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Estimation of Parameters of Topp-Leone Inverse Lomax Distribution in Presence of Right Censored Samples

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Highlights

- Topp-Leone inverse Lomax is proposed as a new model.
- Structure properties of the new model are provided.
- Reliability and parameter estimators are obtined from complete and Type II censored samples.
- Simulation as well as data analysis is worked out.

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Abstract

In this paper, we deal with a three-parameter inverse Lomax cited as the Topp-Leone inverse Lomax (TLIL) distribution depend on Topp-Leone-G family. Expressions of its density and distribution functions are explored. The structure properties of suggested model are provided like quantile function, moments, incomplete moments and Rényi entropy. Maximum likelihood estimators of the TLIL distribution parameters along with reliability estimator are worked out via complete and type II censored samples. To investigate the statistical properties of estimates we present numerical illustration along with two real data.

1. INTRODUCTION

The inverse Lomax (**IL**) distribution is one of the notable lifetime models that gives an application in economics and actuarial sciences (see [1]). Kleiber [2] employed the IL to get Lorenz ordering relationship among ordered statistics. McKenzie et al. [3] applied the IL model on geophysical databases.

The IL distribution is a particular model from generalized beta distribution of the second kind. The IL distribution is the inverse of Lomax distribution. The probability density function (**pdf**) of the IL distribution with shape parameter γ and scale parameter φ is

$$g(x;\gamma,\varphi)=\gamma\varphi^{-\gamma}x^{\gamma-1}(1+\frac{x}{\varphi})^{-(\gamma+1)}; \qquad x>0,\gamma,\varphi>0. \tag{1}$$

The associated cumulative distribution function (**cdf**) is

$$G(x;\gamma,\varphi)=(1+\frac{x}{\varphi})^{-\gamma};\ x>0,\ \gamma,\varphi>0. \tag{2}$$

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Rahman and Aslam [4] discussed Bayesian estimation of unknown shape parameter of IL model owing to various loss functions. Further, Bayesian prediction of future ordered observations from IL mixture model was discussed by Rahman and Aslam [5]. Singh et al. [6] discussed reliability estimator of the IL distribution in presence of type II censoring (**TIIC**). Jan and Ahmad [7] considered the behaving shape parameter of IL distribution through distinct approximation manner. Estimation of entropy for IL model was considered from multiple censored samples by Bantan et al. [8]. Recent extensions of the IL distribution for further usage have been provided by several researches, for instance the reader can refer to [9-13].

Recently, several researchers considered various generated families of continuous distributions in order to develop new models as well as provide great flexibility in modelling real data. Many families of distributions were submitted from dissimilar bounded or unbounded distributions. Two essential components, namely generator and a parent distribution, are desired to assign new families (see [14]). Jones [15] provided beta family owing to beta random variable. Alzaarteh et al. [16] proposed the widespread family using any non-negative continuous random variable *T* as the generator, instead of the beta random variable. They defined this class as follows

$$F(x)=\int_0^{W\,(G\,(x))} r(t)dt,$$

where $r(t)$ is the pdf of a non-negative continuous random variable *T*. This generated family is called "T-X distribution". The transformation $W(G(x))$ verify that: $W(G(x))\in[0,\infty)$ and it is a monotonic non-decreasing function. For a random variable *T* has the Topp Leone (**TL**) distribution on [0, 1], the TL-G family has been proposed by AL-Shomrani et al. [17] with cdf given by

$$F(x)=\left(1-(\overline{G}(x))\right)^{\alpha}, \tag{3}$$

where α is the shape parameter. The pdf corresponding to (3) is given by

$$f(x)=2\alpha g(x)\overline{G}(x)\left(1-(\overline{G}(x))\right)^{\alpha-1}, \tag{4}$$

where $G(x)$ is the baseline distribution function and $g(x)$ is the associated density. Further, the TL-G family has been discussed by Rezaei et al. [18].

In many life-testing experiments, censoring is essential according to save the total time on test and to reduce the cost associated with the experiment. Various censoring methods are at hand to experimenter such as type-I in which the test stops at a pre-fixed time, and TIIC in which test stops at predetermined number of failures.

In the present work, we define a new model as a modification of the IL distribution with three-parameter. We provide some comprehensive descriptions of its statistical properties. Then we estimate the reliability function and population parameters of the model using the maximum likelihood (**ML**) method when the available data are drawn from complete and TIIC. The layout of the paper consists of the following sections. In Section 2, we introduce the three-parameter TLIL distribution and the formation of the pdf and cdf of their expansions for stated distribution is given. Statistical properties of TLIL distribution are illustrated in Section 3. In Section 4, ML estimators, approximate confidence intervals (**CIs**) and reliability estimators are derived. Numerical study is given to assess the behaviour of estimates and analysis to real data is considered as appear in Section 5. Finally we come up with concluding remarks in Section 6.

2. TOPP-LEONE INVERSE LOMAX MODEL

In this section, we define a three-parameter TLIL distribution. Expansions for the pdf and cdf, reliability, hazard and reversed hazard functions are also presented.

Inserting (2) in (3) yields TLIL distribution with cdf specified by

$$F(x;\alpha,\gamma,\varphi)=[1-\{1-(1+\frac{\varphi}{x})^{-\gamma}\}^{2}]^{\alpha}; \qquad x,\alpha,\gamma,\varphi>0. \tag{5}$$

The associated pdf corresponding to (5) is given by

$$f(x;\alpha,\gamma,\varphi)=2\alpha\gamma\varphi^{-\gamma}x^{\gamma-1}(1+\frac{x}{\varphi})^{-(\gamma+1)}[1-(1+\frac{\varphi}{x})^{-\gamma}][1-\{1-(1+\frac{\varphi}{x})^{-\gamma}\}^{2}]^{\alpha-1}; x>0;\alpha,\gamma,\varphi>0. \tag{6}$$

Figure 1 shows possible shapes of TLIL pdf for certain values of parameters. The extra shape parameter α is considered as a manner to supply a more flexible TLIL distribution.

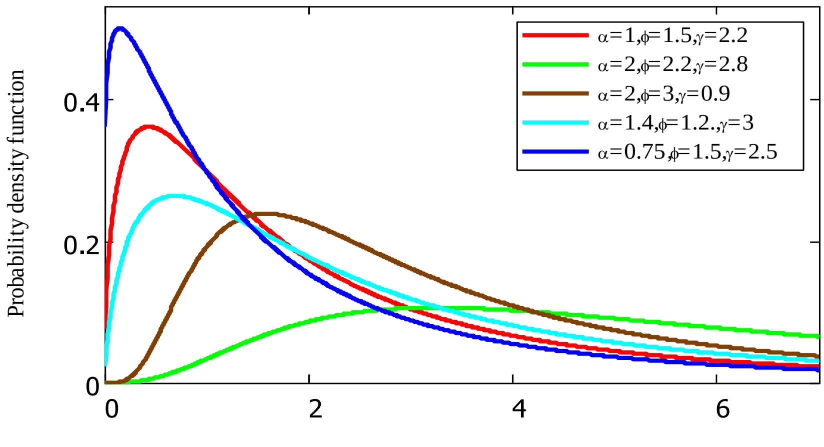

Figure 1. *The pdf plots of TLIL distribution*

2.1. Expansions of pdf and cdf

The binomial expansion, for real non-integer value of α, is given by

$$(1-D)^{\kappa}=\sum_{c=0}^{\infty}\frac{(-1)^{c}\,\Gamma(\kappa+1)}{c!\Gamma(\kappa+1-j)}D^{c}. \tag{7}$$

Then, by applying the previous binomial series for pdf (6), so it can be formed as follows

$$f(x;\alpha,\gamma,\varphi)=\sum_{j=0}^{\infty}\sum_{m=0}^{2j+1}\eta_{j,m}\gamma\varphi^{-\gamma}x^{\gamma-1}(1+\frac{x}{\varphi})^{-(\gamma+1)})(1+\frac{\varphi}{x})^{-\gamma m}, \tag{8}$$

where

$$\eta_{j,m}=\frac{(-1)^{j+m}\,2\Gamma(\alpha+1)}{j!\Gamma(\alpha+1-j)(m+1)}\binom{2j+1}{m}.$$

Also, it can be expressed as follows

$$f(x;\alpha,\gamma,\varphi)=\sum_{j=0}^{\infty}\sum_{m=0}^{2j+1}w_{j,m}k_{m+1}(x),$$

where $w_{j,m}=\dfrac{\eta_{j,m}}{(m+1)}$, and $k_{m+1}(x)=(m+1)g(x)(G(x))^{m}$.

Note that $k_{m+1}(x)$ is the exponentiated IL distribution with power parameter *m*. The expansion of the cdf is produced from expansion (7) for s is a positive integer as follows

$$\left[F(x;\alpha,\gamma,\varphi)\right]^{s}=\sum_{i=0}^{\infty}\sum_{l=0}^{2i}\zeta_{i,l}[G(x)]^{l}, \tag{9}$$

where $\zeta_{i,l}=\dbinom{2i}{l}\dfrac{(-1)^{i+l}\,\Gamma(\alpha s+1)}{\Gamma(\alpha s+1-i)i\,!}$ and *G*(*x*) is the cdf of IL distribution.

2.2. Reliability Analysis

The reliability function; $S(x;\alpha,\gamma,\varphi)$, the hazard rate function (**hrf**); $\pi(x;\alpha,\gamma,\varphi)$, cumulative hrf; $H(x;\alpha,\gamma,\varphi)$, and reversed hrf;TLIL distribution are respectively given bythe of $R(x;\alpha,\gamma,\varphi)$,

$$S(x;\alpha,\gamma,\varphi)=1-[1-\{1-(1+\varphi/x)^{-\gamma}\}^{2}]^{\alpha},$$

$$\pi(x;\alpha,\gamma,\varphi)=\frac{2\alpha\gamma\varphi^{-\gamma}x^{\gamma-1}(1+\frac{x}{\varphi})^{-(\gamma+1)}[1-(1+\varphi/x)^{-\gamma}][1-\{1-(1+\varphi/x)^{-\gamma}\}^{2}]^{\alpha-1}}{1-[1-\{1-(1+\varphi/x)^{-\gamma}\}^{2}]^{\alpha}}.$$

$$H(x;\alpha,\gamma,\varphi)=-\ln(1-[1-\{1-(1+\frac{\varphi}{x})^{-\gamma}\}^{2}]^{\alpha}).$$

and,

$$R(x;\alpha,\gamma,\varphi)=2\alpha\gamma\varphi^{-\gamma}x^{\gamma-1}[1-(1+\frac{\varphi}{x})^{-\gamma}](1+\frac{x}{\varphi})^{-(\gamma+1)}[1-\{1-(1+\frac{\varphi}{x})^{-\gamma}\}^{2}]^{-1}.$$

Figure 2 shows a variety of possible shapes of hrf of TLIL distribution for specific values of parameters.

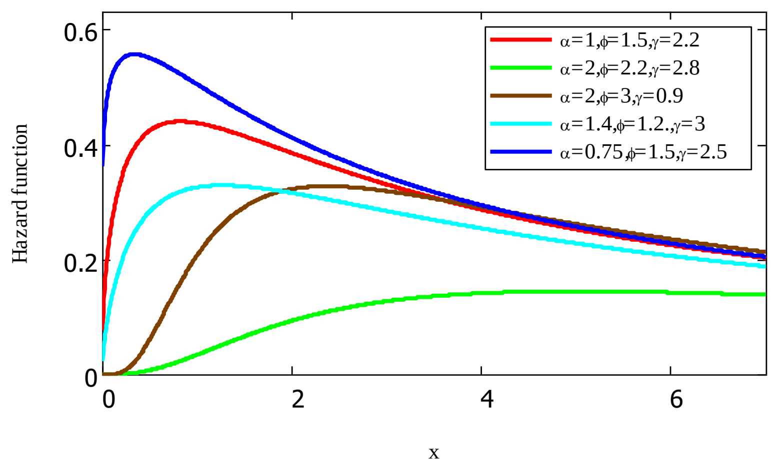

***Figure 2**. Plots of hrf of TLIL distribution*

3. PRINCIPLE PROPERTIES

We derive some significant characteristics of the TLIL distribution, specifically; the r^{th} moment, incomplete moments, moments of residual life function and Rényi entropy.

3.1. Moments

Moments in statistical analysis are important in study characteristics and shapes of distribution such that spread and dispersion which measured by mean and variance. It can study the flatness or peakedess of distribution which measured by kurtosis, also it can be used to study the symmetry of shape of distribution which measured by skewness. The r^{th} moment for the TLIL can be obtained from pdf (8) as follows

$$\mu_r' = \sum_{j=0}^{\infty}\sum_{m=0}^{2j+1}\eta_{j,m}\int_0^{\infty}\gamma\varphi^{-\gamma}x^{r+\gamma-1}(1+\frac{x}{\varphi})^{-(\gamma+1)}(1+\frac{\varphi}{x})^{-\gamma m}\,\mathrm{d}x.$$

Hence the r^{th} moment of TLIL is obtained as follows

$$\mu_r' = \sum_{j=0}^{\infty}\sum_{m=0}^{2j+1}(-1)^{j+m}\eta_{j,m}\gamma\varphi^{r}\frac{\Gamma(1-r)\Gamma(r+\gamma+\gamma m)}{\Gamma(1+\gamma+\gamma m)}. \quad (10)$$

Fisher and Kılıcman [19] proved the following

$$\Gamma(0)=-\varepsilon,\ \Gamma(-r)=\frac{(-1)^r}{r!}\delta(r)-\frac{(-1)^r}{r!}\varepsilon\ for\ r=1,2,...,$$

where ε denotes Euler's constant, and $\delta(r)=\sum_{i=1}^{r}\frac{1}{i}.$

Setting r =1, 2, 3 and 4 in (10), we can obtain the first four moments about zero. The mean and (μ_1') variance (σ^2) of the TLIL distribution for some selected values of the parameters which can be calculated numerically in Table 1.

Table 1. *Mean and variance of TLIL distribution*

α	φ	$\gamma=1$		$\gamma=1.5$		$\gamma=2$	
		μ_1'	σ^2	μ_1'	σ^2	μ_1'	σ^2
1.5	0.02	0.027	0.042	0.044	0.093	0.062	0.164
	0.5	0.687	26.14	1.106	58.175	1.541	102.59
	1	1.356	104.65	2.212	232.70	3.082	410.36
2.5	0.02	0.039	0.069	0.062	0.154	0.086	0.272
	0.5	0.973	43.295	1.558	96.316	2.148	169.82
	1	1.945	173.18	3.115	385.26	4.295	679.27
3.5	0.02	0.049	0.097	0.077	0.215	0.106	0.379
	0.5	1.218	60.347	1.931	134.23	2.647	236.64
	1	2.436	241.39	3.861	536.91	5.295	946.55

Next, we derive a simple formula for the r^{th} incomplete moment of X defined by $\Lambda_r(y)=E(X^r|X<y)$. So, the quantity $\Lambda_r(y)$ comes from (8) as

$$\Lambda_r(y)=\sum_{j=0}^{\infty}\sum_{m=0}^{2j+1}\eta_{j,m}\gamma\varphi^r B(1-r,r+\gamma+\gamma m,\frac{\gamma}{\gamma+y}),$$

where $B(.,.,x)$ is the incomplete beta function. The incomplete moments are useful in fields like economics, reliability, demography, insurance and medicine.

3.2. Skewness and Kurtosis

The effect of each shape parameters α and γ on the skewness and kurtosis of the TLIL distribution is considered here based on quantiles. The quantile function of the TLIL distribution say $Q(u)$ can be obtained as

$$Q(u)=\varphi\left([1-\sqrt{1-(u)^{\frac{1}{\alpha}}}]^{\frac{1}{-\gamma}}-1\right)^{-1},\tag{11}$$

where u is a number between $(0,1)$. The Bowley' skewness and Moors' kurtosis based on Equation (11) are plotted for certain values of γ as function of α, and α as function of γ (see Figures 3 and 4). These plots show that the skewness decreases when α gets larger for fixed φ and when γ increases for fixed φ. Figure 4 reveal that there is great flexibility of kurtosis curves.

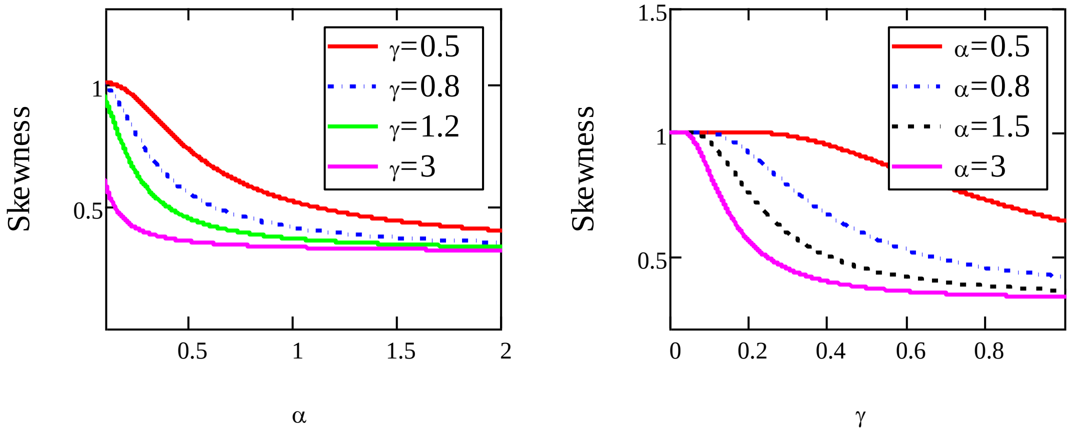

***Figure 3**. The Bowley' skewness of the TLIL distribution as a function of* α *and* γ

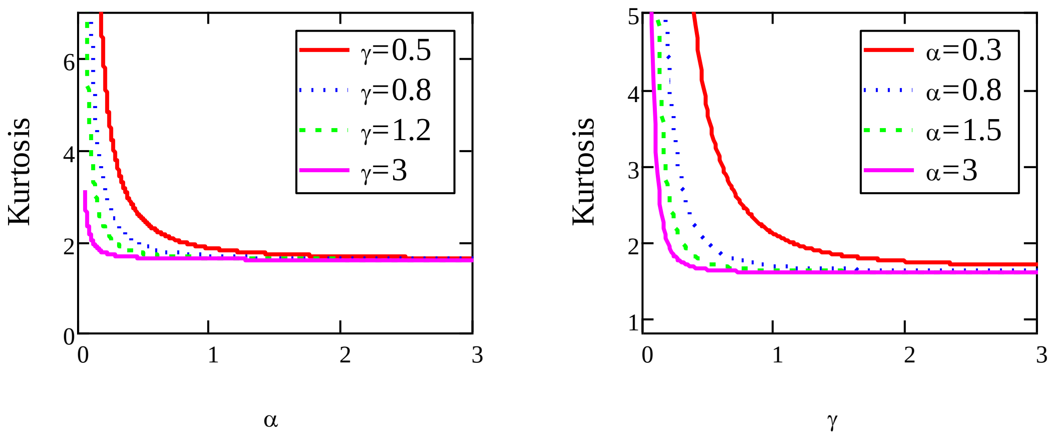

***Figure 4**. The Moors' kurtosis of the TLIL distribution as a function of* α *and* γ

3.3. Moments of Residual Life Function

The n^{th} moment of the residual life (RR) of TLIL model is given by

$$m_n(y)=\frac{1}{S(y)}\int_t^\infty (x-y)^n f(x)dx.$$

Hence, n^{th} moment of RR of TLIL distribution is yielded by applying binomial expansion of $(x-y)^n$ as follows

$$m_n(y)=\frac{1}{S(y)}\sum_{j=0}^{\infty}\sum_{m=0}^{2j+1}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\gamma\varphi^{-\gamma}\eta_{j,m}\int_y^\infty x^k x^{\gamma-1}y^{n-k}(1+\frac{x}{\varphi})^{-(\gamma+1)})(1+\frac{\varphi}{x})^{-\gamma m}dx,$$

which leads to

$$m_n(y)=\frac{1}{S(y)}\sum_{j=0}^{\infty}\sum_{m=0}^{2j+1}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}y^{n-k}\gamma\varphi^{k}\eta_{j,m}\,\mathrm{B}(1-k,k+\gamma+\gamma m,\frac{\varphi}{\varphi+y}),$$

where $\mathrm{B}(.,.,x)$ is the incomplete beta function. For n =1, we get the mean residual life of TLIL distribution which has many applications.

3.4. Rényi Entropy

The Rényi entropy of X for continuous random variable with range R is a measure of uncertainty. It is defined as follows

$$\upsilon_R(X)=\frac{1}{1-\vartheta}\log\{\int_R f(x)^{\vartheta}dx\}, \qquad \vartheta\neq 1, \vartheta>0.$$

To obtain Rényi entropy of the TLIL distribution, we must obtain explicit expression for $(f(x;\alpha,\gamma,\varphi))^{\vartheta}$, as follows

$$(f(x;\alpha,\gamma,\varphi))^{\vartheta}=\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{j+m}}{j!}\binom{2j+\vartheta}{m}\frac{(2\alpha\gamma\varphi^{-\gamma})^{\vartheta}x^{\vartheta(\gamma-1)}\Gamma(\vartheta(\alpha-1)+1)}{\Gamma(\vartheta(\alpha-1)+1-j)}(1+\frac{\varphi}{x})^{-\gamma m}(1+\frac{x}{\varphi})^{-\vartheta(\gamma+1)}.$$

Therefore, the Rényi entropy of TLIL distribution is given by

$$\upsilon_R(x)=\frac{1}{1-\vartheta}\log\{\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}L_{j,m}\frac{\Gamma(2\vartheta-1)\Gamma(\vartheta(\gamma-1)+\gamma m+1)}{\Gamma(\vartheta m+\vartheta\gamma+\vartheta)}\},$$

where

$$L_{j,m}=\frac{(-1)^{j+m}}{j!}\binom{2j+\vartheta}{m}(2\alpha\gamma)^{\vartheta}\frac{\Gamma(\vartheta(\alpha-1)+1)}{\Gamma(\vartheta(\alpha-1)+1-j)}\varphi^{1-\vartheta}.$$

Furthermore, the $\varpi-$ entropy, where $\varpi>0,\ \varpi\neq 1,$ is given by

$$H_{\varpi}(x)=\frac{1}{\varpi-1}\left(1-\left\{\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}L_{j,m}\frac{\Gamma(2\varpi-1)\Gamma(\varpi(\gamma-1)+\gamma m+1)}{\Gamma(\varpi m+\varpi\gamma+\varpi)}\right\}\right).$$

4. PARAMETER ESTIMATION

This section deals with parameter and reliability function estimators for TLIL distribution from complete and TIIC samples. ML estimators are obtained as well as the approximate CIs are constructed.

The TIIC scheme is observed when n units are placed on test, and the test is stopped at the time of the h^{th} failure. It has the advantage that the number of observed failures is fixed to be h which ensures reasonable information is available for statistical analysis.

Suppose that n items in which their lifetimes follow TLIL distribution (6) are put on test and the test is stopped at fixed value of h failure. The likelihood function for the observed samples $X_{(1)}<X_{(2)}<...<X_{(h)}$ is given by

$$L(\alpha,\gamma,\varphi;\underline{x})=\frac{n!}{(n-h)!}\prod_{i=1}^{h}\frac{2\alpha\gamma}{\varphi^{\gamma}}\frac{x_{(i)}^{\gamma-1}}{(1+\frac{x_{(i)}}{\varphi})^{\gamma+1}}\left[1-\left(1+\frac{\varphi}{x_{(i)}}\right)^{-\gamma}\right][1-\{1-(1+\frac{\varphi}{x_{(i)}})^{-\gamma}\}^2]^{\alpha-1}(1-[1-\{1-(1+\frac{\varphi}{x_{(h)}})^{-\gamma}\}^2]^{\alpha})^{n-h}.$$

For simplicity, write x_i instead of $x_{(i)}$. The logarithm of the previous equation, denoted by, ln *L*, takes the following form

$$\ln L \propto h\ln\alpha + h\ln\gamma - h\gamma\ln\varphi + (\gamma-1)\sum_{i=1}^{h}\ln(x_i) - (\gamma+1)\sum_{i=1}^{h}\ln(1+\frac{x_i}{\varphi}) + \sum_{i=1}^{h}\ln z_i + (\alpha-1)\sum_{i=1}^{h}\ln(1-z_i^{\,2}) + (n-h)\ln(1-(1-z_h^{\,2})^{\alpha}).$$

where $z_i = 1-(1+\frac{\varphi}{x_i})^{-\gamma}, z_h = 1-(1+\frac{\varphi}{x_h})^{-\gamma}$.

The components of the score vector ($\partial\ln L/\partial\varphi, \partial\ln L/\partial\alpha, \partial\ln L/\partial\gamma$) are given below

$$\frac{\partial\ln L}{\partial\varphi} = \frac{-h\gamma}{\varphi} + (\gamma+1)\sum_{i=1}^{h}\frac{x_i}{\varphi(\varphi+x_i)} + \sum_{i=1}^{h}\frac{1}{z_i}\frac{\partial z_i}{\partial\varphi} - 2(\alpha-1)\sum_{i=1}^{h}\frac{z_i}{1-z_i^{\,2}}\frac{\partial z_i}{\partial\varphi} + (n-h)\frac{2z_h\,\alpha(1-z_h^{\,2})^{\alpha-1}}{1-(1-z_h^{\,2})^{\alpha}}\frac{\partial z_h}{\partial\varphi}, \tag{12}$$

$$\frac{\partial\ln L}{\partial\alpha} = \frac{h}{\alpha} + \sum_{i=1}^{h}\ln(1-z_i^{\,2}) - (n-h)\frac{(1-z_h^{\,2})^{\alpha}\ln(1-z_h^{\,2})}{1-(1-z_h^{\,2})^{\alpha}}, \tag{13}$$

and

$$\frac{\partial\ln L}{\partial\gamma} = \frac{h}{\gamma} - h\ln\varphi + \sum_{i=1}^{h}\ln x_i - \sum_{i=1}^{h}\ln(1+\frac{x_i}{\varphi}) + (\alpha-1)\sum_{i=1}^{h}\frac{2z_i}{1-z_i^{\,2}}\frac{\partial z_i}{\partial\gamma} + \sum_{i=1}^{h}\frac{1}{z_i}\frac{\partial z_i}{\partial\gamma} + (n-h)\frac{2\alpha\, z_h(1-z_h^{\,2})^{\alpha-1}}{1-(1-z_h^{\,2})^{\alpha}}\frac{\partial z_h}{\partial\gamma}, \tag{14}$$

$$\frac{\partial z_i}{\partial\varphi} = \gamma(1+\frac{\varphi}{x_i})^{-\gamma-1}(\frac{1}{x_i}), \frac{\partial z_i}{\partial\gamma} = (1+\frac{\varphi}{x_i})^{-\gamma}\ln(1+\frac{\varphi}{x_i}).$$

The ML estimators of the model parameters are produced after solving the non-linear Equations (12) - (14) numerically. Also, the ML estimators in case of complete sample are obtained by setting *h* = *n* in previous equations. However, it is difficult to find a closed form solution for the above equations; so an iterative procedure is applied to obtain ML estimates (**MLEs**).

Based on invariance property of ML estimation, we obtain the ML estimator of *S*(*x*) by replacing the parameters from (12) - (14) by their ML estimators as follows:

$$\hat{S}(x;\hat{\alpha},\hat{\gamma},\hat{\varphi}) = 1-[1-\{1-(1+\hat{\varphi}/x)^{-\hat{\gamma}}\}^2]^{\hat{\alpha}}.$$

For interval estimation of the parameters, the observed information matrix $I(\Psi)=\{I_{uv}\}$ for $(u,v)=(\alpha,\gamma,\varphi)$ must be obtained. Under the regularity conditions, asymptotic variance–covariance matrix of the MLEs of α,φ and γ can obtained by inverting the following observed information matrix,

$$I^{-1}(\hat{\Psi})=\begin{bmatrix}\operatorname{var}(\hat{\alpha}) & \operatorname{cov}(\hat{\alpha},\hat{\varphi}) & \operatorname{cov}(\hat{\alpha},\hat{\gamma})\\ \operatorname{cov}(\hat{\alpha},\hat{\varphi}) & \operatorname{var}(\hat{\varphi}) & \operatorname{cov}(\hat{\varphi},\hat{\gamma})\\ \operatorname{cov}(\hat{\alpha},\hat{\gamma}) & \operatorname{cov}(\hat{\varphi},\hat{\gamma}) & \operatorname{var}(\hat{\gamma})\end{bmatrix}=\frac{1}{|I|}\begin{bmatrix}\partial^2\ln L/\partial\alpha^2 & \partial^2\ln L/\partial\alpha\partial\varphi & \partial^2\ln L/\partial\alpha\partial\gamma\\ \partial^2\ln L/\partial\alpha\partial\varphi & \partial^2\ln L/\partial\varphi^2 & \partial^2\ln L/\partial\varphi\partial\gamma\\ \partial^2\ln L/\partial\alpha\partial\gamma & \partial^2\ln L/\partial\varphi\partial\gamma & \partial^2\ln L/\partial\gamma^2\end{bmatrix}_{\alpha=\hat{\alpha},\varphi=\hat{\varphi},\gamma=\hat{\gamma}}.$$

The asymptotic normality of ML estimation can be used to compute the asymptotic $100(1-\nu)\%$, $0<\nu<1$, confidence limits for α,γ and φ as follows

$$\hat{\alpha}\pm z_{\nu/2}\sqrt{\operatorname{var}(\hat{\alpha})},\ (\hat{\gamma}\pm z_{\nu/2}\sqrt{\operatorname{var}(\hat{\gamma})},\ \text{and}\ (\hat{\varphi}\pm z_{\nu/2}\sqrt{\operatorname{var}(\hat{\varphi})},$$

where $z_{\nu/2}$, standard normal percentile and $(1-\nu)$ is the confidence coefficient. The asymptotic $100(1-\nu)\%$ confidence limits for reliability function is given by

$$\hat{S}(x)\pm z_{\nu/2}\sqrt{\operatorname{var}(\hat{S}(x))}.$$

5. SIMULATION STUDY

We present numerical study to examine the behaviour of the ML and reliability function estimates. Measures like mean square errors (**MSEs**), relative bias (**RB**), standard errors (**SEs**), lower bound (**LB**) of CIs, upper bound (**UB**) of CIs, and average length (**Le**) of 95% CIs are calculated. We perform the following algorithm.

- 1000 random sample of sizes 50, 200 and 300 are generated from the TLIL distribution.
- The number of failure items**;** *h*, is chosen as 90% (censoring scheme) and 100% (complete sample).
- Parameters values are specified as (i) ($\alpha=1,\gamma=0.8,\varphi=2$), (ii) ($\alpha=1.25,\gamma=0.8,\varphi=2$), (iii) ($\alpha=0.8,\gamma=1.5,\varphi=2$), (iv) ($\alpha=1.3,\gamma=1,\varphi=1$), (v) ($\alpha=1,\gamma=1.5,\varphi=0.5$), (vii) ($\alpha=1.6,\gamma=0.8,\varphi=2$).
- The MSEs, RB, SEs for all samples sizes and for all selected set of parameters are computed.
- The LB, UB and Le atfor all samples sizes and for all selected sets of parameters are $\nu=0.05$ computed.
- Reliability estimates and the associated 95% CI for reliability function at different mission time t_0 where t_0= 0.1, 0.3, 0.5 for different sample sizes are presented.

5.1. Numerical Results

The observed outcomes are reported in Tables 2 to 9, we detect the following about the performance of estimates:

- SE of all estimates decreases as *n* increases. Also, it has the smallest values in complete sample (see Tables 2 and 3).
- The MSEs and RBs of α,γ and φ estimates decrease as *n* increases for all cases. Also, the MSEs and RBs at *h* = 0.9 *n* for all estimates are greater than the corresponding at *h* = *n* (see Tables 2 and 3).
- The MSEs and SEs of γ estimates are less than the corresponding for α and φ estimates in almost all cases (see Tables 2 and 3).
- For all estimates of parameters, as the value of failure items; *h* increases the MSEs, RBs and SEs decrease.
- For fixed value of φ=2, sestimate α of sand RB sincreases, the MSE α =0.8 and as the value of γ decrease at the same sample size (see Table 2).
- For all cases, it is clear that the Le of CIs for MLEs decrease as *n* increases (see Tables 4 and 5).

- The MSE for α estimates gets the smallest values at $\alpha=1.3, \gamma=1$ and $\varphi=1$ compared to other cases. The MSE for φ estimates at $\alpha=0.8, \gamma=1.5$ andhas the smallest value compared to $\varphi=2$ other cases. The MSE of γ estimates at $\alpha=1, \gamma=0.8$ andables Thas the smallest value (see $\varphi=2$ 2 and 3).
- For all cases, reliability estimates decrease as the mission time's increase. Also, the Le of CIs gets shorter as *n* increases (see Tables 6-8).

Table 2. *The MSEs RBs and SEs of the estimates for sets (i), (ii) and (iii)*

n	*h*	Criteria Measures	$\alpha=1, \gamma=0.8, \varphi=2$			$\alpha=1.25, \gamma=0.8, \varphi=2$			$\alpha=0.8, \gamma=1.5, \varphi=2$		
			$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\varphi}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\varphi}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\varphi}$
50	45	MSE	0.241	0.005	0.247	0.113	0.010	0.184	0.560	0.303	0.552
		RB	0.453	0.038	0.089	0.184	0.048	0.043	0.855	0.349	0.273
		SE	0.004	0.001	0.009	0.005	0.002	0.008	0.006	0.003	0.010
	50	MSE	0.231	0.005	0.202	0.109	0.011	0.153	0.553	0.303	0.542
		RB	0.443	0.033	0.066	0.060	0.047	0.026	0.846	0.348	0.259
		SE	0.004	0.001	0.009	0.005	0.001	0.008	0.006	0.003	0.010
200	180	MSE	0.036	0.004	0.109	0.024	0.010	0.095	0.331	0.278	0.397
		RB	0.151	0.007	0.035	0.034	0.078	0.007	0.686	0.345	0.248
		SE	0.001	0.000	0.002	0.001	0.000	0.002	0.001	0.001	0.002
	200	MSE	0.031	0.004	0.109	0.022	0.009	0.094	0.327	0.276	0.396
		RB	0.152	0.001	0.033	0.033	0.074	0.004	0.682	0.344	0.226
		SE	0.001	0.000	0.002	0.001	0.000	0.002	0.001	0.001	0.002
300	270	MSE	0.021	0.003	0.085	0.015	0.009	0.068	0.317	0.275	0.343
		RB	0.109	0.009	0.040	0.012	0.086	0.012	0.680	0.345	0.242
		SE	0.000	0.000	0.001	0.000	0.000	0.001	0.000	0.000	0.001
	300	MSE	0.021	0.003	0.083	0.015	0.009	0.067	0.301	0.271	0.333
		RB	0.108	0.006	0.025	0.011	0.084	0.012	0.661	0.343	0.237
		SE	0.000	0.000	0.001	0.000	0.000	0.001	0.000	0.000	0.001

***Table 3**. The MSEs, RBs and SEs of the estimates for sets (iv), (v) and (vi)*

n	h	Criteria Measures	$\alpha=1.3,\gamma=1,\varphi=1$			$\alpha=1,\gamma=1.5,\varphi=0.5$			$\alpha=1.6,\gamma=0.8,\varphi=2$		
			$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\varphi}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\varphi}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\varphi}$
50	45	MSE	0.341	0.032	0.054	0.216	0.044	0.012	0.101	0.038	0.276
		RB	0.353	0.115	0.127	0.386	0.105	0.148	0.014	0.183	0.047
		SE	0.007	0.003	0.004	0.005	0.003	0.002	0.006	0.003	0.007
	50	MSE	0.335	0.031	0.049	0.205	0.041	0.012	0.099	0.037	0.120
		RB	0.348	0.113	0.121	0.376	0.100	0.145	0.015	0.180	0.044
		SE	0.007	0.003	0.004	0.005	0.003	0.002	0.006	0.003	0.006
200	180	MSE	0.114	0.014	0.029	0.085	0.048	0.008	0.052	0.033	0.092
		RB	0.209	0.082	0.066	0.385	0.048	0.270	0.088	0.196	0.058
		SE	0.001	0.000	0.001	0.001	0.001	0.000	0.001	0.000	0.001
	200	MSE	0.111	0.014	0.029	0.077	0.041	0.008	0.049	0.029	0.091
		RB	0.204	0.080	0.066	0.225	0.105	0.084	0.079	0.183	0.057
		SE	0.001	0.000	0.001	0.001	0.001	0.000	0.001	0.000	0.001
300	270	MSE	0.111	0.015	0.025	0.061	0.036	0.006	0.042	0.025	0.069
		RB	0.222	0.100	0.079	0.206	0.101	0.078	0.082	0.174	0.046
		SE	0.001	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.001
	300	MSE	0.107	0.015	0.025	0.056	0.032	0.006	0.041	0.024	0.068
		RB	0.206	0.092	0.067	0.195	0.093	0.074	0.081	0.170	0.045
		SE	0.001	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.001

***Table 4**. The LB, UB and Le of the estimates for sets (i), (ii) and (iii)*

			$\alpha=1,\gamma=0.8,\varphi=2$			$\alpha=1.25,\gamma=0.8,\varphi=2$			$\alpha=0.8,\gamma=1.5,\varphi=2$		
n	h		LB	UB	Le	LB	UB	Le	LB	UB	Le
50	45	α	1.080	1.524	0.444	0.999	1.599	0.600	0.889	1.665	0.775
		γ	0.641	0.898	0.257	0.654	1.023	0.369	0.646	1.306	0.660
		φ	1.267	3.087	1.820	1.262	2.909	1.647	1.501	3.531	2.029
	50	α	1.092	1.532	0.439	1.005	1.599	0.594	0.889	1.640	0.751
		γ	0.643	0.891	0.248	0.712	1.007	0.295	0.662	1.281	0.619
		φ	1.384	3.099	1.715	1.395	2.590	1.195	1.536	3.557	2.021
200	180	α	0.917	1.166	0.249	1.009	1.333	0.324	1.010	1.408	0.398
		γ	0.687	0.925	0.238	0.716	1.009	0.293	0.781	1.184	0.403
		φ	1.438	2.703	1.265	1.381	2.591	1.209	1.738	3.255	1.518
	200	α	0.927	1.177	0.250	1.009	1.331	0.322	1.009	1.403	0.394
		γ	0.683	0.915	0.232	0.717	1.011	0.294	0.783	1.186	0.403
		φ	1.451	2.707	1.256	1.385	2.585	1.201	1.740	3.294	1.514
300	270	α	0.942	1.155	0.213	1.028	1.294	0.266	1.058	1.385	0.327
		γ	0.688	0.898	0.210	0.743	0.994	0.251	0.816	1.149	0.334
		φ	1.532	2.630	1.098	1.466	2.485	1.019	1.837	3.131	1.294
	300	α	0.941	1.151	0.209	1.034	1.297	0.264	1.048	1.369	0.322
		γ	0.692	0.899	0.206	0.740	0.994	0.253	0.811	1.145	0.334
		φ	1.541	2.625	1.085	1.461	2.477	1.016	1.846	3.136	1.290

Table 5. *The LB, UB and Le of the estimates for sets (iv), (v) and (vi)*

			$\alpha=1.3,\gamma=1,\varphi=1$			$\alpha=1,\gamma=1.5,\varphi=0.5$			$\alpha=1.6,\gamma=0.8,\varphi=2$		
n	*h*		LB	UB	Le	LB	UB	Le	LB	UB	Le
50	45	α	1.050	2.014	0.964	0.879	1.518	0.638	1.008	1.814	0.805
		γ	0.617	1.154	0.537	1.072	1.613	0.540	0.690	1.202	0.512
		φ	0.751	1.484	0.733	0.411	0.737	0.327	1.261	2.553	1.292
	50	α	1.031	1.994	0.964	0.883	1.500	0.681	1.010	1.816	0.806
		γ	0.623	1.151	0.528	1.084	1.616	0.532	0.692	1.196	0.504
		φ	0.756	1.487	0.730	0.408	0.736	0.328	0.937	2.201	1.263
200	180	α	1.214	1.696	0.481	0.907	1.294	0.387	1.105	1.524	0.419
		γ	0.731	1.067	0.335	1.067	1.583	0.517	0.782	1.132	0.350
		φ	0.761	1.384	0.623	0.390	0.699	0.309	1.335	2.435	1.100
	200	α	1.173	1.644	0.472	0.906	1.277	0.371	1.102	1.517	0.415
		γ	0.751	1.088	0.337	1.092	1.592	0.501	0.786	1.135	0.349
		φ	0.759	1.373	0.615	0.389	0.695	0.305	1.339	2.433	1.094
300	270	α	1.263	1.643	0.380	0.939	1.243	0.303	1.163	1.516	0.353
		γ	0.768	1.050	0.282	1.127	1.569	0.442	0.794	1.084	0.290
		φ	0.809	1.348	0.538	0.405	0.672	0.267	1.427	2.388	0.961
	300	α	1.246	1.621	0.376	0.933	1.231	0.298	1.165	1.518	0.353
		γ	0.762	1.041	0.279	1.140	1.582	0.442	0.791	1.082	0.291
		φ	0.809	1.345	0.537	0.405	0.669	0.264	1.429	2.390	0.961

Table 6. *MLEs and 95% CI of reliability estimates for sets (i) and (ii) at h =0.9*

			$\alpha=1,\gamma=0.8,\varphi=2$				$\alpha=1.25,\gamma=0.8,\varphi=2$			
n	*h*	t_0	$\hat{S}$	CI			$\hat{S}$	CI		
				LB	UB	Le		LB	UB	Le
50	45	0.1	0.836	0.824	0.849	0.024	0.895	0.885	0.906	0.020
		0.3	0.645	0.630	0.661	0.031	0.730	0.715	0.744	0.029
		0.5	0.522	0.506	0.538	0.032	0.606	0.590	0.622	0.032
200	180	0.1	0.833	0.830	0.836	0.006	0.894	0.892	0.897	0.004
		0.3	0.646	0.642	0.649	0.007	0.728	0.724	0.733	0.009
		0.5	0.523	0.519	0.527	0.007	0.605	0.601	0.609	0.007
300	270	0.1	0.833	0.831	0.835	0.003	0.894	0.891	0.895	0.003
		0.3	0.647	0.644	0.649	0.004	0.727	0.725	0.730	0.004
		0.5	0.525	0.522	0.527	0.005	0.605	0.601	0.608	0.006

Table 7. *MLEs and 95% CI of reliability estimates for sets (iii) and (iv) at h =0.9*

			$\alpha=0.8,\gamma=1.5,\varphi=2$				$\alpha=1.3,\gamma=1,\varphi=1$			
n	*h*	t_0	$\hat{S}$	CI			$\hat{S}$	CI		
				LB	UB	Le		LB	UB	Le
50	45	0.1	0.957	0.951	0.963	0.012	0.899	0.889	0.909	0.019
		0.3	0.856	0.845	0.867	0.022	0.686	0.671	0.701	0.030
		0.5	0.761	0.748	0.774	0.026	0.530	0.513	0.546	0.034
200	180	0.1	0.955	0.954	0.957	0.003	0.898	0.896	0.900	0.004
		0.3	0.853	0.850	0.855	0.005	0.688	0.684	0.691	0.007
		0.5	0.757	0.754	0.761	0.006	0.534	0.530	0.538	0.008
300	270	0.1	0.956	0.955	0.957	0.002	0.897	0.895	0.899	0.003
		0.3	0.852	0.850	0.854	0.003	0.688	0.685	0.690	0.004
		0.5	0.757	0.755	0.759	0.004	0.534	0.531	0.536	0.005

Table 8. *MLEs and 95% CI of reliability estimates for sets v and vii at h=0.9*

			$\alpha=1,\gamma=1.5,\varphi=0.5$				$\alpha=1.6,\gamma=0.8,\varphi=2$			
n	*h*	t_0	$\hat{S}$	CI			$\hat{S}$	CI		
				LB	UB	Le		LB	UB	Le
50	45	0.1	0.871	0.860	0.882	0.022	0.944	0.937	0.952	0.015
		0.3	0.592	0.576	0.608	0.032	0.812	0.799	0.824	0.025
		0.5	0.415	0.399	0.432	0.033	0.695	0.680	0.709	0.029
200	180	0.1	0.869	0.866	0.872	0.005	0.943	0.941	0.945	0.003
		0.3	0.593	0.589	0.596	0.007	0.812	0.809	0.815	0.006
		0.5	0.416	0.412	0.420	0.008	0.696	0.693	0.700	0.007
300	270	0.1	0.869	0.867	0.871	0.003	0.943	0.942	0.944	0.002
		0.3	0.593	0.591	0.596	0.005	0.811	0.809	0.813	0.004
		0.5	0.418	0.415	0.421	0.005	0.696	0.693	0.698	0.004

5.2. Data Analysis

Two real data are provided to illustrate the importance of the TLIL distribution compared with IL, TL generalized exponential (**TLGE**) and TL inverse Weibull (**TLIW**). Kolmogorov- Smirnov goodness of fit test is obtained for each data set and the p-values in each distribution indicate that the model fits the data very well.

The first data set about time between failures for repairable item (see [20]). The second data reported by Jorgensen [21] represent active repair times (in hours) for airborne communication transceiver. To compare the fitted models, measures include; -2log-likelihood function evaluated at the parameter estimates, Akaike information criterion (**AIC**), Bayesian information criterion (**BIC**), consistent AIC (**CAIC**) and Hannan-Quinn information criterion (**HQIC**) are considered. The better model is corresponding to the smallest values of mentioned measures. Table 9 gives the results of selected measures.

Table 9. *The statistics -2logL, AIC, CAIC, BIC and HQIC for the two real data sets*

Data I						Data Π				
model	-2LogL	AIC	BIC	CAIC	HQIC	-2LogL	AIC	BIC	CAIC	HQIC
TLIL	99.63	105.63	109.84	106.84	106.98	187.78	193.76	198.85	194.45	99.63
TLGE	600.25	606.25	610.46	607.46	607.60	195.39	201.39	206.45	202.05	600.25
TLIW	325.54	331.54	335.74	332.46	332.88	224.82	230.82	235.89	231.49	325.54
IL	199.01	205.01	209.22	205.94	206.36	314.48	320.48	325.55	321.15	199.01

Additionally, MLEs, reliability estimates of TLIL distribution and their SEs for both real data based on TIIC are listed in Table 10.

Table 10. *MLE, reliability estimate and SEs of TLIL distribution based on TIIC for both data*

Real data	*n*	*h*	*estimator*	*Estimate*	*SEs*
I	30	21	$\hat{\alpha}$ $\hat{\lambda}$ $\hat{\beta}$ $\hat{S}(0.1)$	0. 502 7.768 0.369 0.997	0.016 0.307 0.010 0.000
Π	40	28	$\hat{\alpha}$ $\hat{\lambda}$ $\hat{\beta}$ $\hat{S}(0.1)$	0.445 34.602 0.126 1	0.006 0.556 0.001 0.000

6. CONCLUDING REMARKS

In this paper, we introduce a three-parameter new model, so called, Topp-Leone inverse Lomax distribution. Explicit expressions for its density and distribution functions are proposed. Some of its statistical properties are derived. The ML estimators of population parameters are obtained as well as reliability estimator is derived in presence of complete and censored samples. The approximate confidence intervals of parameters together with interval of reliability estimator are provided. Simulation study is implemented to check the behaviour of proposed estimators and recommendations are reported based on simulation outcomes. The applicability and importance of the new model is proved empirically using two real data sets. Indeed, the TLIL model provides a better fit to these data than some other models.

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CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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