



Analyzing the Competition of HIV-1 Phenotypes with Quantum Game Theory

Bilge OZLUER BASER* *Mimar Sinan Fine Arts University, Department of Statistics, İstanbul, Turkey*

Highlights

- This paper focuses on quantum game theory for biological games.
- Using a quantum perspective is proposed for the micro-level HIV-1 phenotypes game in the study.
- The findings obtained by quantum game theory are compared with the decision of classical approach.

Article Info

*Received: 21 July 2020**Accepted: 15 Sep 2021*

Keywords

*Quantum game theory
Evolutionary game
theory,
HIV-1 phenotypes*

Abstract

HIV-1 (Human Immunodeficiency Virus) is a virus that causes AIDS (Acquired Immunodeficiency Syndrome), which damages the immune system by reducing people's resistance to infections and diseases. Antiretroviral treatment methods are based on drug designs developed using inhibitors that suppress the dynamics that enable the maturation of the virus. However, studies are needed to improve treatment methods against infection because HIV-1 is frequently mutated and mutant viruses develop resistance to the treatment used. Therefore, it is important to model the evolutionary development of the virus. For this purpose, the developmental process and spread of HIV-1 are modeled as a game with the players of phenotypes in this study. The related searches known to be made so far have been carried out based on the rules of classical physics. However, games of survival are being played on the molecular level, where the rules of quantum mechanics work. Since the HIV-1 game is being played on the molecular level, the behaviors of the virus phenotypes are examined from the perspective of quantum computation.

1. INTRODUCTION

HIV-1 is a virus widespread in the world that can cause immunodeficiency syndrome or AIDS, if it is not treated. HIV-1 attacks the immune system and spreads through blood and lymph nodes. Like all other viruses, HIV-1 cannot replicate on its own and needs live cells to reproduce itself. The primary target of HIV-1 is lymphocytes called CD4+T cells. In an infected person, while the CD4+T cells rapidly decrease over time, the number of HIV-1 greatly increases. With high viral load and low CD4+T cells, the body's defense mechanism is broken and becomes apparent to many other infections. Such a propensity for rapid replication and high degree of mutation brings diversity and evolutionary advantage for HIV-1. Therefore, if the nature of the virus mutation can be patterned and discoveries on the way of inhibiting the mutation can be obtained, then strongly important developments can be achieved in the therapy of infections. The studies on the developmental process of HIV-1 can be categorized into four main groups such as agent-based modeling [1], differential equations [2], Markov models [3], and game theory [4-5].

This work aims to examine the pattern of reproduction and developmental process of HIV-1 by using decision-making methods and to obtain inspirational results on drug design against infection. In the study, four HIV-1 phenotypes are analyzed, and each phenotype possesses all stages of the evolutionary process such as replication, mutation, and selection. Evolutionary game theory is preferred to examine the behaviors and evolutionary processes of these phenotypes and to monitor their effects on one another in an environment with HIV-1 infection. In this competitive environment, the development of HIV-1 phenotype swarms depends on the number of CD4+T cells that they can infect.

* e-mail: bilge.baser@msgsu.edu.tr

The study divided into four sections is made up of this scheme: In the first section, the developmental process of HIV-1 is provided. In the second section, the game of HIV-1 phenotypes is constructed, and stable phenotype is obtained by using evolutionary game theory principles and quantum game theory concepts are explained. In the next section, the game is analyzed with the quantum perspective and the stable states of different scenarios are discussed. In the last section, the results are presented and references to further studies are given.

2. GAME OF INFECTIOUS PHENOTYPES

As a threshold matter, four HIV-1 phenotypes are considered in this study following Harada [2];

- An infectious virus with a low production rate (v)
- An infectious virus with a high production rate (V)
- A defective virus with a low production rate (d)
- A defective virus with a high production rate (D).

HIV-1 adheres to the targeted CD4+T cell through its molecules called gp120. During HIV-1 replication, sometimes the gp120 molecule is mutated and defective viruses (d, D) appear that have lost the infectious capability. Defective viruses cannot adhere to the host cell and therefore cannot carry out replication and mutation processes. Therefore, in this study, viruses (v, V) that are capable of infecting are investigated. These viruses can be transformed into a different phenotype after a probable mutation. *Table 1* shows the phenotypes that viruses (v, V) can be transformed after possible mutations.

Table 1. The phenotypes (v, V) that can be transformed by a mutation

HIV-1 Phenotype	Phenotypes After a Mutation
v	V, d, ϕ
V	v, D

A virus is called the provirus which is the presence of the virus in the host cell after replication by using its DNA. Sometimes a viral DNA cannot satisfy the conditions of being a provirus because of some mutations. In this case, the phenotype indicated by the symbol (ϕ) appears.

To investigate the behavioral and evolutionary processes such as replication, mutation, and selection of phenotypes in an environment with HIV-1 infection and to observe their effects on each other, the game theory approach is adopted.

Competition between phenotypes begins when the common resource sharing is made between the two phenotypes discussed in the study, and the situations that may arise are as follows: When the phenotype v encounters its species, the potential to share the existing CD4+T cells equally will emerge. Therefore, both viruses earn an equal outcome. Similarly, the V phenotype will share CD4+T cells equally when faced with its species. Again, they will have equal outcomes. However, their gains will be smaller than those of the phenotype v . Because the V phenotype is growing faster than the v phenotype, and that will lead to rapid depletion of resources. That will cause V to have a shorter life in terms of evolutionary continuity. If two different phenotypes meet, the gain of V will be greater than v , as V will reproduce faster and adhere to more CD4+T cells. *Table 2* represents the outcomes of these possible matches with a game matrix.

Table 2. Game matrix of infectious phenotypes

Phenotypes	v	V
v	(α, α)	(β, γ)
V	(γ, β)	(θ, θ)

So, in terms of survival and reproductive capacity, the relationship between outcomes is as follows: $\gamma > \alpha > \theta > \beta$.

2.1. Evolutionary Game Theory

Evolutionary game theory emerged from the utilization of game theory to the lifestyles of populations evolving in biology. Its theory is relying on the Darwinian process of natural selection [6]. It merges the fundamental structure of game theory and evolution with its dynamical systems to reveal the distribution of distinct phenotypes in biological populations [7]. According to Smith [8], evolutionary game theory is a way of thinking about their evolution at the phenotypic level in which the suitability of phenotypes depends on their frequency in the population.

In evolutionary game theory context, the strategies or behaviors are involuntarily chosen by the genes of the players, not by the rational reasoning of the players, and accordingly, there are situations where the more successful one among these strategies grows and the less successful is extinct. What is open to research here is the question of which strategies to choose in an evolutionary environment. This question triggered biologists to conduct research on animal behavior and led to game theory for modeling and simulating [9].

The evolutionary approach adopts stable situations in which the strategies of all individuals in the population are balanced. In this respect, it has similar principles to the Nash equilibrium.

In classical game theory, a player who has agreed to play the Nash equilibrium does not want to deviate from his equilibrium strategy. This is because, no player can increase his payoff as a result of choosing any other strategy, under the assumption of each player does not deviate his choice. The analog of this in evolutionary game theory is the concept of evolutionarily stable strategy. If almost all elements of a population have adopted a strategy \hat{s} of the game's strategy set S and any mutant strategy cannot pervade the population under the effect of natural selection, then the strategy is evolutionarily stable. The mathematical definition of evolutionarily stable strategy was first described by J. M. Smith and G. R. Price [10,11]. For a two-player symmetric game, *Definition 1* describes the evolutionarily stable strategy from a biological aspect whereas *Definition 2* identifies from the game theory perspective [9].

Definition 1. The strategy \hat{s} , belonging to the strategy set S , is evolutionarily stable in pure strategies if it satisfies the inequality (1) for all possible deviations to $s' \in S$ in consequence of the mutation size ε in the population, where $\forall \varepsilon < \bar{\varepsilon}$ and $\bar{\varepsilon} > 0$ ($\bar{\varepsilon}$: small number)

$$(1 - \varepsilon)u(\hat{s}, \hat{s}) + \varepsilon u(\hat{s}, s') > (1 - \varepsilon)u(s', \hat{s}) + \varepsilon u(s', s'). \quad (1)$$

The left side of the inequality is the expected payoff of the strategy \hat{s} in a mixed population that mutates with the size of ε playing s' and the rest $(1 - \varepsilon)$ playing \hat{s} . On the other hand, the right side of the inequality is the expected payoff of the strategy s' in the same mixed population and if the pure strategy \hat{s} is evolutionarily stable, then the incumbent gets a strictly better payoff.

Definition 2. The strategy \hat{s} is evolutionarily stable in pure strategies if it satisfies the conditions given below.

- (1) (\hat{s}, \hat{s}) is a Nash equilibrium ($u(\hat{s}, \hat{s}) \geq u(s', \hat{s})$ for each s'),
- (2) If (\hat{s}, \hat{s}) is not a strict Nash equilibrium (for some $s' \neq \hat{s}$, $u(\hat{s}, \hat{s}) = u(s', \hat{s})$, then $u(\hat{s}, s') > u(s', s')$).

Similar evolutionarily stable strategy definitions can be made for mixed strategies.

According to *Definition 2*, a strategy is evolutionarily stable, if it is strict Nash equilibrium. If case (2) is valid, the mutation gets payoff against the incumbent as well as the incumbent gets against itself, then the incumbent should get a better payoff against the mutant than the mutant gets against itself.

An evolutionarily stable strategy is obtained by using those definitions on the game of infectious phenotypes. *Table 3* represents the evolutionarily stable strategy for the game in bold color.

Table 3. The evolutionarily stable phenotype of the game

Phenotypes	v	V
v	(α, α)	(β, γ)
V	(γ, β)	(θ, θ)

Although it is not Pareto optimal, the equilibrium point occurs at the point of $(V - V)$ combination with the payoffs (θ, θ) . So, V is the evolutionarily stable phenotype. According to these results of classical aspects, one can interpret as “give the priority to the phenotype V while developing pharmaceutical ingredients.” However, Eisert et al [12] indicated by referring to Dawkins [13] that “if the *Selfish Genes* are reality, then the games of survival are being played already on a molecular level where quantum mechanics dictates the rules.” This argument leads the biological games to a quantum perspective.

2.2. Quantum Game Theory

Game theory, being a branch of applied mathematics, defines games without reference to the physical universe. However, quantum mechanics is a physical theory and quantum game theory investigates the behavior of players with access to quantum randomization. Such technology can be employed in randomization devices and/or in communication devices. So, a quantum strategy can be thought of as a strategy conditioned on the value of some quantum mechanical observable [14].

The foundations of quantum game theory began to develop formally about forty years ago. In 1980, Blaquiere analyzed wave mechanics as a two-player game with a connection between dynamic programming and the theory of differential games, on one hand, and wave mechanics on the other hand [15].

In 1998, Meyer introduced the basic ideas of game theory from the perspective of quantum algorithms and the first quantized coin-tossing game [16]. He theorized that a player can increase his payoff by choosing quantum strategy against his opponents’ classical strategy. After his work, numerous new quantum games with varying levels of complexities and outcomes have been written about by anyone from physicists to statisticians to game theorists [17].

Another important work on quantum games was developed by Eisert et al in 1999. They constructed a physical model of the *Prisoner’s Dilemma* and showed that the players can escape the dilemma if they both resort to quantum strategies. Moreover, they demonstrated that there exists a particular pair of quantum strategies that always gives reward and is a Nash equilibrium, and there exists a particular quantum strategy that always gives at least reward if played against any classical strategy [12].

After these significant developments, Marinatto and Weber extended the concept of a classical two-person static game to the quantum domain, by giving a Hilbert structure to the space of classical strategies and studying the *Battle of the Sexes* game [18].

Du et. al. generalized the quantum prisoner’s dilemma to the case where the players share nonmaximally entangled states and realized the quantum game on nuclear magnetic resonance quantum computer [19].

Iqbal and Toor introduced the quantized evolutionary game concepts by integrating quantum mechanics into biology. They explored the aftereffects when a limited group of mutants using quantum strategies attempt to pervade a classical evolutionarily stable strategy in a biological population operated in the symmetric game of *Prisoner’s Dilemma* [20], and equilibria of replicator dynamics [21].

By following these recent works, it is aimed to construct a quantized game model of HIV-1 phenotypes’ evolutionary process in this study.

Definition 3. A quantum game Q is defined as $Q = Q(H, \Lambda, \{s_i\}_j, \{\pi_i\}_j)$ where H is a Hilbert space, Λ is the initial state of the game, $\{s_i\}_j$ is the set of moves of player j , $\{\pi_i\}_j$ is the set of payoffs to player j and the object of the game is determining the strategies that maximize the payoffs to player j [22].

Quantum computation uses the Dirac notation. The Dirac notation labels a quantum state ψ by the ket $|\psi\rangle$ which belongs to Hilbert space. A quantum state is a complex combination of an n -dimensional orthonormal basis $(|\omega_i\rangle, i = 1, 2, \dots, n)$ for Hilbert space as $|\psi\rangle = \sum_{i=1}^n c_i |\omega_i\rangle$, where the c_i are complex numbers.

The bit is the fundamental unit of information in classical computation, taking the values 0 or 1. Its quantum analog is the quantum bit or qubit. Amongst its possible values, $|0\rangle$ or $|1\rangle$, known as the computational basis states. However, a qubit may also be a convex linear combination or superposition $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, with the normalization condition $|\alpha|^2 + |\beta|^2 = 1$ [23].

Unitary operators are used to transforming states in quantum game theory. An operator is unitary if its Hermitian conjugate or adjoint is equal to its inverse. For a two-state system, it is useful to do this with the Pauli spin matrices σ_x (the not operator), σ_y , σ_z , and the Hadamard matrix (H) [22] which are given in below

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (2)$$

$\sigma_x, \sigma_y, \sigma_z$ and the identity matrix (I) form a basis for the space of 2×2 unitary matrices. On the other part, the Hadamard matrix is a superposition halfway between $|0\rangle$ and $|1\rangle$.

Quantum game theory requires an initial state for the game to start searching for the optimal strategies. For this game, the initial state of the game is defined as the competition of the phenotypes v and v . The qubits of the strategies mapped as $|0\rangle$ for v and $|1\rangle$ for V . Then, the initial state of the game becomes,

$$\Lambda = U|vv\rangle = U|00\rangle \quad (3)$$

where U is a unitary operator. U_v is the set of moves (s_v) of the phenotype v , and the U_V is the set of moves (s_V) of the phenotype V . Following the initial state $U|00\rangle$ and after the phenotypes (v, V) made their moves the state of the game becomes,

$$(U_v \otimes U_V)U|00\rangle. \quad (4)$$

Then the phenotypes forward their qubits for final measurement. U^\dagger (the inverse of the unitary operator U) is applied, to bring the game to the state:

$$|\psi_f\rangle = U^\dagger(U_v \otimes U_V)U|00\rangle. \quad (5)$$

The expected payoffs of the players (π_v, π_V) can be obtained by using the outcomes of the game from Table 2 as follows,

$$\pi_v = \alpha|\langle\psi_f|00\rangle|^2 + \beta|\langle\psi_f|01\rangle|^2 + \gamma|\langle\psi_f|10\rangle|^2 + \theta|\langle\psi_f|11\rangle|^2 \quad (6)$$

and

$$\pi_V = \alpha|\langle\psi_f|00\rangle|^2 + \gamma|\langle\psi_f|01\rangle|^2 + \beta|\langle\psi_f|10\rangle|^2 + \theta|\langle\psi_f|11\rangle|^2. \quad (7)$$

For this game, the unitary matrix U is defined as

$$U = \frac{1}{\sqrt{2}}(I^{\otimes 2} + i\sigma_x^{\otimes 2}) \quad (8)$$

and the inverse of unitary operator U^\dagger ,

$$U^\dagger = \frac{1}{\sqrt{2}}(I^{\otimes 2} - i\sigma_x^{\otimes 2}) \tag{9}$$

to give equal probability to each quantum move where $\otimes 2$ denotes the tensor product 2 times. After the implementation of U on the initial state Λ , the state of the system turns into,

$$U|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle). \tag{10}$$

3. THE QUANTUM GAME OF PHENOTYPES

To start with, it is assumed that the phenotypes can play the quantum moves I and σ_x ($s_v, s_V = \{I, \sigma_x\}$). If the phenotype plays I , the state remains the same, and if the phenotype plays σ_x (the not operator), the state transforms into applied the not state. *Table 4* represents the quantum payoffs that can be adapted from *Table 2*.

Table 4. Quantum payoffs of the game for the first scenario

Quantum moves	I	σ_x
I	(α, α)	(β, γ)
σ_x	(γ, β)	(θ, θ)

Then the quantum states that can discoverable related to these states and the expected payoffs are obtained by using the Equations (8) - (9) as follows,

- if both phenotypes play their I operator the state of the system becomes,

$$(I \otimes I)U|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle) \tag{11}$$

$$U^\dagger(I \otimes I)U|00\rangle = U^\dagger \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle) = |00\rangle \tag{12}$$

with the probability 1. The outcomes of the phenotypes for this state can be obtained from *Table 4* as $\pi_v = \alpha, \pi_V = \alpha$.

- if the first phenotype plays σ_x and the second phenotype plays I ,

$$(\sigma_x \otimes I)U|00\rangle = \frac{1}{\sqrt{2}}(|10\rangle + i|01\rangle) \tag{13}$$

$$U^\dagger(\sigma_x \otimes I)U|00\rangle = U^\dagger \frac{1}{\sqrt{2}}(|10\rangle + i|01\rangle) = |10\rangle \tag{14}$$

with the probability 1 and the outcomes are $\pi_v = \gamma, \pi_V = \beta$.

- if the first phenotype plays I and the second phenotype plays σ_x ,

$$(I \otimes \sigma_x)U|00\rangle = \frac{1}{\sqrt{2}}(|01\rangle + i|10\rangle) \tag{15}$$

$$U^\dagger(I \otimes \sigma_x)U|00\rangle = U^\dagger \frac{1}{\sqrt{2}}(|01\rangle + i|10\rangle) = |01\rangle \tag{16}$$

with the probability 1 and the outcomes are $\pi_v = \beta, \pi_V = \gamma$.

- if both phenotypes play σ_x ,

$$(\sigma_x \otimes \sigma_x)U|00\rangle = \frac{1}{\sqrt{2}}(|11\rangle + i|00\rangle) \tag{17}$$

$$U^\dagger(\sigma_x \otimes \sigma_x)U|00\rangle = U^\dagger \frac{1}{\sqrt{2}}(|11\rangle + i|00\rangle) = |11\rangle \tag{18}$$

with the probability 1 and the outcomes are $\pi_v = \theta, \pi_V = \theta$.

These four states are the classical outcomes of the game. Therefore, this result indicates that the classical game is a specific form of the quantum game.

Then it is assumed that a *mutant H* comes into the population that can play the Hadamard matrix. In such a case, if the first phenotype plays *I* and the second phenotype plays *H*, the expected payoffs are obtained as follows,

$$(I \otimes H)U|00\rangle = \frac{1}{2}[|00\rangle + |01\rangle + i|10\rangle - i|11\rangle] \tag{19}$$

$$U^\dagger(I \otimes H)U|00\rangle = \frac{1}{\sqrt{2}}(|01\rangle - i|11\rangle) \tag{20}$$

$$\pi_v = \beta|\langle\psi_f|01\rangle|^2 + \theta|\langle\psi_f|11\rangle|^2 = \beta\left|\frac{1}{\sqrt{2}}\right|^2 + \theta\left|\frac{-i}{\sqrt{2}}\right|^2 = \frac{\beta+\theta}{2} \tag{21}$$

and

$$\pi_V = \gamma|\langle\psi_f|01\rangle|^2 + \theta|\langle\psi_f|11\rangle|^2 = \gamma\left|\frac{1}{\sqrt{2}}\right|^2 + \theta\left|\frac{-i}{\sqrt{2}}\right|^2 = \frac{\gamma+\theta}{2}. \tag{22}$$

In this decision environment, the possible expected payoffs of phenotypes can be calculated in the same way, and they are represented in *Table 5* for all quantum move combinations.

Table 5. Quantum payoffs of the game for the second scenario

Quantum moves	<i>I</i>	σ_x	<i>Mutant H</i>
<i>I</i>	(α, α)	(β, γ)	$(\frac{\beta + \theta}{2}, \frac{\gamma + \theta}{2})$
σ_x	(γ, β)	(θ, θ)	$(\frac{\beta + \theta}{2}, \frac{\gamma + \theta}{2})$
<i>Mutant H</i>	$(\frac{\gamma + \theta}{2}, \frac{\beta + \theta}{2})$	$(\frac{\gamma + \theta}{2}, \frac{\beta + \theta}{2})$	$(\frac{\alpha + \beta + \theta + \gamma}{4}, \frac{\alpha + \beta + \theta + \gamma}{4})$

The evolutionarily stable state and the Nash equilibrium of this game occur at the quantum moves combination (*H – H*). But still, it is not Pareto optimal.

Then it is assumed that a mutant comes into the population which can play σ_z operator. For instance, if both phenotypes play σ_z , the expected payoffs are calculated as shown below,

$$(\sigma_z \otimes \sigma_z)U|00\rangle = \frac{1}{\sqrt{2}}(|01\rangle - i|11\rangle) \tag{23}$$

$$U^\dagger(\sigma_z \otimes \sigma_z)U|00\rangle = |00\rangle \tag{24}$$

with the probability 1 and the outcomes are $\pi_v = \alpha, \pi_V = \alpha$.

The other possible payoffs can be calculated similarly. For this scenario, the payoff matrix is constructed as given in *Table 6* for all quantum moves combinations.

Table 6. Quantum payoffs of the game for the third scenario

Quantum moves	<i>I</i>	σ_x	<i>Mutant H</i>	<i>Mutant σ_z</i>
<i>I</i>	(α, α)	(β, γ)	$(\frac{\beta + \theta}{2}, \frac{\gamma + \theta}{2})$	(θ, θ)
σ_x	(γ, β)	(θ, θ)	$(\frac{\beta + \theta}{2}, \frac{\gamma + \theta}{2})$	(β, γ)
<i>Mutant H</i>	$(\frac{\gamma + \theta}{2}, \frac{\beta + \theta}{2})$	$(\frac{\gamma + \theta}{2}, \frac{\beta + \theta}{2})$	$(\frac{\alpha + \beta + \theta + \gamma}{4}, \frac{\alpha + \beta + \theta + \gamma}{4})$	$(\frac{\alpha + \beta}{2}, \frac{\alpha + \gamma}{2})$
<i>Mutant σ_z</i>	(θ, θ)	(γ, β)	$(\frac{\alpha + \gamma}{2}, \frac{\alpha + \beta}{2})$	(α, α)

The evolutionarily stable state and the Nash equilibrium are achieved at the quantum moves combination *Mutant* (σ_z, σ_z), with the payoff (α, α) that is also Pareto optimum. Thus, Pareto optimum evolutionarily stable strategies are obtained by the advantage of using quantum randomization.

4. CONCLUSION

Quantum games put a new complexion on the classical problems and dilemmas in game theory. Studies in recent years show that the adaptation of quantum mechanics to game theory provides more practical results in analyzing biological processes.

Consistent with previous findings, in this study, it is shown that quantum game theory obtains a different equilibrium point from the classical game theory that is also Pareto optimum. According to the solution of this problem, it is not a good action to give priority to just one phenotype that has a high production rate in the treatment process as classical game theory suggests. In terms of the quantum approach, there may emerge different mutant strategies of phenotypes, and they may find an equilibrium point by using quantized strategies.

This result also suggests that there could be evolutionary pressures to develop quantum strategies in micro-level biological games and in future research decision-making environments in which the rules of quantum mechanics work should be designed for such games. Because computational resources that the universe is made available to us allow you to do things that you cannot do classically.

CONFLICTS OF INTEREST

No conflict of interest was declared by the author.

REFERENCES

- [1] Castiglione, F., Pappalardo, F., Bernasch, M., and Motta, S., "Optimization of haart with genetic algorithms and agent-based models of hiv infection", *Bioinformatics*, 23(24): 3350-3355, (2007).
- [2] Harada, K., "Self-destruction Dynamics of HIV-1 Quasi-species Population in the Presence of Mutagenic Activities", *Procedia Computer Science*, 22: 1259-1265, (2013).
- [3] Goshu, A.T., and Dessie, Z.G., "Modelling Progression of HIV/AIDS Disease Stages Using Semi-Markov Processes", *Journal of Data Science*, 11: 269-280, (2013).

- [4] Wu, Y., Zhang, M., Wu, J., Zhao, X., and Xia, L., “Evolutionary game theoretic strategy for optimal drug delivery to influence selection pressure in treatment of HIV-1”, *Journal of Mathematical Biology*, 3(64): 495-512, (2012).
- [5] Khazaei, B., Sartakhti, J.S., Manshaei, M.H., Zhu, Q., Sadeghi, M., and Mousavi, S.R., “HIV-1 infected T-cells dynamics and prognosis: An evolutionary game theory model”, *Computer Methods and Programs in Biomedicine*, 152: 1-14, (2017).
- [6] Fisher, R. A., *The Genetical Theory of Natural Selection*, 1st edition, Oxford Clarendon Press, Oxford, (1930).
- [7] Hidalgo, E.G., “Quantum Games and the Relationships between Quantum Mechanics and Game Theory”, arXiv:0803.0292v2 [quant-ph], (2016).
- [8] Smith, J.M., *Evolution and the Theory of Games*, 1st edition, Cambridge University Press, Oxford, (1982).
- [9] https://oyc.yale.edu/sites/default/files/ess_handout_0_0.pdf. Access date: 05.12.2018
- [10] Smith, J.M., *On Evolution*, 1st edition, Edinburgh University Press, Edinburgh, (1972).
- [11] Smith, J.M., and Price, G.R., “The Logic of Animal Conflict”, *Nature*, 246: 15-18, (1973).
- [12] Eisert, J., Wilkens, M. and Lewenstein, M., “Quantum Games and Quantum Strategies”, *Physical Review Letters*, 83(15): 3077-3080, (1999).
- [13] Dawkins, R., *The Selfish Gene*, 4th edition, Oxford University Press, Oxford, (2016).
- [14] Dahl, G.B., and Landsburg S.E., “Quantum Strategies”, arXiv:1110.4678v1 [math.OC], (2011).
- [15] Blaquiére, A., “Wave Mechanics as a Two-Player Game”, *Dynamical Systems and Microphysics*, 33-69, (1980).
- [16] Meyer, D.A., “Quantum Strategies”, *Physical Review Letters*, 82: 1052-1055, (1999).
- [17] <http://hamilton.uchicago.edu/~sethi/Teaching/P243-W2020/final-papers/price.pdf>. Access date: 14.07.2021
- [18] Marinatto, L., and Weber, T., “A Quantum Approach to Static Games of Complete Information”, *Physics Letters, A* 272: 291-303, (2000).
- [19] Du, J., Li, H., Xu, X., Shi, M., Wu, J., Zhou, X., and Han, R., “Experimental Realization of Quantum Games on a Quantum Computer”, *Physical Review Letters*, 88, 137902, (2002).
- [20] Iqbal, A., and Toor, A.H., “Evolutionarily Stable Strategies in Quantum Games”, *Physics Letters, A* 280/5-6: 249-256, (2001).
- [21] Iqbal, A., and Toor, A.H, “Equilibria of Replicator Dynamics in Quantum Games”, quant-ph/0106135v2, (2004).
- [22] Grabbe, J.O., “An Introduction to Quantum Game Theory”, arXiv:quant-ph/0506219v1, (2005).
- [23] Flitney, A.P., *Game Theory: Strategies, Equilibria and Theorems*, 1st edition, Haugen I.N. and Nielsen A.S., Nova Science Publishers, New York, 1-40, (2009).