Conference Proceeding Science and Technology, 3(1), 2020, 110–114

*3nd International E-Conference on Mathematical Advances and Applications (ICOMAA 2020).*

ISSN: 2651-544X
http://dergipark.gov.tr/cpost

# Investigation of $\Gamma$-Invariant Equivalence Relations of Modular Groups and Subgroups

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**Abstract:** In [2], graphs and permutation groups and in [4], permutation groups releated with combinatorial sets were studied. In [3]-[5], the modular group $\boldsymbol{\Gamma}$, the movement of an element of the modular group on $\widehat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$ (extended set of rational numbers), Farey graph and suborbital graphs $G_{u,n}$ and $F_{u,n}$ were investigated. Furthermore, it is indicated that any two fixed points is conjugated in $\Gamma$ and the element of the modular group that fixes an element on $\widehat{\mathbb{Q}}$ is infinite period. Hence, the element of the modular group that fixes $\infty$ is symbolized as $\Gamma_\infty$. In the same study, H, the subgroups of $\Gamma$ of containing $\Gamma_\infty$ are obtained and its invariant equivalence relations are generated on $\widehat{\mathbb{Q}}$. Taking these points into account, in this study, we show that, the element that fixes $\frac{x}{y}$ in modular group based on the choice of $\frac{x}{y}$ for $x, y \in \mathbb{Z}$ and $(x, y) = 1$, instead of a special value of set $\widehat{\mathbb{Q}}$, such as $\infty$. Similarly, we study subgroup H containing $\Gamma_{\frac{x}{y}}$ and we examine its invariant equivalence relations on $\widehat{\mathbb{Q}}$.

**Keywords:** Infinite period, Invariant equivalence relations, Modular group.

## 1 Introduction

**Definition 1.1.** [3] Modular group is division group of $SL(2, \mathbb{Z})$ by $\{\mp I\}$ . So,

$$\Gamma = PSL(2, \mathbb{Z}) \cong SL(2, \mathbb{Z})/\{\mp I\}$$

$$\Gamma = \left\{ \mp \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Thus, the elements of the $\Gamma$ Modular group consist of the following matrices as

$$\mp \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a, b, c, d \in \mathbb{Z}, ad - bc = 1. \tag{1}$$

Each matrix is considered to be equivalent by its negative. Therefore, we will ignore the $\mp$ difference. With elements of set $\Gamma$ in $H^+ = \{z \in C : Im(z) > 0\}$ the upper half plane

$$z \longrightarrow \frac{az + b}{cz + d}. \tag{2}$$

It is a group that acts with Möbiüs transformations.

**Lemma 1.2.** [3]

i. The movement of $\Gamma$ on $\widehat{\mathbb{Q}}$ is transitive.
ii. The fixed of a point is infinitely period.

For example, let $\Omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ . We find that $\Omega$ such that $\Omega(\infty) = \infty$. If $\infty$ is taken as $\frac{1}{0}$, since $\Omega(\infty) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. So, $a = 1$ and $c = 0$. Since $det\Omega = 1$ by the definition, $d = 1$ is found for $ad - bc = 1$. But $b$ is provided for all $\mathbb{Z}$. Then for all $b \in \mathbb{Z}$, $\Omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty \subset \Gamma$. Thus $\Gamma_\infty$ is a group that infinitely period that produced by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

 

**Proposition 1.3.** [3] Let $(G,\Omega)$ is an transitive permutation group. In this case $(G,\Omega)$ is primitive$\Leftrightarrow G_\alpha$, the stabilizer of a point $\alpha\in\Omega$ is a maximal subgroup of $G$ for $\forall\alpha\in\Omega$.

In accordance with the proposition given above, the following features are provided:

i. $(G,\Omega)$ is transitive $\Leftrightarrow$ There is $\exists g\in G$ such that $g(x)=y$ for $\forall x,y\in\Omega$.
ii. $(G,\Omega)$ permutation group is not impritive, for $G_\alpha\lneqq H\lneqq G,\ \alpha\in\Omega$.
iii. $G_\alpha$ is a maximal subgroup of $G\Leftrightarrow G_\alpha=H$ or $H=G$ when $G_\alpha\le H\le G$ .
iv. Let assume that $G_\alpha<H<G$. Since G transitive, each element of set $\Omega$ is in the form of $g(\alpha)$ for a $g\in G$.
v. Let show that $\Omega=\{g(\alpha):g\in G\}$=$[\alpha]$ (So there is an only one orbid). Since $G$ transitive on $\Omega$, there is an $\exists g\in G$ such that $g(\alpha)=\beta$ for $\forall\alpha,\beta\in\Omega$. From here $\beta\in[\alpha]$. If $g=e$ is taken, $\beta=g(\alpha)=e(\alpha)=\alpha$.So $\beta\in[\alpha]$=$[\beta]$ is $\Omega\subset[\alpha]$. On the contrary, it is obvious that $[\alpha]\subset\Omega$. Because $s:G\times\Omega\longrightarrow\Omega$,$(g,\alpha)$ :=$g\alpha$=$g(\alpha)$. From here $\Omega$=$[\alpha]$ is obtained. So, if the action is transitive, there is only one orbid.

## 2 Some Equivalence Subgroups of $\Gamma$

The basic equivalence subgroup for $\Gamma$ is defined as

$$\Gamma(n)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma: a\equiv d\equiv 1, b\equiv c\equiv 0(modn)\right\}. \tag{3}$$

Some basic congruence subgroups can be given as follows:

$$\Gamma_1(n)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma: a\equiv d\equiv 1, c\equiv 0(modn)\right\} \tag{4}$$

$$\Gamma_0(n)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma: c\equiv 0(modn)\right\} \tag{5}$$

$$\Gamma^0(n)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma: b\equiv 0(modn)\right\} \tag{6}$$

$$\Gamma_0^0(n)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma: b\equiv c\equiv 0(modn)\right\}. \tag{7}$$

Among these equivalence groups, there is an order as $\Gamma(n)\le\Gamma_1(n)\le\Gamma_0^0\le\Gamma_0(n)(\Gamma^0(n))$ [1].
Let $\Gamma$ is an element of Modular group that acting on $\widehat{\mathbb{Q}}$. If there is a relation other than $\alpha\approx\beta\Leftrightarrow\alpha$=$\beta$ (Identity Relation) for all $\alpha,\beta\in\widehat{\mathbb{Q}}$ and $\alpha\approx\beta$ (Universal Relation) for all $\alpha,\beta\in\widehat{\mathbb{Q}}$, $(\Gamma,\widehat{\mathbb{Q}})$ is imprimitive, otherwise primitive.
Let $\Gamma_\alpha<H<\Gamma$ such that the stabilizer $\Gamma_\alpha$ of $\alpha$. By finding subgroups $H$ covering $\Gamma_\alpha$ equivalence groups on $\Gamma$ were found.
For $g,g^{'}\in\Gamma_\alpha$, "$\approx$" equivalence relation given by $g(\alpha)\approx g^{'}(\alpha)\Leftrightarrow g^{'}\in gH$ is well defined [3].
Let $g$=$\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ and $g^{'}$=$\begin{pmatrix} e & f\\ g & h\end{pmatrix}\in\Gamma$. For $\begin{pmatrix} a & b\\ c & d\end{pmatrix}:\dfrac{x}{y}\to\dfrac{ax+by}{cx+dy}=u$ and $\begin{pmatrix} e & f\\ g & h\end{pmatrix}:\dfrac{x}{y}\to\dfrac{ex+fy}{gx+hy}=v$ ;
$u\approx v\Leftrightarrow g^{-1}g^{'}\in H$.
$g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\Rightarrow g^{-1}=\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}$ and $g^{-1}g^{'}=\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}\begin{pmatrix} e & f\\ g & h\end{pmatrix}=\begin{pmatrix} de-bg & df-bh\\ ag-ec & ah-cf\end{pmatrix}\in H$
If $g^{-1}g^{'}\in H=\Gamma_0(n), ag-ec\equiv 0(modn)$. So, $\dfrac{a}{c}\equiv\dfrac{e}{g}(modn)$
If $g^{-1}g^{'}\in H=\Gamma^0(n), df-bh\equiv 0(modn)$. So, $\dfrac{d}{b}\equiv\dfrac{h}{f}(modn)$
If $g^{-1}g^{'}\in H=\Gamma_0^0(n), ag-ec\equiv 0(modn), df-bh\equiv 0(modn)$. So, $\dfrac{a}{c}\equiv\dfrac{e}{g}(modn),\ \dfrac{d}{b}\equiv\dfrac{h}{f}(modn)$

**Theorem 2.1.** [3] For each positive integer $n\neq 2,5$ there is a $\Gamma$-invariant equivalence relation on $\widehat{\mathbb{Q}}$ with $n$ blocks.

## 3 Results

**Theorem 3.1.** The fixed point of an arbitrary point is infinite period on $\widehat{\mathbb{Q}}$.

*Proof:*

Let the stabilizer of any two points are conjugated. For $\dfrac{x}{y}\in\widehat{\mathbb{Q}}$ and $(x,y)=1$;$a,b,c,d\in\mathbb{Z}$ , from here

$c(ax+by)-a(cx+dy)=cax+cby-acx-ady=(cb-ad)y=-y$

$d(ax+by)-b(cx+dy)=dax+dby-bcx-bdy=(ad-bc)x=x.$

So, we find $(ax+by,cx+dy)=1$. Let assume that $\dfrac{ax+by}{cx+dy}$ is in reduced form.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \to \frac{ax+by}{cx+dy}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of modular group that leaves $\frac{x}{y} \in \widehat{\mathbb{Q}}$ constant. So, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \to \frac{x}{y}$ . For $\frac{ax+by}{cx+dy} = \frac{x}{y}$;

1. $ax + by = x \Rightarrow (a-1)x + by = 0$

$cx + dy = y \Rightarrow cx + (d-1)y = 0$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ identity matrix is obtained for especially $x, y \neq 0, a = 1, b = 0, c = 0$ and $d = 1$.

2. Let $b, c \neq 0$. For $a = 1$ and $d = 1$, $\begin{pmatrix} 1 & b \\ c & 1 \end{pmatrix} : \frac{x}{y} \to \frac{x+by}{cx+y} = \frac{x}{y}$,
If $x = 0$,$\begin{pmatrix} 1 & b \\ c & 1 \end{pmatrix} : \frac{0}{y} \to \frac{0+by}{y} = \frac{0}{y} \to b = 0$,
$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in \Gamma$ that leaves fixed $\frac{x}{y} \in \widehat{\mathbb{Q}}$ for $x = 0$ and for all $c \in \mathbb{Z}$ is infinite period.
If $y = 0$,$\begin{pmatrix} 1 & b \\ c & 1 \end{pmatrix} : \frac{x}{0} \to \frac{x}{cx} = \frac{x}{0} \to c = 0$
$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \Gamma$ that leaves fixed $\frac{x}{y} \in \widehat{\mathbb{Q}}$ for $y = 0$ and for all $b \in Z$ is infinite period.
3. Let $b, c \neq 0$. For $a = 1$ and $d = 1$, $\begin{pmatrix} 1 & b \\ c & 1 \end{pmatrix} : \frac{x}{y} \Rightarrow \frac{x+by}{cx+y} = \frac{x}{y}$
$xy + by^2 = cx^2 + xy$ , $by^2 = cx^2 \Rightarrow \frac{x}{y} = \sqrt{\frac{b}{c}}$.
4. Let $b, c \neq 0$. For $a \neq 1$ and $d \neq 1$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \Rightarrow \frac{ax+by}{cx+dy} = \frac{x}{y}$
$axy + by^2 = cx^2 + dxy$

$y = \frac{\mp x\sqrt{a^2 - 2ad + 4bc + d^2} - ax + dx}{2b}$

**Proposition 3.2.**Let $(\Gamma, \widehat{\mathbb{Q}})$ is a transitive permutation group. In this case $(\Gamma, \widehat{\mathbb{Q}})$ is primitive$\Leftrightarrow \Gamma_\alpha$, the stabilizer of $\alpha \in \widehat{\mathbb{Q}}$, is a maximal subgroup of $\Gamma$ for all $\alpha \in \widehat{\mathbb{Q}}$.

In accordance with the proposition given above, the following features are provided:

i. $(\Gamma, \widehat{\mathbb{Q}})$ is transitive $\Leftrightarrow$ There is $g \in \Gamma$ such that $g(x) = y$ for $\forall x, y \in \widehat{\mathbb{Q}}$.

Since $\Gamma$ acts as transitive on $\widehat{\mathbb{Q}}$, there is $g \in \Gamma$ such that $g(x) = y$ for $\forall x, y \in \widehat{\mathbb{Q}}$.

ii. $(\Gamma, \widehat{\mathbb{Q}})$ permutation group is not primitive for $\Gamma_\alpha \gneqq H \gneqq \Gamma, \alpha \in \widehat{\mathbb{Q}}$.

Since $\Gamma_\alpha \gneqq H$, the relation given is not identity or universal relation.

Suppose there is an identity relation.

$g(\alpha) \approx g(\alpha^{'}) \Leftrightarrow g(\alpha) = g(\alpha^{'})$. From here, $g^{'} g^{-1}(\alpha) = \alpha \Rightarrow g^{'} \in g\Gamma_\alpha$. Then $\Gamma_\alpha \gneqq H$,$\exists h_0 \in H$ such that $h_0 \notin H$. So $h_0\alpha \neq \alpha = e(\alpha)$ ve $e(\alpha) \approx h_0\alpha$. Because, $h_0 \in eH = H$. From here, $e(\alpha) = h_0\alpha \Rightarrow h_0 \in e\Gamma_\alpha$, but contradicts with $h_0 \notin \Gamma_\alpha$.

Suppose there is an universal relation.

Since $H \gneqq \Gamma$, there is $\exists g_0 \in \Gamma$ such that $g_0 \notin H$. So $e(\alpha) \approx g_0$. But this is only possible with $g_0 \in eH = H$. This is a contradiction. Hence it is not an universal relation.

Hence $(\Gamma, \widehat{\mathbb{Q}})$ permutation group is imprimitive.

iii. $\Gamma_\alpha$ is a maximal subgroup of $\Gamma \Leftrightarrow \Gamma_\alpha = H$ or $H = \Gamma$ when $\Gamma_\alpha \leq H \leq \Gamma$ .

iv. Let assume that $\Gamma_\alpha < H < \Gamma$. Since $\Gamma$ transitive, each element of set $\widehat{\mathbb{Q}}$ is in the form of $g(\alpha)$ for a $g \in \Gamma$.

v. Let show that $\widehat{\mathbb{Q}} = \{g(\alpha) : g \in \gamma\}$=$[\alpha]$ (So there is an only one orbid). Since $\Gamma$ transitive on $\widehat{\mathbb{Q}}$, there is $g \in \Gamma$ such that $g(\alpha) = \beta$ for all $\alpha, \beta \in \widehat{\mathbb{Q}}$. From here $\beta \in [\alpha]$. If $g = e$,then $\beta = g(\alpha) = e(\alpha) = \alpha$. So, $\beta \in [\alpha]$=$[\beta]$ is $\widehat{\mathbb{Q}} \subset [\alpha]$. On the contrary, it is obvious that $[\alpha] \subset \widehat{\mathbb{Q}}$. Because $s : \Gamma \times \widehat{\mathbb{Q}} \longrightarrow \widehat{\mathbb{Q}}$,$s(g, \alpha)$ :=$g\alpha$=$g(\alpha)$. From here $\widehat{\mathbb{Q}}$=$[\alpha]$ is obtained. So, if the action is transitive, there is only one orbid.

vi. "$\approx$" equivalence relation on $\widehat{\mathbb{Q}}$ given by $g(\alpha) \approx g^{'}(\alpha) \Leftrightarrow g^{'} \in gH$ is well defined $G$-invariant relation.

Let $h \in H$ be arbitrary. First we have to show that

$g(\alpha) \approx g^{'}(\alpha) \Leftrightarrow h(g(\alpha)) \approx h(g^{'}(\alpha))$.

$g(\alpha) \approx g^{'}(\alpha) \Leftrightarrow g^{'} \in gH \Leftrightarrow hg^{'} \in hgH$

$h(g(\alpha)) \approx h(g^{'}(\alpha)) \Leftrightarrow hg(\alpha) \approx hg^{'}(\alpha) \Leftrightarrow hg^{'} \in hgH$.

vii. If $\beta \in \widehat{\mathbb{Q}}$, there is $g \in \Gamma$ such that $\beta = g(\alpha)$. So $[\beta]$ block containing $\beta$ given by $M = \{gh(\alpha) : h \in H\}$ set.

$[\beta] = \{\gamma \in \widehat{\mathbb{Q}} : \gamma \approx \beta\}$ ve $\beta = g(\alpha)$ for $g \in \Gamma$. Then $\gamma \in \widehat{\mathbb{Q}}$, there is $\exists s \in \Gamma$ such that $\gamma = s(\alpha)$.

$\gamma \approx \beta \Leftrightarrow s(\alpha) \approx g(\alpha) \Leftrightarrow g \in sH$

$\exists h \in H : g = sh \Rightarrow s = \dfrac{g}{h} \Rightarrow s = gh^{-1}$

$\gamma = s(\alpha) = gh^{-1}(\alpha) \in M \Rightarrow [\beta] \subset M$.

Conversely, we have to show that $gh(\alpha) \approx \beta$ for $gh(\alpha) \in M$.

Then $\beta = g(\alpha)$, $gh(\alpha) \approx g(\alpha) \Leftrightarrow g \in ghH = gH$

$g \in gH \Rightarrow gh(\alpha) \in [\beta] \Rightarrow M \subset [\beta]$. Consequently, $M = [\beta]$.

Especially, $[\alpha]$ block is $H(\alpha) = \{h(\alpha) : h \in H\}$ orbit.

If $\alpha = e(\alpha)$

$[\alpha] = [e(\alpha)] = \{eh(\alpha) : h \in H\} = \{h(\alpha) : h \in H\} = H(\alpha)$.

viii. The fixed of any two points in $\widehat{\mathbb{Q}}$ is also conjugate in $\Gamma$.

Let us $p, q \in \widehat{\mathbb{Q}}$. We have to show $S_p$ and $S_q$ conjugate in $\Gamma$, where

$S_p = \{T_1 \in \Gamma : T_1 p = p\}$, $S_q = \{T_2 \in \Gamma : T_2 q = q\}$.

There is $T_3 \in \Gamma$ such that $S_p = T_3 S_q T_3^{-1}$. $p, q \in \widehat{\mathbb{Q}}$ and since $\Gamma$ transitive on $\widehat{\mathbb{Q}}$, there is $L \in \Gamma$ such that $L_p = q$.

Let us $T \in S_p$. We find $S \in S_q$ such that $T = LSL^{-1}$. Then $LTL^{-1}(q) = q$, $LTL^{-1} \in S_q$ and $L^{-1}TL = S$.

From here $LSL^{-1} = T \in LS_qL^{-1}$, $S_p \subset LS_qL^{-1}$.

Similarly, there is $T_3 \in \Gamma$ such that $S_q = T_3 S_p T_3^{-1}$. $p, q \in \widehat{\mathbb{Q}}$ and since $\Gamma$ transitive on $\widehat{\mathbb{Q}}$, there is $L \in \Gamma$ such that $L_q = p$.

Let us $T \in S_q$. We find $S \in S_p$ such that $T = LSL^{-1}$. Then $LTL^{-1}(p) = p$, $LTL^{-1} \in S_p$ and $L^{-1}TL = S$.

From here $LSL^{-1} = T \in LS_pL^{-1}$, $S_q \subset LS_pL^{-1}$.

**Lemma 3.3.** *[3]* $\psi(n) = n \prod(1 + \dfrac{1}{p})$, where the product is over the distinct primes $p$ dividing $n$.

**Example 3.4.** 2 and 3 are primes dividing 6, $\psi(6) = 6.(1 + \dfrac{1}{2})(1 + \dfrac{1}{3}) = 6.\dfrac{3}{2}.\dfrac{4}{3} = 12$.

Especially, if $n$ is a $p$ prime number, there is $\psi(p) = p + 1$ blocks. These blocks are

$[0], [1], ..., [p-1], [\infty]$,where

$[j] = \{\dfrac{x}{y} \in \mathbb{Q} : x \equiv jy(modp)\}, j \neq \infty$

$[\infty] = \{\dfrac{x}{y} \in \mathbb{Q} : y \equiv 0(modp)\}$,

In this study, we examined Modular group and its subgroups. Elements of the Modular group can be represented as Möbius transformations. For example;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \Rightarrow \frac{az+b}{cz+d} \in \text{Möb}.$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma \Rightarrow \frac{1z+1}{0z+1} = z+1 \in \text{Möb}.$$

As a result of the transitive action on $\widehat{\mathbb{Q}}$, an element of Modular group permutate vertices transitively. If we take the first two vertices as $\infty$ and $\frac{u}{n}$ respectively, graph is denoted by $G_{u,n}$. Especially if we take $u = n = 1$, we find Farey graph.

In [3], $\approx_n$ non-trivial equivalence relation on $\widehat{\mathbb{Q}}$ defined;

$v \approx_n w \Leftrightarrow x \equiv ur(modn), y \equiv us(modn)$, where $v = \frac{r}{s}$, $w = \frac{x}{y}$ and $(u, n) = 1$.

**Lemma 3.5.** [5] $G_{u,n} = G_{u',n'} \Leftrightarrow n = n'$ and $u \equiv u'(modn)$.

**Lemma 3.6.** [3] $G_{u,n}$ is self-paired $\Leftrightarrow u^2 \equiv -1(modn)$.

**Lemma 3.7.** [3] The suborbital graph paired with $G_{u,n}$ is $G_{-\overline{u},n}$, where $u\overline{u} \equiv 1(modn)$.

**Lemma 3.8.** [3] $\frac{r}{s} \rightarrow \frac{x}{y} \in G_{u,n} \Leftrightarrow x \equiv \mp ur(modn), y \equiv \mp us(modn), ry - sx = \mp n.$

**Lemma 3.9.** [3] $\frac{r}{s} \rightarrow \frac{x}{y} \in F_{u,n} \Leftrightarrow x \equiv \mp ur(modn), ry - sx = \mp n.$

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