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CHEBYSHEV INEQUALITY ON CONFORMABLE DERIVATIVE

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Abstract. Integral inequalities are very important in applied sciences. Chebyshev's integral inequality is widely used in applied mathematics. First of all, some necessary definitions and results regarding conformable derivative are given in this article. Then we give Chebyshev inequality for simultaneously positive (or negative) functions using the conformable fractional derivative. We used the Gronwall inequality to prove our results, unlike other studies in the literature.

1. Introduction

Various definitions are given in the literature for fractional derivatives [8, 14, 17, 20]. Some of which are Riemann-Liouville, Caputo, Grünwald-Letnikov, Riesz, Weyl fractional derivatives. Having more than one definition of derivative in fractional analysis ensures that the most suitable one is used according to the type of the problem and thus the best solution is obtained.

In [12], a new fractional derivative that is known as conformable derivative has been defined by Khalil. This new fractional derivative based on classical limit definition. Authors gave linearity condition, the product rule, the division rule, Rolle theorem and mean value theorem for this new definition of fractional derivative. They also defined the fractional integral of order $0 < \alpha \leq 1$ only.

In [1], definition of left and right conformable fractional integrals of any order $\alpha > 0$ has been given by Abdeljawad. He also gave chain rule, linear differential systems, Laplace transforms and exponential functions on a fractional version.

Conformable fractional derivative has been formulated in [1, 12] as

$$D^{\alpha}F(t) = \lim_{\epsilon \to 0} \frac{F(t + \epsilon t^{1-\alpha}) - F(t)}{\epsilon},$$

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or in [11] as

$$D^{\alpha}F(t)=\lim_{\epsilon\to 0}\frac{F(te^{\epsilon t^{-\alpha}})-F(t)}{\epsilon},\ D^{\ \alpha}F(0)=\lim_{t\to 0^{+}}D^{\alpha}F(t),$$

provided the limit exist; in both we have

$$D^{\alpha}F(t)=t^{1-\alpha}D^{'}(t),$$

where $F^{'}(t)=\lim_{\epsilon\to 0}[F(t+\epsilon)-F(t)]/\epsilon$.

In [2], Anderson and Ulness present an exact definition of a conformable fractional derivative of order α for $0\le\alpha\le 1$ and $t\in\mathbb{R}$, where D^0 is the identity operator and D^1 is the classical differential operator.

Monotonicity is an important part of applications of derivatives. The monocity of a function gives an idea about behaviour of the function. Monotonic function is defined as a function that is either completely non-increasing or completely non-decreasing.

For monotonicity and convexity results for fractional integrals and some of their application we recommend the readers to refer the literature [18, 13, 7, 5, 19].

2. Preliminaries

Main definitions and results of conformable derivatives from [2] will be presented as follows:

Definition 1. *Let $\alpha\in[0,1]$. A differential operator D^{α} is conformable if and only if*

$$D^0F(t)=F(t)\ \text{and}\ D^1F(t)=\frac{d}{dt}F(t)=F'(t),\tag{1}$$

where D^0 is the identity operator and D^1 is the classical differential operator.

Definition 2. *Let $\alpha\in[0,1]$ and let the functions $\kappa_0,\kappa_1:[0,1]\times\mathbb{R}\to[0,\infty)$ be continuous such that*

$$\lim_{\alpha\to 0^{+}}\kappa_1(\alpha,t)=1,\qquad \lim_{\alpha\to 0^{+}}\kappa_0(\alpha,t)=0,\ \forall t\in\mathbb{R},$$

$$\lim_{\alpha\to 1^{-}}\kappa_1(\alpha,t)=0,\qquad \lim_{\alpha\to 1^{-}}\kappa_0(\alpha,t)=1,\forall t\in\mathbb{R},$$

$$\kappa_1(\alpha,t)\neq 0\ ,\ \alpha\in[0,1),\quad \kappa_0(\alpha,t)\neq 0,\ \alpha\in(0,1]\ ,\ \forall t\in\mathbb{R}.$$

Then the following differential operator D^{α}, defined via

$$D^{\alpha}F(t)=\kappa_1(\alpha,t)F(t)+\kappa_0(\alpha,t)F'(t),\tag{3}$$

is conformable provided the function F is differentiable at t and $F^{'}=\frac{d}{dt}F$.

For more information on conformable derivative and integral, we refer [1, 2, 12, 11, 4].

Definition 3. ***(Partial Conformable Derivatives).*** *Let $\alpha \in [0,1]$, and let the functions $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy (2). Given a function $F : \mathbb{R}^2 \to \mathbb{R}$ such that $\frac{\partial}{\partial t} F(t,s)$ exists for each fixed $s \in \mathbb{R}$, define the partial differential operator D_t^α via*

$$D_t^\alpha F(t,s) = \kappa_1(\alpha,t)F(t,s) + \kappa_0(\alpha,t)\frac{\partial}{\partial t}F(t,s). \tag{4}$$

Definition 4. ***(Conformable Exponential Function).*** *Let $\alpha \in (0,1]$, the points $s,t \in \mathbb{R}$ with $s \leq t$, and the function $\rho : [s,t] \to \mathbb{R}$ be continuous. Let $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy (2), with ρ/κ_0 and κ_1/κ_0 Riemann integrable on $[s,t]$. After that the exponential function with respect to D^α in (3) is defined as follows*

$$e_\rho(t,s) = e^{\int\limits_s^t \frac{\rho(\tau)-\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)} d\tau}, \quad e_0(t,s) = e^{-\int\limits_s^t \frac{\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)} d\tau}. \tag{5}$$

Lemma 5. ***(Basic Derivatives).*** *Let the conformable differential operator D^α be given as in (3), where $\alpha \in [0,1]$, $\rho : [s,t] \to \mathbb{R}$ be continuous. Let $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy (2), with ρ/κ_0 and κ_1/κ_0 Riemann integrable on $[s,t]$. Assume the functions F and H are differentiable as needed. Then*

(i) $D^\alpha[aF(t) + bH(t)] = aD^\alpha[F(t)] + bD^\alpha[H(t)]$ for all $a,b \in \mathbb{R}$**;**
(ii) $D^\alpha[c] = c\kappa_1(\alpha,t)$ for all constants $c \in \mathbb{R}$;
(iii) $D^\alpha[F(t)H(t)] = F(t)D^\alpha[H(t)] + H(t)D^\alpha[F(t)] - F(t)H(t)\kappa_1(\alpha,t)$;
(iv) $D^\alpha[F(t)/H(t)] = \frac{H(t)D^\alpha[F(t)]-F(t)D^\alpha[H(t)]}{H^2(t)} + \frac{F(t)}{H(t)}\kappa_1(\alpha,t)$;
(v) for $\alpha \in (0,1]$ and fixed $s \in \mathbb{R}$, the exponential function satisfies

$$D_t^\alpha[e_\rho(t,s)] = \rho(t)e_\rho(t,s), \tag{6}$$

for $e_\rho(t,s)$ given in (5);
(vi) for $\alpha \in (0,1]$ and for the exponential function e_0 given in (5), we have

$$D^\alpha[\int\limits_a^t \frac{F(s)e_0(t,s)}{\kappa_0(\alpha,s)} ds] = F(t). \tag{7}$$

Definition 6. *Let $\alpha \in (0,1]$ and $t_0 \in \mathbb{R}$. In light of (5) and Lemma 1 (v) and (vi), define the antiderivative via*

$$\int D^\alpha F(t) d_\alpha t = F(t) + ce_0(t,t_0),\ c \in \mathbb{R}$$

Similarly, define the integral of F over [a,b] as

$$\int\limits_a^t F(s)e_0(t,s)d_\alpha s = \int\limits_a^t \frac{F(s)e_0(t,s)}{\kappa_0(\alpha,s)} ds, \quad d_\alpha s = \frac{1}{\kappa_0(\alpha,s)}, \tag{8}$$

recall that

$$e_0(t,s) = e^{-\int_s^t \frac{\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)} d\tau} = e^{-\int_s^t \kappa_1(\alpha,\tau) d_\alpha \tau}$$

from (5).

Lemma 7. *Let the conformable differential operator D^α be given as in (3), the integral be given as in (8) with $\alpha \in (0,1]$. Let the functions κ_0, κ_1 be continuous and satisfy (2), and let F and H be differentiable as needed. Then*

(i) the derivative of the definite integral of F is given by

$$D^\alpha[\int_a^t F(s) e_0(t,s) d_\alpha s] = F(t);$$

(ii) the definite integral of the derivative of F is given by

$$\int_a^t D^\alpha[F(s)] e_0(t,s) d_\alpha s = F(s) e_0(t,s) \mid_{s=a}^t = F(t) - F(a) e_0(t,a);$$

(iii) an integration by parts formula is given by

$$\begin{aligned}\int_a^b F(t) D^\alpha[H(t)] e_0(b,t) d_\alpha t &= F(t) H(t) e_0(b,t) \mid_{t=a}^b \\ &\quad - \int_a^b H(t)(D^\alpha[F(t)] - \kappa_1(\alpha,,t) F(t)) e_0(b,t) d_\alpha t;\end{aligned}$$

(iv) a version of the Leibniz rule for differentiation of an integral is given by

$$D^\alpha[\int_a^t F(t,s) e_0(t,s) d_\alpha s] = \int_a^t (D_t^\alpha[F(t,s)] - \kappa_1(\alpha,t) F(t,s)) e_0(t,s) d_\alpha s + F(t,t),$$

using (4); or, if e_0 is absent,

$$D^\alpha(\int_a^t F(t,s) d_\alpha s) = F(t,t) + \int_a^t D_t^\alpha[F(t,s)] d_\alpha s.$$

Lemma 8. ***(Variation of Constants).*** *Assume κ_0, κ_1 satisfy (2). Let $f, \rho : [t_0, \infty] \to \mathbb{R}$ be continuous, let e_ρ be as in (5), and let $x_0 \in \mathbb{R}$. Then the unique solution of the initial value problem*

$$D^\alpha x(t) - \rho(t) x(t) = f(t), \; x(t_0) = x_0,$$

is given by

$$x(t) = x_0 e_\rho(t,t_0) + \int_{t_0}^t e_\rho(t,s) f(s) d_\alpha s, \; t \in [t_o, \infty). \tag{9}$$

Theorem 9. ***(Gronwall's Inequality).*** *Let ρ, x, f be continuous functions on $[t_0, \infty)$, with $\rho \geq 0$. Then*

$$x(t) \leq f(t) + \int_{t_0}^{t} \rho(s)x(s)e_0(t,s)d_\alpha s \quad \text{for all } t \in [t_0, \infty),$$

implies

$$x(t) \leq f(t) + \int_{t_0}^{t} \rho(s)f(s)e_0(t,s)d_\alpha s \quad \text{for all } t \in [t_0, \infty).$$

Corollary 10. *Let ρ, x be continuous functions on $[t_0, \infty)$, with $\rho \geqslant 0$. Then*

$$x(t) \leqslant \int_{t_0}^{t} \rho(s)x(s)e_0(t,s)d_\alpha s \text{ for all } t \in [t_0, \infty),$$

implies $x(t) \leqslant 0$ for all $t \in [t_0, \infty)$.

3. Main Result

It is new to refer to inequalities as a mathematics discipline. A very small portion of these inequalities originated from the ancient traditions. In the 18th and early 19th century names such as Newton, Cauchy and Maclaurin started to work in this field. In this period, only Bernoulli and Cauchy-Schwarz-Bunyakovsky inequalities, which are mentioned with their own name, can be given as an example [9].

Towards the end of the 19th century, original products started to be given in the field of inequalities. Hölder and Minkovski could be shown among their pioneers. But the milestone in this area is the Chebyshev's paper [6]. Chebyshev submitted his paper to the Han'kovshov University's Editorial Committee in order to be published in the journal for the volumes in 1883. But the mentioned committee extremely impressed from this paper that they published it in the last volume of 1882 [9].

Theorem 11. ***(Chebyshev Inequality)***. *Let f and g be two integrable functions on the $[0,1]$. If both functions are simultaneously increasing or decreasing for the same $x \in [0,1]$, then*

$$\int_0^1 f(x)g(x)dx \geqslant \int_0^1 f(x)dx \int_0^1 g(x)dx.$$

If one of the functions is increasing, the other is decreasing for the same $x \in [0,1]$ values, then

$$\int_0^1 f(x)g(x)dx \leq \int_0^1 f(x)dx \int_0^1 g(x)dx.$$

(Chebyshev 1882).

Belarbi and Dahmani gave results on Chebyshev's inequality using the Riemann-Liouville integral in 2009 [4]. E.Set gave results on Chebyshev's inequality using conformable fractional integrals in 2019 [21]. For the background and summary on inequalities, we refer the readers to the references [3, 9, 10, 15].

Before giving Chebyshev inequality using conformable derivative, mentioning about following results [16] that play a key role in our proof will provide a better understanding:

Monotonicity

Let $a > 0$ and $F : [a, b] \to \mathbb{R}$ be $\alpha-$differentiable on an interval $[a, b]$.

i. If $F^{\alpha}(x) \geq 0$ for all $x \in [a, b]$, then F is nondecreasing on $[a, b]$.

ii. If $F^{\alpha}(x) > 0$ for all $x \in [a, b]$, then F is increasing on $[a, b]$.

iii. If $F^{\alpha}(x) \leq 0$ for all $x \in [a, b]$, then F is nonincreasing on $[a, b]$.

iv. If $F^{\alpha}(x) < 0$ for all $x \in [a, b]$, then F is decreasing on $[a, b]$.

v. If $F^{\alpha}(x) = 0$ for all $x \in [a, b]$, then F is constant on $[a, b]$.

Theorem 12. *Let f and g be two integrable functions on $[a, b]$. If both functions are simultaneously positive or negative for the same $x \in [a, b]$ values then*

$$\int_a^b f(x)g(x)e_0(t,x)d_\alpha x \geqslant \int_a^b f(x)e_0(t,x)d_\alpha x \int_a^b g(x)e_0(t,x)d_\alpha x.$$

If one of the functions for the same $x \in [a, b]$ values is positive and the other is negative then

$$\int_a^b f(x)g(x)e_0(t,x)d_\alpha x \leqslant \int_a^b f(x)e_0(t,x)d_\alpha x \int_a^b g(x)e_0(t,x)d_\alpha x.$$

Proof. Let f and g be two integrable functions on [a,b]. Let define

$$F(x) = \int_a^x f(t)g(t)e_0(x,t)d_\alpha t - \int_a^x f(t)e_0(x,t)d_\alpha t \int_a^x g(t)e_0(x,t)d_\alpha t.$$

If we take the derivative of both sides, we have

$$\begin{aligned} D^{\alpha}F(x) &= f(x)g(x) - f(x)\int_a^x g(t)e_0(x,t)d_\alpha t - g(x)\int_a^x f(t)e_0(x,t)d_\alpha t \\ &\quad + \kappa_1(\alpha,t)\int_a^x f(t)e_0(x,t)d_\alpha t \int_a^x g(t)e_0(x,t)d_\alpha t. \end{aligned}$$

$$D^{\alpha}F(x) = \frac{f(x)g(x)}{2} - f(x)\int_a^x g(t)e_0(x,t)d_\alpha t + \frac{f(x)g(x)}{2} - g(x)\int_a^x f(t)e_0(x,t)d_\alpha t$$
$$+\kappa_1(\alpha,t)\int_a^x f(t)e_0(x,t)d_\alpha t\int_a^x g(t)e_0(x,t)d_\alpha t.$$

$$D^{\alpha}F(x) = \frac{f(x)}{2}[g(x) - 2\int_a^x g(t)e_0(x,t)d_\alpha t] + \frac{g(x)}{2}[f(x) - 2\int_a^x f(t)e_0(x,t)d_\alpha t]$$
$$+\kappa_1(\alpha,t)\int_a^x f(t)e_0(x,t)d_\alpha t\int_a^x g(t)e_0(x,t)d_\alpha t.$$

a) (i) Let $g(x) > 0$, assume that

$$g(x) - 2\int_a^x g(t)e_0(x,t)d_\alpha t \leqslant 0,$$

then

$$g(x) \leqslant 2\int_a^x g(t)e_0(x,t)d_\alpha t.$$

From Corollory 10 $g(x) \leqslant 0$. This is the contradiction. Then;

$$g(x) > 0,\ g(x) - 2\int_a^x g(t)e_0(x,t)d_\alpha t > 0.$$

Using similar arguments, we can write

$$f(x) > 0,\ f(x) - 2\int_a^x f(t)e_0(x,t)d_\alpha t > 0.$$

(ii) Let $g(x) < 0, -g(x) = G(x), G(x) > 0$, assume that

$$g(x) - 2\int_a^x g(t)e_0(x,t)d_\alpha t \geq 0,$$

then

$$-G(x) + 2\int_a^x G(t)e_0(x,t)d_\alpha t \geq 0,$$

this implies

$$G(x) \leq 2 \int_a^x G(t) e_0(x,t) d_\alpha t.$$

From Corollory 10 $G(x) \leq 0$. This is the contradiction. Consequently,

$$g(x) < 0, \; g(x) - 2 \int_a^x g(t) e_0(x,t) d_\alpha t < 0.$$

Using similar arguments, we can write

$$f(x) < 0, \; f(x) - 2 \int_a^x f(t) e_0(x,t) d_\alpha t < 0.$$

Also we can say $\kappa_1(\alpha,t) \int_a^x f(t) e_0(x,t) d_\alpha t \int_a^x g(t) e_0(x,t) d_\alpha t \geq 0$. As a result of this part we have

$$\begin{aligned} D^\alpha \digamma(x) &= \frac{f(x)}{2}[g(x) - 2 \int_a^x g(t) e_0(x,t) d_\alpha t] + \frac{g(x)}{2}[f(x) - 2 \int_a^x f(t) e_0(x,t) d_\alpha t] \\ &\quad + \kappa_1(\alpha,t) \int_a^x f(t) e_0(x,t) d_\alpha t \int_a^x g(t) e_0(x,t) d_\alpha t, \end{aligned}$$

is positive. So, the function $\digamma(x)$ is increasing on $[a,b]$. Then,

$$\digamma(b) \geq \digamma(a) = 0.$$

This implies the first inequality in theorem is proved.

b)Let $f(x) > 0$, assume that

$$f(x) - 2 \int_a^x f(t) e_0(x,t) d_\alpha t \leq 0,$$

then

$$f(x) \leq 2 \int_a^x f(t) e_0(x,t) d_\alpha t.$$

From Corollory 10 $f(x) \leq 0$. This is the contradiction. Hence,

$$f(x) > 0, \; f(x) > 2 \int_a^x f(t) e_0(x,t) d_\alpha t.$$

Now, from part a, if

$$g(x) < 0,\ g(x) - 2\int_a^x g(t)e_0(x,t)d_\alpha t < 0.$$

As a result of this part we have

$$\begin{aligned} D^\alpha F(x) &= \frac{f(x)}{2}[g(x) - 2\int_a^x g(t)e_0(x,t)d_\alpha t] + \frac{g(x)}{2}[f(x) - 2\int_a^x f(t)e_0(x,t)d_\alpha t] \\ &\quad +\kappa_1(\alpha,t)\int_a^x f(t)e_0(x,t)d_\alpha t \int_a^x g(t)e_0(x,t)d_\alpha t, \end{aligned}$$

is negative. So the function $F(x)$ is decreasing on [a,b]. Then,

$$F(b) \leq F(a) = 0.$$

This implies the second inequality in theorem is proved. □

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References

[1] Abdeljawad, T., On conformable fractional calculus, *Journal of Computational and Applied Mathematics,* 279 (2015), 57-66. https://doi:101016/j.cam.2014.10.016

[2] Anderson, D. R., Ulness, D. J., Newly defined conformable derivatives, *Advances in Dynamical Systems and Applications*, 10(2) (2015), 109-137.

[3] Beckenbach, E. F., Bellman, R., Inequalities, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1961.

[4] Belarbi, S., Dahmani, Z., On some new fractional integral inequalities, *Journal of Inequalities in Pure and Applied Mathematics*, 10(3) (2009), Article 86, 5 pp.

[5] Butt, S. I., Umar, M., Rashid, S., Akdemir, A. O., Chu, Y., New Hermite-Jensen-Mercer-type inequalities via k-fractional integrals, *Advances in Difference Equations*, (2020), Article 635, 24 pp. https://doi:10.1186/s13662-020-03093-y

[6] Chebyshev, P. L., Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites, *Proc. Math. Soc. Charkov*, 2 (1882), 93-98.

[7] Chen, S., Rashid, S., Hammouch, Z., Noor, M. A., Ashraf, R., Chu, Y., Integral inequalities via Raina's fractional integrals operator with respect to a monotone function, *Advances in Difference Equations*, Volume 2020 (2020), Article 647, 20 pp. https://doi:10.1186/s13662-020-03108-8

[8] Diethelm, K., The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type, Springer-Verlag, Berlin, 2010.

[9] Fink, A. M., An essay on the history of inequalities, *Journal of Mathematical Analysis and Applications*, 249(1) (2000), 118-134. https://doi:10.1006/jmaa.2000.6934

[10] Hardy G. H., Littlewood J. E., Polya, G., Inequalities, Cambridge University Press, Cambridge, 1952.

[11] Katugampola, U. N., A new fractional derivative with classical properties, (2014), 8 pp. arXiv:1410.6535v2

[12] Khalil, R., Al Horani, M., Yousef, A., Sababheh, M., A new definition of fractional derivative, *Journal of Computational and Applied Mathematics*, 264 (2014), 65-70. https://doi:10.1016/j.cam.2014.01.002

[13] Khan, Z. A., Rashid, S., Ashraf, R., Baleanu, D., Chu, Y., Generalized trapezium-type inequalities in the settings of fractal sets for functions having generalized convexity property, *Advances in Difference Equations*, Volume 2020 (2020), Article 657, 24 pp. https://doi:10.1186/s13662-020-03121-x

[14] Miller, K. S., Ross, B., An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.

[15] Mitrinovic, D. S., Analytic Inequalities. Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[16] Musraini, M., Efendi, R., Lily, E., Hidayah, P., Classical properties on conformable fractional calculus, *Pure and Applied Mathematics Journal,* 8(5) (2019), 83-87. https://doi:10.11648/j.pamj.20190805.11

[17] Podlubny, I., Fractional Differential Equations, Academic Press, San Diego, 1999.

[18] Rashid, S., Jarad, F., Noor, M. A., Kalsoom, H., Chu, Y., Inequalities by means of generalized proportional fractional integral operators with respect to another function, *Mathematics*, 7(12) (2019), Article 1225, 18 pp. https://doi:10.3390/math7121225

[19] Rashid, S., Ahmad, H., Khalid, A., Chu, Y., On discrete fractional integral inequalities for a class of functions, *Hindawi*, Volume 2020 (2020), Article ID 8845867, 13 pp. https://doi:10.1155/2020/8845867

[20] Samko, S. G., Kilbas, A. A., Marichev, O. I., Fractional Integrals and Derivatives, Gordon and Breach Science Publishers, Yverdon, 1993.

[21] Set E., Akdemir A. O., Mumcu İ., Chebyshev type inequalities for conformable fractional integrals, *Miskolc Mathematical Notes*, 20(2) (2019), 1227-1236. https://doi:10.18514/MMN.2019.2766