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Existence Results for a Nonlinear Fourth Order Ordinary Differential Equation with Four-Point Boundary Value Conditions

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Abstract

The aim of this paper is to study the more accurate existence results of positive solution for a nonlinear fourth order ordinary differential equation (for short NLFOODE) using four-point boundary value conditions (for short BVCs). The upper and lower solution method and Schauder's fixed point theorem have been applied to demonstrate the obtained existence results. First, the Green's function of the corresponding linear boundary value problem (for short BVP) has been constructed and then it is used to solve the considered BVP of this paper. An example has also been included at the end of this paper to support the analytic proof.

Keywords: NLFOODE with four-point BVCs; Upper and lower solution method; Schauder's fixed point theorem; Existence of positive solution.
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1. Introduction

It is well established that the fixed point technique and upper and lower solution method are most important techniques for checking the existence of positive solutions of ordinary differential equations using various BVCs. Boundary value problems (for short BVPs) for fourth order ordinary differential equations (for short ODEs) are used to describe a huge number of physical, biological and chemical phenomena, see for instance [1, 11, 13, 22, 23, 27] and references therein. In the last few decades, positive solution of two-point, three-point and four-point boundary value problems for second order, third order, fourth

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order as well as higher order has extensively been studied by using various techniques, see for instance [2, 3, 4, 5, 8, 9, 10, 14, 15, 19, 24, 25, 28] and references therein. Inspiring by the above-mentioned works, we have interested to check the existence of positive solutions of a four-point BVP for NLFOODE by applying upper and lower solution method [10] and SchauderâĂŹs fixed point theorem [20]. For brevity, here we only described the most recent analogous literature about the existence of positive solutions of four-point BVP for NLFOODE. Although, literature may contain some more general results on BVPs for nonlinear fourth order differential equation, for instance we may refer [6, 7, 16, 17, 18, 26, 29], but for the better correction of some previously developed results sometimes we have to reconsider some fewer general results. From this point of view, here we reconsider a fewer general result of Chen *et al.* [9].

In 2006, Chen *et al.* [9] checked the existence of positive solutions of following four-point BVP for NLFOODE by applying upper and lower solution method and Schauder's fixed point theorem:

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), \ \ t \in (0, 1), \\ u(0) = u(1) = 0, \\ au''(\xi_1) - bu'''(\xi_1) = 0, \ cu''(\xi_2) + du'''(\xi_2) = 0, \end{cases} \tag{1}$$

where a, b, c, d are nonnegative constants satisfying, $ad + bc + ac(\xi_2 - \xi_1) > 0$, $0 \leq \xi_1 < \xi_2 \leq 1$ and $f \in C([0, 1] \times \mathbf{R})$. They established their main result on basis the following lemma:

Lemma 1.1. *(See Lemma 2.2 of Chen et al.[9]). Suppose* a, b, c, d, ξ_1, ξ_2 *are nonnegative constants satisfying* $0 \leq \xi_1 < \xi_2 \leq 1, b - a\xi_1 \geq 0,\ d - c + c\xi_2 \geq 0$ *and* $\delta = ad + bc + ac(\xi_2 - \xi_1) \neq 0$. *If* $u(t) \in C^4[0, 1]$ *satisfies*

$$\begin{cases} u^{(4)}(t) \geq 0, \ t \in (0, 1), \\ u(0) \geq 0, \ u(1) \geq 0, \\ au''(\xi_1) - bu'''(\xi_1) \leq 0, \ cu''(\xi_2) + du'''(\xi_2) \leq 0, \end{cases}$$

then $u(t) \geq 0$ *and* $u''(t) \leq 0$ *for* $t \in [0, 1]$.

Unfortunately, this lemma is incorrect.

Now, we provide a counter example to demonstrate it.

Counter example to the Lemma 1.1(Lemma 2.2 of Chen *et al.* [9]). Let $u(t) = \frac{1}{6}t^4 - \frac{1}{3}t^3 + \frac{15}{196}t^2 + \frac{7}{64}$, $\xi_1 = \frac{5}{8}$, $\xi_2 = \frac{8}{15}$ and a, b, c, d are four positive constants such that $a = b$, and $c = d$. Then we have

$$\begin{cases} u^{(4)}(t) \geq 0, \ t \in (0, 1), \\ u(0) = \frac{7}{64} \geq 0, \ u(1) = \frac{181}{9408} \geq 0, \\ au''(\xi_1) - bu'''(\xi_1) = -a\frac{1279}{1568} \leq 0 \\ and\, cu''(\xi_2) + du'''(\xi_2) = -c\frac{4661}{22050} \leq 0, \ implies\, u(t) \geq 0\, for\, all\, t \in [0, 1]. \end{cases}$$

But, $u''\left(\frac{1}{16}\right) = \frac{225}{6272} > 0$, which means that the Lemma 2.2 of Chen *et al.* [9] is not correct.

Therefore, the results of Chen *et al.* [9] should be reconsidered. From this ground, here again we considered the fourth order four-point BVP given by (1) and establish the existence result of positive solutions of this BVP by applying upper and lower solution method [10] and Schauder's fixed point theorem [20].

The considered BVP of this paper relates to the classical bending theory of flexible elastic beams on a nonlinear basis. If we put $f(t, u(t)) = p(t)g(u(t))$ in the considered BVP given by (1), then it will refer as the beam equation and further physical interpretation of the beam equation can be found in the work of Zill and Cullen ([27], pp. 237-243).

Rest of this paper has been arranged as follows:

Section 2 is used to introduce some preliminaries facts. In Section 3, we state and prove our main result and verify it by a particular example. Finally, in Section 4 we conclude this paper.

2. Preliminaries Notes

In this section, we give some definitions, lemmas, and state Schauder's fixed point theorem which are crucial to establish our main result.

Definition 2.1. *(See [10]) A function $\alpha(t)$ is said to be a lower solution of the BVP given by (1), if it belongs to $C^4[0,\,1]$ and satisfies*

$$\begin{cases} \alpha^{(4)}(t) \le f(t,\alpha(t)),\ t\in(0,1),\\ \alpha(0)\le 0, \qquad \alpha(1)\le 0,\\ a\alpha''(\xi_1)-b\alpha'''(\xi_1)\ge 0,\ c\alpha''(\xi_2)+d\alpha'''(\xi_2)\ge 0. \end{cases}$$

Definition 2.2. *(See [10]) A function $\beta(t)$ is said to be an upper solution of the BVP given by (1), if it belongs to $C^4[0,\,1]$ and satisfies*

$$\begin{cases} \beta^{4}(t) \le f(t,\beta(t)), t\in(0,\,1),\\ \beta(0)\ge 0, \qquad \beta(1)\ge 0,\\ a\beta''(\xi_1)-b\beta'''(\xi_1)\le 0,\ c\beta''(\xi_2)+d\beta'''(\xi_2)\le 0. \end{cases}$$

Definition 2.3. *A function $u(t)$ is said to be a solution of the BVP given by (1), if it is both lower and upper solutions of that BVP.*

Lemma 2.1. *Assume a, b, c, d, ξ_1, ξ_2 are nonnegative constants satisfying $0\le\xi_1<\xi_2\le 1$, and $\delta = ad+bc+ac(\xi_2-\xi_1)\ne 0$. If $h(t)\in C[\xi_1,\,\xi_2]$, then the BVP*

$$\begin{cases} u''(t)=h(t),\ t\in[\xi_1,\,\xi_2],\\ au(\xi_1)-bu'(\xi_1)=0,\ cu(\xi_2)+du'(\xi_2)=0, \end{cases} \tag{2}$$

has a unique solution

$$u(t) = -\int_{\xi_1}^{\xi_2} G(t,\,s)h(s)ds,$$

where

$$G(t,\,s)=\frac{1}{\delta}\begin{cases} (a(t-\xi_1)+b)(d+c(\xi_2-s)), & \xi_1\le t<s\le\xi_2,\\ (a(s-\xi_1)+b)(d+c(\xi_2-t)), & \xi_1\le s\le t\le\xi_2 \end{cases}$$

is the Green's function of the linear BVP given by

$$\begin{cases} u''(t)=0,\ t\in[0,1],\\ au(\xi_1)-bu'(\xi_1)=0,\ cu(\xi_2)+du'(\xi_2)=0. \end{cases} \tag{3}$$

Proof. Here first we solve the BVP (3) by using Green's function.

The general solution of (3) is

$$u(t)=At+B. \tag{4}$$

Using the boundary conditions of (3), we obtain $A=B=0$. Hence (4) yields only trivial solution $u(t)=0$. Therefore, the unique GreenâĂŹs function exists for BVP (3) and is given by

$$G(t,s)=\begin{cases} a_1t+a_2, & \xi_1\le t<s\le\xi_2,\\ b_1t+b_2, & \xi_1\le s\le t\le\xi_2. \end{cases} \tag{5}$$

Now, by the properties of GreenâĂŹs function, we have

$$(b_1-a_1)s+(b_2-a_2)=0. \tag{6}$$

$$b_1 - a_1 = -1 \Rightarrow b_1 = a_1 - 1. \tag{7}$$

$$aG(\xi_1,\, s) - bG'(\xi_1,\, s) = 0 \Rightarrow (a\xi_1 - b)a_1 + aa_2 = 0. \tag{8}$$

$$cG(\xi_2,\, s) + dG'(\xi_2,\, s) = 0 \Rightarrow (c\xi_2 + d)b_1 + cb_2 = 0. \tag{9}$$

Solving (6),(7), (8) and (9), we obtain

$$a_1 = \frac{(ac(\xi_2 - s) + ad)}{\delta},\;\; a_2 = -\frac{(a\xi_1 - b)(c(\xi_2 - s) + d)}{\delta},$$
$$b_1 = \frac{(ac(\xi_1 - s) - bc)}{\delta}\; and\, b_2 = -\frac{(c\xi_2 + d)(a(\xi_1 - s) - b)}{\delta},$$

where $\delta = ad + bc + ac(\xi_2 - \xi_1)$.

Putting the values of a_1, a_2, b_1, and b_2 in (5), we obtain the unique Green's function

$$G(t,\, s) = \frac{1}{\delta}\begin{cases}(a(t-\xi_1) + b)(d + c(\xi_2 - s)), & \xi_1 \le t < s \le \xi_2,\\ (a(s-\xi_1) + b)(d + c(\xi_2 - t)), & \xi_1 \le s \le t \le \xi_2.\end{cases}$$

Therefore, the unique solution of BVP given by (3) is

$$u(t) = -\int_{\xi_1}^{\xi_2} G(t,\, s)ds,$$

where

$$G(t,\, s) = \frac{1}{\delta}\begin{cases}(a(t-\xi_1) + b)(d + c(\xi_2 - s)), & \xi_1 \le t < s \le \xi_2,\\ (a(s-\xi_1) + b)(d + c(\xi_2 - t)), & \xi_1 \le s \le t \le \xi_2,\end{cases}$$

and this solution ensure that the BVP given by (2) has a unique solution and which is

$$u(t) = -\int_{\xi_1}^{\xi_2} G(t,\, s)h(s)ds.$$

This completes the lemma. □

Remark 2.1. *Considering*

$$R(t) = \frac{1}{\delta}((a(t-\xi_1) + b)x_3 + (c(\xi_2 - t) + d)x_2), \tag{10}$$

$$G_1(t,\, s) = \begin{cases} t(1-s), & 0 \le t < s \le 1,\\ s(1-t), & 0 \le s \le t \le 1,\end{cases} \tag{11}$$

$$G_2(t,\, s) = \frac{1}{\delta}\begin{cases}(a(t-\xi_1) + b)(d + c(\xi_2 - s)), & \xi_1 \le t < s \le \xi_2,\\ (a(s-\xi_1) + b)(d + c(\xi_2 - t)), & \xi_1 \le s \le t \le \xi_2,\end{cases} \tag{12}$$

in the Lemma 2.2 of Chen et al. [9], they claimed that

$$\begin{aligned} u(t) =& tx_1 + (1-t)x_0 - \int_0^1 G_1(t,\, \xi)R(\xi)d\xi \\ &+ \int_0^1 G_1(t,\, \xi)\int_{\xi_1}^{\xi_2} G_2(\xi,\, s)h(s)dsd\xi, \end{aligned} \tag{13}$$

is a solution of the following BVP

$$\begin{cases} u^{(4)}(t) = h(t),\;\; t \in (0,\, 1),\\ u(0) = x_0,\;\; u(1) = x_1,\\ au''(\xi_1) - bu'''(\xi_1) = x_2,\;\; cu''(\xi_2) + du'''(\xi_2) = x_3,\end{cases}$$

where $x_0 \geq 0,\ x_1 \geq 0,\ x_2 \leq 0,\ x_3 \leq 0,\ \ h(t) \in C[0,\, 1]$ *and* $h(t) \geq 0$.

But the solution defined by (13) *is not correct. Definitely, by our Lemma 2.1,* (13) *should be replaced as follows:*

$$u(t) = tx_1 + (1-t)x_0 - \int_0^1 G_1(t,\, \xi)R(\xi)d\xi - \int_0^1 G_1(t,\, \theta)v(\theta)d\theta,$$

where,

$$v(\theta) = \int_{\xi_1}^{\theta} (\theta - s)h(s)ds + \frac{1}{\delta}\int_{\xi_1}^{\xi_2} (a(\xi_1 - \theta) - b)(c(\xi_2 - s) + d)h(s)ds.$$

Remark 2.2. *In Theorem 3.1 of Chen et al. [9], the operator* $A : C[0,\, 1] \to C[0,\, 1]$ *is defined as*

$$Au(t) = \int_0^1 G_1(t,\, \theta)\int_{\xi_1}^{\xi_2} G_2(\theta,\, s)f(s,\, u(s))dsd\theta,$$

where, $G_1(t,\, \theta)$ *and* $G_2(\theta,\, s)$ *are as in Remark 2.1. But, according to our Lemma 2.1 and Remark 2.1, this definition is not accurate. So, according to our Lemma 2.1 and Remark 2.1, the operator* $A : C[0,1] \to C[0,1]$ *should be defined as follows:*

$$\begin{aligned} Au(t) = &\int_0^1 G_1(t,\, \theta)\int_{\xi_1}^{\theta} (s - \theta)f(s,\, u(s))dsd\theta, \\ &+ \frac{1}{\delta}\int_0^1 G_1(t,\, \theta)\int_{\xi_1}^{\xi_2} (b - a(\xi_1 - \theta))(c(\xi_2 - s) + d)f(s,\, u(s))dsd\theta. \end{aligned} \tag{14}$$

Remark 2.3. *From the discussion of our counter example on the Lemma 2.2 of Chen et al. [9], it is clear that the Lemma 2.2 of Chen et al. [9] is partially true, i.e., this lemma should be considered in the following form:*

Suppose $a,\, b,\, c,\, d,\, \xi_1,\, \xi_2$ *are nonnegative constants satisfying* $0 \leq \xi_1 < \xi_2 \leq 1,\ b - a\xi_1 \geq 0,\ d - c + c\xi_2 \geq 0$ *and* $\delta = ad + bc + ac(\xi_2 - \xi_1) \neq 0$*. If* $u(t) \in C^4[0,\, 1]$ *satisfies*

$$\begin{aligned} &u^{(4)}(t) \geq 0,\ \ t \in (0,\, 1), \\ &u(0) \geq 0,\ \ u(1) \geq 0, \\ &au''(\xi_1) - bu'''(\xi_1) \leq 0,\ \ cu''(\xi_2) + du'''(\xi_2) \leq 0, \end{aligned}$$

then $u(t) \geq 0$ *for all* $t \in [0,\, 1]$.

Now, we state a lemma of Chen *et al.* [9], which will be needed to establish our main result.

Lemma 2.2. *(See Lemma 2.3 of Chen et al. [9]). If* $u(t) \in C^2[0,\, 1],\ \ u(0) = u(1) = 0$ *and* $u''(t) \leq 0$ *for* $t \in [0,\, 1]$*, then* $p_1 t(1-t) \leq u(t) \leq p_2 t(1-t)$*, where* $p_1 = \frac{1}{2}min_{t\in[0,\,1]}[-u''(t)],\ \ p_2 = \frac{1}{2}max_{t\in[0,\,1]}[-u''(t)]$.

We end this section by stating the Schauder's fixed point theorem [20], which will be needed to establish our main result.

Theorem 2.3. *(See Theorem 2.1 of [20]). Let* B *be a Banach space and* C *be a nonempty closed convex subset of* B*. If* $T : C \to C$ *is a compact operator, then* T *has a fixed point in* B.

3. Main Results

In this section, we state and prove our main result and justify it by a suitable example of NLFOODE with four-point BVCs.

Theorem 3.1. *If the following assumptions are satisfied:*
(A_1) *Let* a, b, c, d, ξ_1, ξ_2 *are nonnegative constants satisfying*

$$0 \le \xi_1 < \xi_2 \le 1,\ b - a\xi_1 \ge 0,\ d - c + c\xi_2 \ge 0,\ and$$
$$\delta = ad + bc + ac(\xi_2 - \xi_1) \ne 0;$$

(A_2) *Let* $f(t,\, u(t)) \in C([0,\, 1] \times [0,\, +\infty),\, \boldsymbol{R}^+)$ *be non-decreasing relative to*

$$u,\ f(t,\, t(1-t)) \ne 0\ for\ t \in (\xi_1,\, \xi_2)\ and\ there\ exists\ a\ positive$$
$$constant\ \lambda < 1\ such\ that\ m^\lambda f(t,\, u(t)) \le f(t,\, mu(t))\ for\ any$$
$$0 \le m \le 1;$$

then the BVP given by (1) *has a positive solution* $u \in C^2[0,\, 1]$.

Proof. We will prove our theorem by three major steps.
Step 1.
In this step, we will prove that the functions $\alpha(t) = m_1 p(t)$ and $\beta(t) = m_2 p(t)$ are lower and upper solutions of BVP given by (1) respectively, where

$$0 < m_1 \le min\left\{\frac{1}{c_2},\, (c_1)^{\frac{\lambda}{1-\lambda}}\right\},$$
$$m_2 \ge max\left\{\frac{1}{c_1},\, (c_2)^{\frac{\lambda}{1-\lambda}}\right\}\ and$$
$$c_1 = min\left\{1,\, \frac{1}{2}\cdot\frac{1}{\delta}\int_{\xi_1}^{\xi_2}(b - a(\xi_1 - s))(c(\xi_2 - s) + d)f(s,\, s(1-s))ds\right\} > 0,$$
$$c_2 = max\left\{1,\, \frac{1}{2}max_{t\in[0,1]}\frac{1}{\delta}\int_{\xi_1}^{\xi_2}(b - a(\xi_1 - t))(c(\xi_2 - s) + d)f(s, s(1-s))ds\right\},$$

$$\begin{aligned} p(t) = &\int_0^1 G_1(t,\, \theta)\int_{\xi_1}^{\theta}(s-\theta)f(s,\, u(s))dsd\theta,\\ &+ \frac{1}{\delta}\int_0^1 G_1(t,\, \theta)\int_{\xi_1}^{\xi_2}(b - a(\xi_1 - \theta))(c(\xi_2 - s) + d)f(s,\, u(s))dsd\theta. \end{aligned} \tag{15}$$

By Lemma 2.1, we have

$$\begin{aligned} p''(t) &= -\int_{\xi_1}^{s}(s-s)f(s,\, s(1-s))ds\\ &\quad - \frac{1}{\delta}\int_{\xi_1}^{\xi_2}(b - a(\xi_1 - s))(c(\xi_2 - s) + d)f(s,\, s(1-s))ds\\ &= -\frac{1}{\delta}\int_{\xi_1}^{\xi_2}(b - a(\xi_1 - s))(c(\xi_2 - s) + d)f(s,\, s(1-s))ds. \end{aligned} \tag{16}$$

$$\begin{cases} p^{(4)}(t) = f(t,\, t(1-t)),\\ p(0) = p(1) = 0,\\ ap''(\xi_1) - bp'''(\xi_2) = 0,\ cp''(\xi_2) + dp'''(\xi_2) = 0. \end{cases} \tag{17}$$

Now, we noting that

$$\begin{aligned} &min_{t\in[0,\,1]}\frac{1}{\delta}\int_{\xi_1}^{\xi_2}(b - a(\xi_1 - t))(c(\xi_2 - s) + d)f(s,\, s(1-s))ds\\ &= \frac{1}{\delta}\int_{\xi_1}^{\xi_2}(b - a(\xi_1 - s))(c(\xi_2 - s) + d)f(s,\, s(1-s))ds > 0. \end{aligned} \tag{18}$$

Combining the conclusion of Lemma 2.2 and (18), we get

$$c_1 t(1-t) \le p(t) \le c_2 t(1-t), \ t \in [0, 1]. \tag{19}$$

According to the assumption and from (19), we obtain

$$\begin{aligned} & m_1 c_1 t(1-t) \le \alpha(t) \le m_1 c_2 t(1-t) \\ \Rightarrow\ & m_1 c_1 \le \frac{\alpha(t)}{t(1-t)} \le m_1 c_2 \le 1, \end{aligned} \tag{20}$$

$$\begin{aligned} & m_2 c_1 t(1-t) \le \beta(t) \le m_2 c_2 t(1-t) \\ \Rightarrow\ & m_2 c_1 \le \frac{\beta(t)}{t(1-t)} \le m_2 c_2 \\ \Rightarrow\ & \frac{1}{m_2 c_2} \le \frac{t(1-t)}{\beta(t)} \le \frac{1}{m_2 c_1} \le 1, \end{aligned} \tag{21}$$

and

$$(m_1 c_1)^{\lambda} \ge m_1, \ \textit{and} \ (m_2 c_2)^{\lambda} \le m_2. \tag{22}$$

Thus, from assumption (A_2) and (19) to (22), we have

$$\begin{aligned} f(t, \alpha(t)) &= f\left(t, \frac{\alpha(t)}{t(1-t)} \cdot t(1-t)\right) \\ &\ge \left(\frac{\alpha(t)}{t(1-t)}\right)^{\lambda} f(t, t(1-t)) \\ &\ge (m_1 c_1)^{\lambda} f(t, t(1-t)) \ge m_1 f(t, t(1-t)), \end{aligned} \tag{23}$$

and

$$\begin{aligned} m_2 f(t, t(1-t)) &= m_2 f\left(t, \frac{t(1-t)}{\beta(t)} \cdot \beta(t)\right) \\ &\ge m_2 \left(\frac{t(1-t)}{\beta(t)}\right)^{\lambda} f(t, \beta(t)) \\ &\ge m_2 (m_2 c_2)^{-\lambda} f(t, \beta(t)) \ge f(t, \beta(t)). \end{aligned} \tag{24}$$

The inequalities (23) and (24), lead to

$$\begin{cases} \alpha^{(4)}(t) = m_1 f(t, t(1-t)) \le f(t, \alpha(t)), \\ \beta^{(4)}(t) = m_2 f(t, t(1-t)) \ge f(t, \beta(t)), \ t \in (0, 1)). \end{cases} \tag{25}$$

Hence, $\alpha(t) = m_1 p(t)$ and $\beta(t) = m_2 p(t)$ satisfies the BVP given by (1). Therefore, $\alpha(t) = m_1 p(t)$ and $\beta(t) = m_2 p(t)$ are lower and upper solutions of BVP given by (1), respectively.

Step 2.

In this step, we will prove that the fourth order four-point BVP

$$\begin{cases} u^{(4)}(t) = p(t, u(t)), \ t \in (0, 1), \\ u(0) = u(1) = 0, \\ au''(\xi_1) - bu'''(\xi_1) = 0, \ cu''(\xi_2) + du'''(\xi_2) = 0, \end{cases} \tag{26}$$

has a solution, where

$$p(t, u(t)) = \begin{cases} f(t, \alpha(t)), & \textit{when } u(t) < \alpha(t), \\ f(t, u(t)), & \textit{when } \alpha(t) \le u(t) \le \beta(t), \\ f(t, \beta(t)), & \textit{when } u(t) > \beta(t). \end{cases} \tag{27}$$

Now, we consider the operator $T : C[0,\,1] \to C[0,\,1]$ defined by

$$
\begin{aligned}
Tu(t) = & \int_0^1 G_1(t,\,r) \int_{\xi_1}^{r} (s-r)p(s,\,u(s))dsdr, \\
& + \frac{1}{\delta}\int_0^1 G_1(t,\,r) \int_{\xi_1}^{\xi_2} (b-a(\xi_1-r))(c(\xi_2-s)+d)p(s,\,u(s))dsdr,
\end{aligned} \tag{28}
$$

where $G_1(t,r)$ is as in (11). It is clear that the operator T is continuous in $C[0,\,1]$. Since, according to the assumption the function $f(t,\,u(t))$ is non-decreasing in u, and we know that, for any $u(t) \in C[0,\,1]$,

$$f(t,\,\alpha(t)) \leq p(t,\,u(t)) \leq f(t,\,\beta(t)),\;\; for\, t \in [0,\,1].$$

Hence, there exists a positive constant M such that $\mid p(t,\,u(t)) \mid \leq M$ for any $u(t) \in C[0,\,1]$, which implies that the operator T is uniformly bounded. Moreover, for all $u(t) \in C[0,\,1]$ and $0 \leq t_1 < t_2 \leq 1$, we have

$$
\begin{aligned}
|Tu(t_1) - Tu(t_2)| \leq & \int_0^1 G_1(t_1,\,r) - G_1(t_2,\,r) \mid \int_{\xi_1}^{r} (s-r)p(s,\,u(s))dsdr, \\
& + \frac{1}{\delta}\Bigg[\int_0^1 \mid G_1(t_1,\,r) - G_1(t_2,\,r) \mid \\
& \int_{\xi_1}^{\xi_2} (b-a(\xi_1-r))(c(\xi_2-s)+d)p(s,\,u(s))dsdr\Bigg] \\
& \leq 2 \mid t_1 - t_2 \mid \Bigg[\int_0^1 \int_{\xi_1}^{r} (s-r)p(s,\,u(s))dsdr \\
& + \frac{1}{\delta}\int_0^1 \int_{\xi_1}^{\xi_2} (b-a(\xi_1-r))(c(\xi_2-s)+d)p(s,\,u(s))dsdr\Bigg] \\
& \leq 2 \mid t_1 - t_2 \mid \Bigg[\int_0^1 \int_{\xi_1}^{r} (s-r)f(s,\,\beta(s))dsdr \\
& + \frac{1}{\delta}\int_0^1 \int_{\xi_1}^{\xi_2} (b-a(\xi_1-r))(c(\xi_2-s)+d)f(s,\beta(s))dsdr\Bigg],
\end{aligned}
$$

which implies that the operator T is equicontinuous. So, by the well-known Arzela-Ascoli theorem [12, 21], we can say that the operator T is compact. Consequently, by Theorem 2.3 (Schauder's fixed point theorem [20]), the operator T must have a fixed point and this ensure that the BVP given by (26) has a solution. This completes the step-2.

Step-3.

In this step, we will prove that BVP given by (1) has a positive solution. Let $u^+(t)$ be a solution of the BVP given by (26). Since, according to assumption the function $f(t,\,u(t))$ is non-decreasing in u and we know that,

$$f(t,\,\alpha(t)) \leq p(t,\,u^+(t)) \leq f(t,\,\beta(t)),\;\; for\, t \in [0,\,1].$$

Hence, if we apply Remark 2.1 and Remark 2.3 in the following BVP

$$
\begin{cases}
v^{(4)}(t) = f(t,\,\beta(t)) - p(t,\,u^+(t)), \\
v(0) = v(1) = 0, \\
av''(\xi_1) - bv'''(\xi_1) = 0,\;\; cv''(\xi_2) + dv''(\xi_2) = 0,
\end{cases}
$$

then we have, $\beta(t) - u^+(t) = v(t) \geq 0$, but $v''(t) \nleq 0$, i.e., $u^+(t) \leq \beta(t)$ for all $t \in [0,\,1]$. Similarly, we can prove that $\alpha(t) \leq u^+(t)$ for $t \in [0,\,1]$. Therefore $u^+(t)$ is a positive solution of the BVP given by (1).

This completes the proof. □

The Theorem 3.1 leads the following corollary:

Corollary 3.1. *If the assumption (A_2) of Theorem 3.1 is satisfied and we replace the BVCs of BVP given by (1) by the Lidstone BVCs*

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

then the BVP given by (1) has at least one positive solution.

Now we give an example to justify our Theorem 3.1.

Example 3.1. *Consider a NLFOODE with four-point BVCs as follows:*

$$\begin{cases} u^{(4)}(t) = e^{\cos t}(\sqrt{u} + 1), \ t \in (0, 1), \\ u(0) = u(1) = 0, \\ u''\left(\frac{1}{3}\right) - u'''\left(\frac{1}{3}\right) = 0, \ u''\left(\frac{2}{3}\right) + u'''\left(\frac{2}{3}\right) = 0. \end{cases} \tag{29}$$

For proving that the BVP given by (29) has a positive solution, we apply our Theorem 3.1 with

$$f(t, u) = e^{\cos t}(\sqrt{u} + 1), \ p(t) = e^{cost}, \ g(u) = \sqrt{u} + 1, \ a = b = c = d = 1,$$
$$\xi_1 = \frac{1}{3}, \ \xi_2 = \frac{2}{3}, \ m = .85 \in [0, 1], \ t = .5 \in (0, 1) \ and \ \lambda = .5 < 1.$$

Clearly, assumption (A_1) of Theorem 3.1 is satisfied with $\delta = \frac{7}{3} \neq 0$.
It is also clear that the function $f(t, u)$ is non-decreasing and $f(t, t(1-t)) \neq 0$ for $t \in (\xi_1, \xi_2)$.
Now, we have

$$\begin{aligned} m^\lambda f(t, u) &= m^\lambda e^{\cos t}(\sqrt{u} + 1) \\ &= (.85)^{.5} e^{\cos(.5)}(\sqrt{u} + 1) \\ &= 2.506\sqrt{u} + 2.506 > 0 \ for\, all\, positive\, u, \end{aligned} \tag{30}$$

$$\begin{aligned} f(t, mu) &= e^{\cos t}(\sqrt{mu} + 1) \\ &= e^{\cos(.5)}\sqrt{.85}\left(\sqrt{u} + \frac{1}{\sqrt{.85}}\right) \\ &= 2.506\sqrt{u} + 2.718 > 0 \ for\, all\, positive\, u. \end{aligned} \tag{31}$$

Thus, from (30) and (31), we yield

$$m^\lambda f(t, u) \leq f(t, mu).$$

Hence, assumption (A_2) of Theorem 3.1 is satisfied. Therefore, Theorem 3.1 assure that the BVP given by (29) has at least one positive solution $u \in C^2[0, 1]$.

4. Conclusion

In this study, we established the existence result of positive solution for a NLFOODE with four-point BVCs by the help of Upper and Lower solution method and Schauder's fixed point theorem (Theorem 2.3). Here we improved the result of Chen *et al.* [9] by correcting their key lemma (Lemma 2.2 of Chen *et al.* [9]) and we made this correction by using Remark 2.1, Remark 2.2 and Remark 2.3. As the considered NLFOODE with four-point BVCs of this paper represents a beam equation, so we can conclude that the result of this paper will play a vital role to check the existence of positive solution of this type of beam equations. A justifying example also discussed here.

Competing Interests

The author declares that he has no any competing interests.

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