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Riemann Solutions on (LCS)n Manifolds Admitting Different Semi-Symmetric Structures

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Abstract

The object of the present paper is to study the Riemannian solitons on $(LCS)_n$-manifolds and we observed in this case the Riemann soliton on M is shrinking, steady or expanding according to $\alpha^2 - \rho$ being positive, zero or negative respectively. Here also we discussed the Riemann solitons in $(LCS)_n$-manifold admitting (i) $R \cdot C = 0$, $R \cdot K = 0$, (ii) $E \cdot C = 0$, $E \cdot K = 0$, (iii) $R \cdot R = 0$, $R \cdot P = 0$, $R \cdot E = 0$, $R \cdot P^* = 0$, $R \cdot \mathcal{M} = 0$, $R \cdot \mathcal{W}_i = 0$, $R \cdot \mathcal{W}_i^* = 0$, (iv) $E \cdot R = 0$, $E \cdot P = 0$, $E \cdot E = 0$, $E \cdot P^* = 0$, $E \cdot \mathcal{M} = 0$, $E \cdot \mathcal{W}_i = 0$ and $E \cdot \mathcal{W}_i^* = 0$.(for all $i = 1, 2, ...9$). We found that the Riemann soliton on M is shrinking, steady or expanding according to the conditions (i) $\alpha^2 - \rho$ being positive, zero or negative respectively, (ii) $[k(n-1)(n-2)(1+\alpha^2-\rho) - kr - r]$ being positive, zero or negative respectively and (iv) $\alpha^2 - \rho$ being negative, zero or positive. But for the condition (iii) the Riemann soliton on M is always steady.

1. Introduction

In 1982, R. S. Hamilton [1] first introduced the notion of Ricci flow. This concept generalized to the idea of Riemann flow ([2], [3]). Keeping the tune with Ricci soliton, Hirica and Udriste [4] introduced and discussed Riemann soliton. Recently various geometer like Venkatesha, Devaraja and Aruna ([5] and [6]) have studied Riemann soliton. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined ([2], [3]) as follows

$$\frac{\partial}{\partial t} G(t) = -2R(g(t)), \quad t \in [0, I], \tag{1}$$

where $G = \frac{1}{2} g \circledast g$, $\circledast$ is the Kularni-Nomizu product and R is the Riemann curvature tensor associated to the metric g. For $(0,2)$-tensors α and β, the kKulkarni-Nomizu product $(\alpha \circledast \beta)$ is given by

$$\begin{aligned}(\alpha \circledast \beta)(Y,U,V,Z) &= \alpha(Y,V)\beta(U,Z) + \alpha(U,Z)\beta(Y,V) \\ &\quad -\alpha(Y,Z)\beta(U,V) - \alpha(U,V)\beta(Y,Z).\end{aligned} \tag{2}$$

The authors Stepanov and Tsyganok [7] characterize the Riemann soliton in terms of infinitesimal harmonic transformation. The Riemann soliton is a smooth manifold M together with Riemannian metric g that satisfies

$$2R + \lambda(g \circledast g) + (g \circledast \pounds_W g) = 0, \tag{3}$$

where W is a potential vector field, $\pounds_V$ denotes the Lie-derivative along the vector field W and λ is a constant. The Riemann soliton also corresponds to the Riemann flow as a fixed point, and on the space of Riemannian

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metric modulo diffeomorphism they can be seen as a dynamic system. A Riemann soliton is called expanding, steady and shrinking when $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ respectively.

Throughout our manuscript, we denote by Q, S and r the Ricci operator, the Ricci curvature tensor and the scalar curvature, respectively.

Definition 1 *Let T and D be two tensors of type (0, 4). A Riemannian (or semi-Riemannian) manifold is said to be D-semisymmetric type if $T(X,Y) \cdot D = 0$ for all X, $Y \in \chi(M)$, the set all vector fields of the manifold M where $T(X,Y)$ acts on D as derivation of tensor algebra. The above condition is often written as $T \cdot D = 0$. Especially, if we consider $T = D = R$, then the manifold is called semisymmetric [8]. Details about the semisymmetry and other conditions of semisymmetry type are available in : [9], [10], [11], [12], [13], [14] and also references therein.*

In 2013, Kundu and Shaikh [15] investigated the equivalency of various geometric structures. They have established the following conditions:

i) $R \cdot C = 0$ and $R \cdot K = 0$ are equivalent and named such a class by C_1;

ii) $E \cdot C = 0$ and $E \cdot K = 0$ are equivalent and named such a class by C_2,

iii) $R \cdot R = 0$, $R \cdot P = 0$, $R \cdot E = 0$, $R \cdot P^* = 0$, $R \cdot \mathcal{M} = 0$, $R \cdot \mathcal{W}_i = 0$ and $R \cdot \mathcal{W}_i^* = 0$ (for all $i = 1, 2, ..., 9$) are equivalent and named such a class by C_3;

iv) $E \cdot R = 0$, $E \cdot P = 0$, $E \cdot E = 0$, $E \cdot P^* = 0$, $E \cdot \mathcal{M} = 0$, $E \cdot \mathcal{W}_i = 0$ and $E \cdot \mathcal{W}_i^* = 0$ (for all $i = 1, 2, ..., 9$) are equivalent and named such a class by C_4; where the concircular curvature tensor E [16], conformal curvature tensor C [17], conharmonic curvature tensor K [18], projective curvature tensor P [16], M-projective curvature tensor $\mathcal{M}$ [19], $\mathcal{W}_i$-curvature tensor, $i = 1, 2, ..., 9$([19], [20], [21]) and $\mathcal{W}_i^*$-curvature tensor, $i = 1, 2, ..., 9$ [19] are defined respectively by

$$E(U_1, V_1) = R(U_1, V_1) - \frac{r}{n(n-1)}(U_1 \wedge_g V_1), \tag{4}$$

$$\begin{aligned} C(U_1, V_1) &= R(U_1, V_1) - \frac{1}{n-2}[(U_1 \wedge_g QV_1) + (QU_1 \wedge_g V_1) \\ &\quad + \frac{r}{(n-1)}(U_1 \wedge_g V_1)], \end{aligned} \tag{5}$$

$$K(U_1, V_1) = R(U_1, V_1) - \frac{1}{n-2}[(U_1 \wedge_g QV_1) + (QU_1 \wedge_g V_1)], \tag{6}$$

$$P(U_1, V_1) = R(U_1, V_1) - \frac{1}{n-1}(U_1 \wedge_S V_1), \tag{7}$$

$$\mathcal{M}(U_1, V_1) = R(U_1, V_1) - \frac{1}{2(n-1)}[(U_1 \wedge_g QV_1) + (QU_1 \wedge_g V_1)], \tag{8}$$

$$\mathcal{W}_0(U_1, V_1) = R(U_1, V_1) - \frac{1}{(n-1)}(U_1 \wedge_g QV_1), \tag{9}$$

$$\mathcal{W}_0^*(U_1, V_1) = R(U_1, V_1) + \frac{1}{(n-1)}(U_1 \wedge_g QV_1), \tag{10}$$

$$\mathcal{W}_1(U_1, V_1) = R(U_1, V_1) - \frac{1}{(n-1)}(U_1 \wedge_S V_1), \tag{11}$$

$$\mathcal{W}_1^*(U_1, V_1) = R(U_1, V_1) + \frac{1}{(n-1)}(U_1 \wedge_S V_1), \tag{12}$$

$$\mathcal{W}_2(U_1, V_1) = R(U_1, V_1) - \frac{1}{(n-1)}[(QU_1 \wedge_g V_1) \tag{13}$$

$$\begin{aligned} \mathcal{W}_2^*(U_1, V_1) &= R(U_1, V_1) + \frac{1}{(n-1)}[(QU_1 \wedge_g V_1) \\ &\quad + (U_1 \wedge_g QV_1) - (U_1 \wedge_S V_1)], \end{aligned} \tag{14}$$

$$\mathcal{W}_3\left(U_1,V_1\right) = R\left(U_1,V_1\right) - \frac{1}{(n-1)}\left(V_1 \wedge_g QU_1\right), \tag{15}$$
$$\mathcal{W}_3^*\left(U_1,V_1\right) = R\left(U_1,V_1\right) + \frac{1}{(n-1)}\left(V_1 \wedge_g QU_1\right), \tag{16}$$
$$\mathcal{W}_5\left(U_1,V_1\right) = R\left(U_1,V_1\right) - \frac{1}{(n-1)}\left[\left(U_1 \wedge_g QV_1\right) - \left(U_1 \wedge_S V_1\right)\right], \tag{17}$$
$$\mathcal{W}_5^*\left(U_1,V_1\right) = R\left(U_1,V_1\right) + \frac{1}{(n-1)}\left[\left(U_1 \wedge_g QV_1\right) - \left(U_1 \wedge_S V_1\right)\right], \tag{18}$$

$$\mathcal{W}_7\left(U_1,V_1\right) = R\left(U_1,V_1\right) + \frac{1}{(n-1)}\left[\left(QU_1 \wedge_g V_1\right) - \left(U_1 \wedge_S V_1\right)\right], \tag{19}$$
$$\mathcal{W}_7^*\left(U_1,V_1\right) = R\left(U_1,V_1\right) - \frac{1}{(n-1)}\left[\left(QU_1 \wedge_g V_1\right) - \left(U_1 \wedge_S V_1\right)\right], \tag{20}$$
$$\mathcal{W}_4\left(U_1,V_1\right)Z_1 = R\left(U_1,V_1\right)Z_1 - \frac{1}{(n-1)}[g\left(U_1,Z_1\right)QV_1 - g\left(U_1,V_1\right)QZ_1], \tag{21}$$
$$\mathcal{W}_4^*\left(U_1,V_1\right)Z_1 = R\left(U_1,V_1\right)Z_1 + \frac{1}{(n-1)}[g\left(U_1,Z_1\right)QV_1 - g\left(U_1,V_1\right)QZ_1], \tag{22}$$
$$\mathcal{W}_6\left(U_1,V_1\right)Z_1 = R\left(U_1,V_1\right)Z_1 - \frac{1}{(n-1)}[S\left(V_1,Z_1\right)U_1 - g\left(U_1,V_1\right)QZ_1], \tag{23}$$

where
$$\left(U_1 \wedge_B V_1\right)Z_1 = B(V_1,Z_1)U_1 - B(U_1,Z_1)V_1. \tag{24}$$

The present paper is structured as follows. After introduction, in Section 2, we briefly recall some known results for $(LCS)_n$-manifolds. In section 3, we discussed Riemann solitons in $(LCS)_n$-manifolds and we obtained in this case the Riemann soliton on M is shrinking, steady or expanding according to $\alpha^2 - \rho$ being positive, zero or negative respectively. Here also we studied the Riemann solitons $(LCS)_n$-manifold admitting $R \cdot C = 0$, $R \cdot K = 0$, $E \cdot C = 0$, $E \cdot K = 0$, $R \cdot R = 0$, $R \cdot P = 0$, $R \cdot E = 0$, $R \cdot P^* = 0$, $R \cdot \mathcal{M} = 0$, $R \cdot \mathcal{W}_i = 0$, $R \cdot \mathcal{W}_i^* = 0$, $E \cdot R = 0$, $E \cdot P = 0$, $E \cdot E = 0$, $E \cdot P^* = 0$, $E \cdot \mathcal{M} = 0$, $E \cdot \mathcal{W}_i = 0$ and $E \cdot \mathcal{W}_i^* = 0$.

2. Properties of $(LCS)_n$-manifold

Let (M^n, g) be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have
$$g(\xi,\xi) = -1. \tag{25}$$
Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that for
$$g(U_1,\xi) = \eta(U_1), \tag{26}$$
the equation of the following form holds
$$(\nabla_{U_1}\eta)V_1 = \alpha\left[g(U_1,V_1) + \eta(U_1)\eta(V_1)\right], \tag{27}$$
for all vector fields U_1, V_1, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function, which satisfies
$$\nabla_{U_1}\alpha = (U_1\alpha) = d\alpha(U_1) = \rho\eta(U_1), \tag{28}$$
ρ being a certain scalar function. If we put
$$\phi U_1 = \frac{1}{\alpha}\nabla_{U_1}\xi, \tag{29}$$
then from (27) and (29) we get
$$\phi U_1 = U_1 + \eta(U_1)\xi, \tag{30}$$

from which it follows that ϕ is a symmetric $(1,1)$ tensor field. Thus the Lorentzian manifold M^n together with the unit timelike concircular vector field ξ, its associated 1-form η and the $(1,1)$ tensor field ϕ is said to be a Lorentzian concircular structure manifold, in brief, $(LCS)_n$-manifold. In an $(LCS)_n$-manifold, the following relations hold:

$$\eta(\xi) = -1, \;\; \phi \circ \xi = 0, \tag{31}$$

$$\eta(\phi U_1) = 0, \;\; g(\phi U_1, \phi V_1) = g(U_1, V_1) + \eta(U_1)\eta(V_1), \tag{32}$$

$$\eta(R(U_1, V_1)Z_1) = (\alpha^2 - \rho)[g(V_1, Z_1)\eta(U_1) - g(U_1, Z_1)\eta(V_1)], \tag{33}$$

$$R(U_1, V_1)\xi = (\alpha^2 - \rho)[\eta(V_1)U_1 - \eta(U_1)V_1], \tag{34}$$

$$R(\xi, U_1)V_1 = (\alpha^2 - \rho)[g(U_1, V_1)\xi - \eta(V_1)U_1], \tag{35}$$

$$S(U_1, \xi) = (n-1)(\alpha^2 - \rho)\eta(U_1), \tag{36}$$

for any vector fields U_1, V_1, Z_1.

In view of (33), from (4), (5), (6), (7) and (24) one can easily bring out the followings:

$$\begin{aligned} & g(C(U_1, V_1)Z_1, \xi) \\ = \; & \eta(C(U_1, V_1)Z_1) \\ = \; & \left[\frac{r}{(n-1)(n-2)} + (\alpha^2 - \rho) - \frac{(n-1)}{(n-2)}\right][g(V_1, Z_1)\eta(U_1) \\ - \; & g(U_1, Z_1)\eta(V_1)] - \frac{1}{n-2}\left[S(V_1, Z_1)\eta(U_1) - S(U_1, Z_1)\eta(V_1)\right], \end{aligned} \tag{37}$$

$$\begin{aligned} & g(K(U_1, V_1)Z_1, \xi) \\ = \; & \eta(K(U_1, V_1)Z_1)) \\ = \; & \left[(\alpha^2 - \rho) - \frac{(n-1)}{(n-2)}\right][g(V_1, Z_1)\eta(U_1) - g(U_1, Z_1)\eta(V_1)] \\ & - \frac{1}{n-2}\left[S(V_1, Z_1)\eta(U_1) - S(U_1, Z_1)\eta(V_1)\right], \end{aligned} \tag{38}$$

$$\begin{aligned} & g(E(U_1, V_1)Z_1, \xi) \\ = \; & \eta(E(U_1, V_1)Z_1) \\ = \; & \left[(\alpha^2 - \rho) - \frac{r}{n(n-1)}\right][g(V_1, Z_1)\eta(U_1) - g(U_1, Z_1)\eta(V_1)], \end{aligned} \tag{39}$$

$$\begin{aligned} & g(P(U_1, V_1)Z_1, \xi) \\ = \; & \eta(P(U_1, V_1)Z_1) \\ = \; & (\alpha^2 - \rho)[g(V_1, Z_1)\eta(U_1) - g(U_1, Z_1)\eta(V_1)] \\ - \; & \frac{1}{n-1}[S(V_1, Z_1)\eta(U_1) - S(U_1, Z_1)\eta(V_1)]. \end{aligned} \tag{40}$$

Definition 2 *An n-dimensional $(LCS)_n$-manifold is said to be an η-Einstein manifold if the Ricci curvature tensor S is of the form*

$$S = ag + b\eta \otimes \eta,$$

where a and b are smooth functions on M^n and η is a 1-form.

In particular, if $b = 0$, then M^n is said to be an *Einstein manifold*.

3. Riemann solitons admitting semi-symmetric strucres in $(LCS)_n$-manifolds

In this section we consider a $(LCS)_n$-manifold $(M^n, g, \phi, \xi, \eta)$ admits an Riemann soliton. Then taking account (2), into (3), we obtain

$$\begin{aligned} & 0 \\ = & 2R(Y_1, U_1, V_1, Z_1) + 2\lambda \left[g(Y_1, V_1) g(U_1, Z_1) - g(Y_1, Z_1) g(U_1, V_1) \right] \\ & + \left[g(Y_1, V_1) \pounds_\xi g(U_1, Z_1) + g(U_1, Z_1) \pounds_\xi g(Y_1, V_1) - g(Y_1, Z_1) \pounds_\xi g(U_1, V_1) - g(U_1, V_1) \pounds_\xi g(Y_1, Z_1) \right] \end{aligned} \tag{41}$$

Now, we express the Lie derivative along ξ on M as follows:

$$\begin{aligned} & (\pounds_\xi g)(U_1, V_1) \\ = & \pounds_\xi (g(U_1, V_1)) - g(\pounds_\xi U_1, V_1) - g(U_1, \pounds_\xi V_1) \\ = & \pounds_\xi g(U_1, V_1) - g([\xi, U_1], V_1) - g(U_1, [\xi, V_1]). \end{aligned} \tag{42}$$

Now using (29) in the foregoing equation we obtain

$$(\pounds_\xi g)(U_1, V_1) = 2\alpha g(\phi U_1, V_1). \tag{43}$$

By using (43) into (41), we get

$$\begin{aligned} & 0 \\ = & 2R(Y_1, U_1, V_1, Z_1) + 2\lambda \left[g(Y_1, V_1) g(U_1, Z_1) - g(Y_1, Z_1) g(U_1, V_1) \right] \\ & + [2\alpha g(\phi U_1, Z_1) g(Y_1, V_1) + 2\alpha g(\phi Y_1, V_1) g(U_1, Z_1) \\ & - 2\alpha g(\phi U_1, V_1) g(Y_1, Z_1) - 2\alpha g(\phi Y_1, Z_1) g(U_1, V_1)]. \end{aligned} \tag{44}$$

By the suitable contraction of (45), we get

$$\begin{aligned} & 0 \\ = & S(U_1, V_1) + [\lambda(1-n) + \alpha(2-n)] g(U_1, V_1) + \alpha(2-n) \eta(U_1) \eta(V_1). \end{aligned} \tag{45}$$

Taking $U_1 = V_1 = \xi$ in (45), we obtain

$$\lambda = -(\alpha^2 - \rho). \tag{46}$$

Thus we can state

Theorem 1 *If (g, ϕ, ξ, η) is a Riemann soliton on $(LCS)_n$-manifold, then it is shrinking, steady or expanding according to the condition $\alpha^2 - \rho$ being positive, zero or negative respectively.*

3.1. Riemann solitons on $(LCS)_n$-manifolds admitting the class C_1

Here, we consider $(LCS)_n$-manifolds admitting the condition

$$(R(U_1, V_1) \cdot C)(X_1, Y_1) Z_1 = 0,$$

which implies

$$\begin{aligned} & g(R(\xi, V_1) C(X_1, Y_1) Z_1, \xi) - g(C(R(\xi, V_1) X_1, Y_1) Z_1, \xi) \\ & - g(C(X_1, R(\xi, V_1) Y_1) Z_1, \xi) - g(C(X_1, Y_1) R(\xi, V_1) Z_1, \xi) = 0. \end{aligned} \tag{47}$$

Putting $V_1 = X_1 = e_i$ in (47) and taking the summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} & \sum_{i=1}^{n} g(R(\xi, e_i) C(e_i, Y_1) Z_1, \xi) - \sum_{i=1}^{n} g(C(R(\xi, e_i) e_i, Y_1) Z_1, \xi) \\ & - \sum_{i=1}^{n} g(C(e_i, R(\xi, e_i) Y_1) Z_1, \xi) - \sum_{i=1}^{n} g(C(e_i, Y_1) R(\xi, e_i) Z_1, \xi) = 0. \end{aligned} \tag{48}$$

In view of (31)-(36) and (37), (48) reduces to

$$\begin{aligned} & (n+1)\, S(Y_1, Z_1) \\ = \;& \left[r + (\alpha^2 - \rho)(n-1)(n-2) - (n-1)^2\right] g(Y_1, Z_1) \\ & + [r - (n-1)(\alpha^2 - \rho + n + 1)]\eta(Y_1)\eta(Z_1). \end{aligned} \tag{49}$$

In view of (45), (49) takes the form

$$\begin{aligned} & 0 \\ = \;& \left[r + (\alpha^2 - \rho)(n-1)(n-2) - (n-1)^2 + \lambda\,(n+1)\,(1-n) + (n+1)\,\alpha\,(2-n)\right] g(Y_1, Z_1) \\ & + [r - (n-1)(\alpha^2 - \rho + n + 1) + \alpha\,(n+1)\,(2-n)]\eta(Y_1)\eta(Z_1). \end{aligned} \tag{50}$$

Replacing Y_1, Z_1 by ξ in (50), we get

$$\lambda(n+1) = (\alpha^2 - \rho + n + 1) + (\alpha^2 - \rho)(n-2) - (n-1)^2, \tag{51}$$

Thus, we state the following theorem.

Theorem 2 *If $(g,\ \phi,\ \xi,\ \eta)$ is a Riemann soliton on $(LCS)_n$-manifold M^n admitting the class C_1, then it is shrinking, steady or expanding according to the condition $\left[(\alpha^2 - \rho + n + 1) + (\alpha^2 - \rho)(n-2) - (n-1)^2\right]$ being negative, zero or positive respectively.*

3.2. Riemann solitons on $(LCS)_n$-manifolds admitting the class C_2

Here, we consider $(LCS)_n$manifolds admitting the condition

$$(E(U_1, V_1) \cdot K)(X_1, Y_1)Z_1 = 0,$$

which implies

$$\begin{aligned} & g(E(\xi, V_1)K(X_1, Y_1)Z_1, \xi) - g(K(E(\xi, V_1)X_1, Y_1)Z_1, \xi) \\ & - g(K(X_1, E(\xi, V_1)Y_1)Z_1, \xi) - g(K(X_1, Y_1)E(\xi, V_1)Z_1, \xi) = 0. \end{aligned} \tag{52}$$

Putting $V_1 = X_1 = e_i$ in (52) and taking the summation over i, $1 \le i \le n$, we get

$$\begin{aligned} & \sum_{i=1}^{n} g(E(\xi, e_i)K(e_i, Y_1)Z_1, \xi) - \sum_{i=1}^{n} g(K(E(\xi, e_i)e_i, Y_1)Z_1, \xi) \\ & - \sum_{i=1}^{n} g(K(e_i, E(\xi, e_i)Y_1)Z_1, \xi) - \sum_{i=1}^{n} g(K(e_i, Y_1)E(\xi, e_i)Z_1, \xi) = 0. \end{aligned} \tag{53}$$

Using (31)-(36) and (38), (39), in (53), we obtain

$$\begin{aligned} & \frac{n-1}{(n-2)} S(Y_1, Z_1) - [\frac{r}{n-2} + k^2(n-3)]g(Y_1, Z_1) \\ = \;& [k^2(n-3) + k(n-1)(1 + \alpha^2 - \rho) - \frac{kr}{n-2}]\eta(Y_1)\eta(Z_1), \end{aligned} \tag{54}$$

where

$$k = (\alpha^2 - \rho) - \frac{(n-1)}{(n-2)}.$$

In view of (45), (54) takes the form

$$\begin{aligned} & -\left\{[r + k^2(n-3)\,(n-2)] + (n-1)\,[\lambda\,(1-n) + \alpha\,(2-n)]\right\} g(Y_1, Z_1) \\ = \;& [k^2(n-3)\,(n-2) + k(n-1)\,(n-2)\,(1 + \alpha^2 - \rho) - kr + \alpha\,(n-1)\,(2-n)]\eta(Y_1)\eta(Z_1). \end{aligned} \tag{55}$$

Replacing Y_1, Z_1 by ξ in (55)

$$\begin{aligned} & \lambda\,(n-1)^2 \\ = \;& -[k(n-1)\,(n-2)\,(1 + \alpha^2 - \rho) - kr - r]. \end{aligned} \tag{56}$$

Thus, we state the following theorem.

Theorem 3 *If $(g,\ \phi,\ \xi,\ \eta)$ is a Riemann soliton on $(LCS)_n$-manifold M^n admitting the class C_2, then it is shrinking, steady or expanding according to the condition $[k(n-1)(n-2)(1+\alpha^2-\rho)-kr-r]$ being positive, zero or negative respectively.*

3.3. Riemann solitons on $(LCS)_n$-manifolds admitting the class C_3

Here, we consider $(LCS)_n$-manifolds admitting the condition

$$(R(U_1,V_1)\cdot R)(X_1,Y_1)Z_1=0,$$

which implies

$$\begin{aligned} &g(R(\xi,V_1)R(X_1,Y_1)Z_1,\xi)-g(R(R(\xi,V_1)X_1,Y_1)Z_1,\xi) \\ &-g(R(X_1,R(\xi,V_1)Y_1)Z_1,\xi)-g(R(X_1,Y_1)R(\xi,V_1)Z_1,\xi)=0. \end{aligned} \tag{57}$$

Setting $V_1=X_1=e_i$ in (57), where $\{e_1,e_2,e_3,...,e_{n-1},e_n=\xi\}$ is an orthonormal basis of the tangent space at each point of the manifold M^n and taking the summation over i, $1\le i\le n$, we get

$$\begin{aligned} &\sum_{i=1}^{n} g(R(\xi,e_i)R(e_i,Y_1)Z_1,\xi)-\sum_{i=1}^{n} g(R(R(\xi,e_i)e_i,Y_1)Z_1,\xi) \\ &-\sum_{i=1}^{n} g(R(e_i,R(\xi,e_i)Y_1)Z_1,\xi)-\sum_{i=1}^{n} g(R(e_i,Y_1)R(\xi,e_i)Z_1,\xi)=0. \end{aligned} \tag{58}$$

Using (31)-(36) in (58), finally we obtain

$$S(Y_1,Z_1)=-2(\alpha^2-\rho)[g(Y_1,Z_1)+\eta(Y_1)\eta(Z_1)]. \tag{59}$$

In view of (45), (59) takes the form

$$\begin{aligned} &0 \\ =\ &\left[\lambda(1-n)+\alpha(2-n)-2(\alpha^2-\rho)\right]g(Y_1,Z_1)+\left[\alpha(2-n)-2(\alpha^2-\rho)\right]\eta(Y_1)\eta(Z_1). \end{aligned} \tag{60}$$

Setting $Y_1=Z_1=\xi$ in (60), we get

$$\lambda=0. \tag{61}$$

Thus, we state the following theorem.

Theorem 4 *If $(g,\ \phi,\ \xi,\ \eta)$ is a Riemann soliton on $(LCS)_n$-manifold M^n admitting the class C_3, then the Riemann soliton on M is always steady.*

3.4. Riemann solitons on $(LCS)_n$-manifolds admitting the class C_4

Here, we consider $(LCS)_n$-manifolds admitting the condition

$$(E(U_1,V_1)\cdot R)(X_1,Y_1)Z_1=0,$$

which implies

$$\begin{aligned} &g(E(\xi,V_1)R(X_1,Y_1)Z_1,\xi)-g(R(E(\xi,V_1)X_1,Y_1)Z_1,\xi) \\ &-g(R(X_1,E(\xi,V_1)Y_1)Z_1,\xi)-g(R(X_1,Y_1)E(\xi,V_1)Z_1,\xi)=0. \end{aligned} \tag{62}$$

Setting $V_1=X_1=e_i$ in (62) and taking the summation over i, $1\le i\le n$, we get

$$\begin{aligned} &\sum_{i=1}^{n} g(E(\xi,e_i)R(e_i,Y_1)Z_1,\xi)-\sum_{i=1}^{n} g(R(E(\xi,e_i)e_i,Y_1)Z_1,\xi) \\ &-\sum_{i=1}^{n} g(R(e_i,E(\xi,e_i)Y_1)Z_1,\xi)-\sum_{i=1}^{n} g(R(e_i,Y_1)E(\xi,e_i)Z_1,\xi)=0. \end{aligned} \tag{63}$$

Using (31)-(36) and (39) in (63), we obtain

$$S(Y_1, Z_1) = (n-1)(\alpha^2 - \rho)g(Y_1, Z_1). \quad (64)$$

In view of (45), (64) takes the form

$$\begin{aligned} & 0 \\ = & \left[\lambda(1-n) + \alpha(2-n) + (n-1)(\alpha^2 - \rho)\right] g(Y_1, Z_1) + \alpha(2-n)\eta(U_1)\eta(V_1). \end{aligned} \quad (65)$$

Replacing Y_1, Z_1 by ξ in (65)

$$\lambda = (\alpha^2 - \rho). \quad (66)$$

Thus, we state the following theorems.

Theorem 5 *If (g, ϕ, ξ, η) is a Riemann soliton on $(LCS)_n$-manifold M^n admitting the class C_4, then it is shrinking, steady or expanding according to the condition $\alpha^2 - \rho$ being negative, zero or positive respectively.*

4. Conclusion

As the semi-symmetric structures (i) $R \cdot C = 0$ and $R \cdot K = 0$ are equivalent, (ii) $E \cdot C = 0$ and $E \cdot K = 0$ are equivalent, (iii) $R \cdot R = 0$, $R \cdot P = 0$, $R \cdot E = 0$, $R \cdot P^* = 0$, $R \cdot \mathcal{M} = 0$, $R \cdot \mathcal{W}_i = 0$ and $R \cdot \mathcal{W}_i^* = 0$ (for all $i = 1, 2, ..., 9$) are equivalent, (iv) $E \cdot R = 0$, $E \cdot P = 0$, $E \cdot E = 0$, $E \cdot P^* = 0$, $E \cdot \mathcal{M} = 0$, $E \cdot \mathcal{W}_i = 0$ and $E \cdot \mathcal{W}_i^* = 0$ (for all $i = 1, 2, ..., 9$) are equivalent, so in a single stroke the properties of Riemann solitons admitting all the semi-symmetric structures have been discussed easily.

Declaration of Competing Interest

The author declares that there is no competing financial interests or personal relationships that influencethe work in this paper.

Authorship Contribution Statement

Ashoke Das: Validation , Writing- Reviewing.
Ashis Biswas: Conceptualization, Methodology, Resources and Editing and Visualization.
Bappaditya Debnath: Data Curation, Software, Formal analysis, Writing - Originaldraft preparation.

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