Then the inclusion $\left(L^{p(.)}_\omega, l^q\right) \subset \left(L^1, l^\infty\right)$ holds.*

Proof. Take any $f \in \left(L^{p(.)}_\omega, l^q\right)$. It is well known that $f \in \left(L^{p(.)}_\omega, l^q\right)$ if and only if $\left\{\|f\|_{L^{p(.)}_\omega[k,k+1)}\right\}_{k\in\mathbb{Z}} \in l^q$. If we use Proposition 3.5 in [3] and the definition of $\|.\|_{(L^1,l^\infty)}$, then we have $L^{p(.)}_\omega[k,k+1) \hookrightarrow L^1_\omega[k,k+1) \hookrightarrow L^1[k,k+1)$, $l^q \hookrightarrow l^\infty$ for $1 \leq p(.), q < \infty$, $0 < c \leq \omega$ and so

$$\|f\|_{(L^1,l^\infty)} = \sup_{k\in\mathbb{Z}} \|f\|_{L^1[k,k+1)} \leq C \sup_{k\in\mathbb{Z}} \|f\|_{L^{p(.)}_\omega[k,k+1)}$$

$$\leq \quad C\left(\sum_{k\in\mathbb{Z}}\|f\|^q_{L^{p(.)}_\omega[k,k+1)}\right)^{\frac{1}{q}} = C\,\|f\|_{\left(L^{p(.)}_\omega,l^q\right)} < \infty.$$

Hence we obtain that $f \in \left(L^1, l^\infty\right)$ and $\left(L^{p(.)}_\omega, l^q\right) \subset \left(L^1, l^\infty\right).$ □

Theorem 1. *Let $1 \leq q \leq p^- \leq p\,(.) \leq p^+ < \infty$ and $0 < c \leq \omega$. Then the inclusion*

$$\left(L^{p(.)}_\omega, l^q\right) \hookrightarrow L^q_\omega \hookrightarrow L^q,$$

hold for all $f \in \left(L^{p(.)}_\omega, l^q\right)$.

Proof. Let $f \in \left(L^{p(.)}_\omega, l^q\right)$ be given. Then we get $\left(L^{p(.)}_\omega, l^q\right) \hookrightarrow \left(L^{p^-}_\omega, l^q\right) \hookrightarrow \left(L^q_\omega, l^q\right) = L^q_\omega$ by Proposition 3.5 in [3] and [24]. Hence we have that there exists a $C > 0$ such that the inequality

$$\|f\|_{L^q} \leq C\,\|f\|_{\left(L^{p(.)}_\omega,l^q\right)}$$

holds for any $f \in \left(L^{p(.)}_\omega, l^q\right)$. This completes the proof. □

Definition 7. *Let $\theta \in L^1(\mathbb{R})$ be radial function. The θ-means of $f \in \left(L^{p(.)}_\omega, l^q\right)$ is defined by*

$$\sigma^\theta_T f\,(x) := (f * \theta_T)\,(x) = \int_{\mathbb{R}} f\,(x-t)\,\theta_T\,(t)\,dt,$$

where

$$\theta_T\,(t) := T^d\theta\,(Tt)\,,\ (x \in \mathbb{R},\ T > 0).$$

Theorem 2. *Let $1 \leq p\,(.)\,, q < \infty$ and $0 < c \leq \omega$. Assume that θ has radial majorant. Then;*

i) *The limit*

$$\lim_{T\to\infty} \sigma^\theta_T f\,(x) = \int_{\mathbb{R}} \theta\,(y)\,dy.f(x)$$

is valid for any Lebesgue point of $f \in \left(L^{p(.)}_\omega, l^q\right)$.

ii) *If in addition $1 \leq q \leq p^- \leq p\,(.) \leq p^+ < \infty$, then the following limit equality*

$$\lim_{T\to 0^+} \sigma^\theta_T f\,(x) = 0$$

is available for all $f \in \left(L^{p(.)}_\omega, l^q\right)$ and $x \in \mathbb{R}$.

Proof. **i)** Let $x \in \mathbb{R}$ be a Lebesgue point of f. Since there exists the inclusion $\left(L_{\omega}^{p(.)}, l^{q}\right) \subset \left(L^{1}, l^{\infty}\right)$ by Lemma 1, we write that

$$\underset{T\to\infty}{lim}\, \sigma_{T}^{\theta} f(x) = \int\limits_{\mathbb{R}^{n}} \theta(y)\, dy.f(x)$$

for $f \in \left(L_{\omega}^{p(.)}, l^{q}\right)$by Theorem 2.2 in [34].

ii) Take any $f \in \left(L_{\omega}^{p(.)}, l^{q}\right)$ and $x \in \mathbb{R}$. By Theorem 1 and Theorem 2.3 in [34], we have that $f \in L^{q}(\mathbb{R})$ and

$$\underset{T\to 0^{+}}{lim}\, \sigma_{T}^{\theta} f(x) = 0.$$

□

Proposition 1. $C_{c}(\mathbb{R})$*, which consists of continuous functions on* $\mathbb{R}$ *whose support is compact, is dense in* $\left(L_{\omega}^{p(.)}, l^{q}\right)$ *for* $1 \leq p(.), q < \infty$ *(see Proposition 2.9 in [6]).*

Theorem 3. *For all* $f \in \left(L_{\omega}^{p(.)}, l^{q}\right)$ *the following statements are valid :*

i) $\left\|\sigma_{T}^{\theta} f\right\|_{\left(L_{\omega}^{p(.)}, l^{q}\right)} \leq C \left\|f\right\|_{\left(L_{\omega}^{p(.)}, l^{q}\right)}$ $(T > 0)$.

ii) $\underset{T\to\infty}{lim}\, \sigma_{T}^{\theta} f = \int\limits_{\mathbb{R}} \theta(x)\, dx.f$ *in the* $\left(L_{\omega}^{p(.)}, l^{q}\right)$*-norm.*

iii) $\underset{T\to 0^{+}}{lim}\, \sigma_{T}^{\theta} f = 0$ *in the* $\left(L_{\omega}^{p(.)}, l^{q}\right)$*-norm.*

Proof. **i)** It is well known that the maximal operator is bounded in $\left(L_{\omega}^{p(.)}, l^{q}\right)$ [30]. Then we have that

$$\left\|\sigma_{T}^{\theta} f\right\|_{\left(L_{\omega}^{p(.)}, l^{q}\right)} \leq C \left\|f\right\|_{\left(L_{\omega}^{p(.)}, l^{q}\right)} (T > 0)$$

for all $f \in \left(L_{\omega}^{p(.)}, l^{q}\right)$ by Theorem 2.1 in [34].

ii) Also, if we follow Theorem 3.8 in [34], Theorem 2.3 in [9], Theorem 5.11 in [10], and Theorem 8 in [1], then we have that

$$\underset{T\to\infty}{lim}\, \sigma_{T}^{\theta} f = \int\limits_{\mathbb{R}} \theta(x)\, dx.f$$

in the $\left(L_{\omega}^{p(.)}, l^{q}\right)$-norm.

iii) Let $\epsilon > 0$ be given. Using Proposition 1, it is obtained that the following inequality

$$\left\|f - g\right\|_{\left(L_{\omega}^{p(.)}, l^{q}\right)} < \epsilon$$

is valid for $g \in C_c(\mathbb{R})$, whose compact support supp$g$ is K. Using i) and Proposition 2 in [7], we have that

$$\begin{aligned} \left\|\sigma_T^\theta f\right\|_{\left(L_\omega^{p(\cdot)},l^q\right)} &\leq \left\|\sigma_T^\theta (f-g)\right\|_{\left(L_\omega^{p(\cdot)},l^q\right)} + \left\|\sigma_T^\theta g\right\|_{\left(L_\omega^{p(\cdot)},l^q\right)} \\ &< C\epsilon + |S(K)|^{\frac{1}{q}} \left\|\sigma_T^\theta g\right\|_{L_\omega^{p(\cdot)}(K)}. \end{aligned}$$

Also using Theorem 3.8 in [34], we get the limit

$$\underset{T\to 0^+}{lim} \left\|\sigma_T^\theta g\right\|_{L_\omega^{p(\cdot)}(K)} = 0.$$

So this completes the proof. □

4. Convergence of ρ_S and $\rho_{S,T}$

Theorem 4. *Assume that g,γ have radial log-majorants and $\int_{\mathbb{R}} (g^* * \gamma)(x)\,dx = 0$. If $\omega \in A_1$ and $0 < c \leq \omega$, then for all $f \in \left(L_\omega^{p(\cdot)}, l^q\right)$ the following relation holds;*

$$\rho_S f = \sigma_{\frac{1}{S}}^\theta f,\ \ (S > 0)$$

where

$$\theta(y) = \int_1^\infty (g^* * \gamma)\left(\frac{y}{u}\right) \frac{1}{u^{n+1}} \chi_{B(0,1)}(y)\,du - \int_0^1 (g^* * \gamma)\left(\frac{y}{u}\right) \frac{1}{u^{n+1}} \chi_{\mathbb{R}^n \setminus B(0,1)}(y)\,du.$$

Proof. Let $f \in L_\omega^1 \cap \left(L_\omega^{p(\cdot)}, l^q\right)$ and $y \in \mathbb{R}$. Then we have decomposition of $\rho_S f\ (y)$ as

$$\begin{aligned} \rho_S f(y) &= \int_S^\infty \int_{\mathbb{R}} \frac{1}{s^3} \int_{\mathbb{R}} f(t)\, \overline{g\left(\frac{t-x}{s}\right)} \gamma\left(\frac{y-x}{s}\right) dt dx ds \\ &= \int_S^\infty \int_{|y-t|<S} \frac{1}{s^3} \int_{\mathbb{R}} f(t)\, \overline{g\left(\frac{t-x}{s}\right)} \gamma\left(\frac{y-x}{s}\right) dx dt ds \\ &\quad - \int_0^S \int_{|y-t|\geq S} \frac{1}{s^3} \int_{\mathbb{R}} f(t)\, \overline{g\left(\frac{t-x}{s}\right)} \gamma\left(\frac{y-x}{s}\right) dx dt ds \\ &\quad + \int_0^\infty \int_{|y-t|\geq S} \frac{1}{s^3} \int_{\mathbb{R}} f(t)\, \overline{g\left(\frac{t-x}{s}\right)} \gamma\left(\frac{y-x}{s}\right) dx dt ds \\ &= I - II + III \end{aligned}$$

by from [29], [34] . Also it is well known that

$$I=\left(f*\varphi_{\frac{1}{S}}\right)(y) \text{ and } II=\left(f*\psi_{\frac{1}{S}}\right)(y),$$

where

$$\varphi(t)=\int\limits_{1}^{\infty}(g^{*}*\gamma)\left(\frac{t}{u}\right)\frac{1}{u^{n+1}}\chi_{B(0,1)}(t)\,du$$

and

$$\psi(t)=\int\limits_{0}^{1}(g^{*}*\gamma)\left(\frac{t}{u}\right)\frac{1}{u^{n+1}}\chi_{\mathbb{R}^{n}\setminus B(0,1)}(t)\,du$$

by proof of Theorem 1.1 in [29]. On the other hand, Szarvas and Weisz proved that φ and ψ have radial majorants by Theorem 5.1 in [34] in case g and γ have radial log-majorants. Since g,γ have radial log-majorants, $f\in L^{1}_{\omega}$, $\omega\in A_{1}$ and

$$\int\limits_{\mathbb{R}}(g^{*}*\gamma)(x)\,dx=0,$$

then we have

$$\begin{aligned}III &= \int\limits_{0}^{\infty}\int\limits_{|y-t|\geq S}\frac{1}{s^{3}}\int\limits_{\mathbb{R}}f(t)\,\overline{g\left(\frac{t-x}{s}\right)}\gamma\left(\frac{y-x}{s}\right)dxdtds\\ &= \frac{1}{\omega_{0}}\int\limits_{|y-t|\geq S}\frac{f(t)}{|y-t|}\int\limits_{\mathbb{R}}(g^{*}*\gamma)(u)\,dudt=0\end{aligned}$$

by Lemma 2.5 in [29]. Therefore we get

$$\begin{aligned}\rho_{S}f(y) &= \left(f*\varphi_{\frac{1}{S}}\right)(y)-\left(f*\psi_{\frac{1}{S}}\right)(y)+0\\ &= f*\left(\varphi_{\frac{1}{S}}-\psi_{\frac{1}{S}}\right)(y)=f*\theta_{\frac{1}{S}}(y)=\sigma^{\theta}_{\frac{1}{S}}f(y),\end{aligned}$$

where

$$\begin{aligned}\theta(y) &= \varphi(y)-\psi(y)\\ &= \int\limits_{1}^{\infty}(g^{*}*\gamma)\left(\frac{y}{u}\right)\frac{1}{u^{n+1}}\chi_{B(0,1)}(y)\,du-\int\limits_{0}^{1}(g^{*}*\gamma)(y)\,\frac{1}{u^{n+1}}\chi_{\mathbb{R}^{n}\setminus B(0,1)}(y)\,du.\end{aligned}$$

If φ,ψ have radial majorants, then $\theta=\varphi-\psi$ have radial majorant, that is, θ is a non-negative and non-increasing function, and belongs to the space $L^{1}\cap L^{\infty}$. So it is obtained that

$$\|\theta\|_{(L^{\infty},l^{1})}=\sum_{k\in\mathbb{Z}}\left\|\theta\chi_{[k,k+1)}\right\|_{\infty}\leq\sum_{k\in\mathbb{Z}}\theta(k)<\infty$$

and $\theta\in\left(L^{\infty},l^{1}\right)$. Then using Hölder inequality and Lemma 1, we have

$$\begin{aligned} |\rho_S f(y)| &= \left|\sigma^{\theta}_{\frac{1}{S}} f(y)\right| \le \frac{1}{S}\int_{\mathbb{R}} |f(y-t)| \left|\theta\left(\frac{t}{S}\right)\right| dt \\ &\le C\|f\|_{(L^1,l^\infty)} \|\theta\|_{(L^\infty,l^1)} \\ &\le C\|f\|_{\left(L^{p(\cdot)}_\omega,l^q\right)} \|\theta\|_{(L^\infty,l^1)}. \end{aligned}$$

Hence the function ρ_S is linear and bounded from $L^1_\omega \cap \left(L^{p(\cdot)}_\omega, l^q\right)$ to $\mathbb{C}$. Also, it is well known that the inclusion $C_c \subset L^1_\omega \cap \left(L^{p(\cdot)}_\omega, l^q\right) \subset \left(L^{p(\cdot)}_\omega, l^q\right)$. Since C_c is dense in $\left(L^{p(\cdot)}_\omega, l^q\right)$ [6], then we find that $L^1_\omega \cap \left(L^{p(\cdot)}_\omega, l^q\right)$ is dense in $\left(L^{p(\cdot)}_\omega, l^q\right)$. Therefore, from the density principle, the function ρ_s is extended from $\left(L^{p(\cdot)}_\omega, l^q\right)$ to $\mathbb{C}$. This completes the proof. □

Theorem 5. *Let $f \in \left(L^{p(\cdot)}_\omega, l^q\right)$. Moreover, assume that g, γ have radial log-majorants and $\int_{\mathbb{R}} (g^* * \gamma)(x)\,dx = 0$. If $\omega \in A_1$ and $0 < c \le \omega$, then*

i) $\lim\limits_{S\to 0^+} \rho_S f(x) = C'_{g,\gamma} f(x)$
for any Lebesgue point of the function f.
ii) If in addition $1 \le q \le p(.) \le p^+ < \infty$, then
$\lim\limits_{S\to 0^+, T\to\infty} \rho_{S,T} f(x) = C'_{g,\gamma} f(x)$
for any Lebesgue point of the function f .

Proof. **i)** Since $p(.) \in P^{\log}(\mathbb{R})$ and $1 < p^- \le p(.) \le p^+ < \infty$, then $A_1 \subset A_{p(.)}$ [4]. By Theorem 2 and Theorem 4, we deduce that

$$\lim_{S\to 0^+} \rho_S f(x) = \lim_{S\to 0^+} \sigma^{\theta}_{\frac{1}{S}} f(x) = \int_{\mathbb{R}} \theta(y)\,dy f(x)$$

for all Lebesgue points of $f \in \left(L^{p(\cdot)}_\omega, l^q\right)$. On the other hand, using Theorem 5.2 in [34] , we have that $\int_{\mathbb{R}} \theta(y)\,dy = C'_{g,\gamma}$ and

$$\lim_{S\to 0^+} \rho_S f(x) = C'_{g,\gamma} f(x).$$

ii) By Theorem 5.2 in [34] we can write the equality $\rho_{S,T} f(x) = \rho_S f(x) - \rho_T f(x)$ for $x \in \mathbb{R}$. Then using (i), Theorem 2 and Theorem 4, we obtain that

$$\begin{aligned} \lim_{S\to 0^+, T\to\infty} \rho_{S,T} f(x) &= \lim_{S\to 0^+} \rho_S f(x) - \lim_{T\to\infty} \rho_T f(x) \\ &= \lim_{S\to 0^+} \sigma^{\theta}_{\frac{1}{S}} f(x) - \lim_{T\to\infty} \sigma^{\theta}_{\frac{1}{T}} f(x) \\ &= C'_{g,\gamma} f(x) - 0 = C'_{g,\gamma} f(x). \end{aligned}$$

□

Corollary 1. *Assume that g, γ have radial log-majorants, $\int_{\mathbb{R}} (g^* * \gamma)(x)\, dx = 0$. If $\omega \in A_1$ and $0 < c \leq \omega$, then the following statements are valid for any $f \in \left(L_\omega^{p(.)}, l^q\right)$;*

i) $\lim\limits_{S\to 0^+} \rho_s f(x) = C'_{g,\gamma} f(x)$ *a.e.*

ii) *If in addition $1 \leq q \leq p(.) \leq p^+ < \infty$, then* $\lim\limits_{S\to 0^+, T\to\infty} \rho_{s,T} f(x) = C'_{g,\gamma} f(x)$ *a.e.*

Proof. Let $f \in \left(L_\omega^{p(.)}, l^q\right)$. Then by Lemma 1, we have $f \in \left(L^1, l^\infty\right)$. It is known that if $f \in \left(L^1, l^\infty\right)$, then real numbers almost everywhere is a Lebesgue point of f, [16], [17]. Hence by the Theorem 5, we complete the proof. □

Theorem 6. *Assume that g, γ have radial log-majorants and $\int_{\mathbb{R}} (g^* * \gamma)(x)\, dx = 0$. If $\omega \in A_1$ and $0 < c \leq \omega$, then the following results*

i) $\lim\limits_{S\to 0^+} \rho_s f = C'_{g,\gamma} f$,

ii) $\lim\limits_{S\to 0^+, T\to\infty} \rho_{s,T} f = C'_{g,\gamma} f$

are satisfied in the $\left(L_\omega^{p(.)}, l^q\right)$-norm for all $f \in \left(L_\omega^{p(.)}, l^q\right)$.

Proof. **i)** Using $\omega \in A_1 \subset A_{p(.)}$, Theorem 3 and Theorem 4, we have

$$\lim_{S\to 0^+} \rho_S f = \lim_{S\to 0^+} \sigma^\theta_{\frac{1}{S}} = \int_{\mathbb{R}} \theta(y)\, dy f$$

in the $\left(L_\omega^{p(.)}, l^q\right)$-norm for all $f \in \left(L_\omega^{p(.)}, l^q\right)$. On the other hand, since $\int_{\mathbb{R}} \theta(y)\, dy = C'_{g,\gamma}$, then we obtain that

$$\lim_{S\to 0^+} \rho_S f = C'_{g,\gamma} f$$

in the $\left(L_\omega^{p(.)}, l^q\right)$-norm.

ii) Since $\rho_{S,T} f = \rho_S f - \rho_T f$, then we have that

$$\begin{aligned} \lim_{S\to 0^+, T\to\infty} \rho_{s,T} f &= \lim_{S\to 0^+} \rho_S f - \lim_{T\to\infty} \rho_T f \\ &= \lim_{S\to 0^+} \sigma^\theta_{\frac{1}{S}} f - \lim_{T\to\infty} \sigma^\theta_{\frac{1}{T}} f = C'_{g,\gamma} f \end{aligned}$$

in the $\left(L_\omega^{p(.)}, l^q\right)$-norm by (i), Theorem 3 and Theorem 4. □

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