

Coatomic refinable modules

Burcu Nişancı Türkmen¹ , Ergül Türkmen^{1*} 

¹Department of Mathematics, Faculty of Arts and Science, Amasya University,
Amasya/Turkey

Abstract

In this article, we define the concept of (strongly) coatomic refinable modules as a proper generalization of (strongly) refinable modules. It is shown that: (1) every direct summand of a coatomic refinable module is coatomic refinable; (2) over a left max ring a module M is (strongly) coatomic refinable if and only if it is (strongly) refinable; (3) if a coatomic refinable module M is π -projective, then it is strongly coatomic refinable; (4) if coatomic direct summands lift modulo every coatomic submodule of M , then M is coatomic refinable.

1. Introduction

In this paper, all rings are associative with identity and all modules are left modules. Let R be such a ring and let M be an R -module. The notation $N \subset M$ ($N \subseteq M$) means that N is a (proper) submodule of M . A submodule U is called *small* (in M), denoted by $U \ll M$, if $M = U + K$ for every proper submodule K of M (Wisbauer, 1991). Following (Wisbauer, 1991), a module M is called *hollow* if every proper submodule of U is small in M . M is called *local* if it is hollow and finitely generated. By $Rad(M)$, namely *radical*, we will denote the sum of all small submodules of M . It is well known that $Rad(M)$ is the intersection of all maximal submodules of M . It is clear that if a module M is hollow, then $Rad(M) \ll M$ or $M = Rad(M)$.

Let M be a module. M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M (Zöschinger, 1974). Every finitely generated module is coatomic, and coatomic modules have small radical. It can be seen that if a coatomic submodule N of a module M is contained in $Rad(M)$, then N is a small submodule of M . Also, over a Dedekind domain every small submodule of a module is coatomic. The class of coatomic modules is closed under factor modules and extensions. In general, a submodule of a coatomic module

Article History

Received 28.07.2020

Accepted 21.08.2020

Keywords

Refinable module,
Coatomic module,
Coatomic refinable module

¹Correspondence: ergulturkmen@hotmail.com

need not be coatomic. Over a commutative noetherian ring, every submodule of a coatomic module is coatomic. A ring R is called a *left max ring* if every nonzero left R -module have a maximal submodule. Left perfect rings (over which every module has a projective cover) are left max rings. It is known that a ring R is a left max ring if and only if every nonzero left R -module is coatomic.

For any ring R , an R -module M is called *supplemented* if every submodule N of M has a *supplement*, that is a submodule K minimal with respect to $N + K = M$. K is a supplement of N in M if and only if $N + K = M$ and $N \cap K \ll K$ (Wisbauer, 1991). Every direct summand of a module M is a supplement submodule of M , and supplemented modules are a proper generalization of Artinian modules.

Mohamed and Muller (1990) call a module M \oplus -*supplemented* if every submodule N of M has a supplement that is a direct summand of M (Mohamed and Müller, 1990). Every \oplus -supplemented module is supplemented, but a supplemented module need not be \oplus -supplemented in general (see Mohamed and Müller, 1990, Lemma A.4 (2)). It is shown in Mohamed and Müller (1990), 3, Proposition A.7 and Proposition A.8] that if R is a Dedekind domain, every supplemented R -module is \oplus -supplemented. Hollow modules are \oplus -supplemented. Characterizations and the structure of supplemented and \oplus -supplemented modules are extensively studied by many authors.

A module M is *lifting* if every submodule N of M contains a direct summand L of M such that $M = L \oplus K$ and $N \cap K \ll K$ (see Clark et al., 2006). Every projective module over a left Artinian ring is lifting, and lifting modules are \oplus -supplemented (see Wisbauer, 1991, 41.15).

In Wisbauer (1996), the class of (strongly) refinable modules is introduced as a proper generalization of lifting modules. An R -module M is called *refinable* if, for any submodules $U, V \subseteq M$ with $M = U + V$, there exists a direct summand A of M with $A \subseteq U$ and $M = A + V$. Every finitely generated regular module is refinable. A module M is said to be *strongly refinable* if, in the given situation, there exist submodules $A \subseteq U, B \subseteq V$ of M with $M = A \oplus B$. Hollow modules and semisimple modules are strongly refinable.

In this paper, we define the concept of coatomic (strongly) refinable modules. The class of

(strongly) coatomic refinable modules properly contains the class of refinable module. We prove that every direct summand of a coatomic refinable module is coatomic refinable. We show that if a coatomic refinable module M is π -projective, then it is strongly coatomic refinable. We also prove that if coatomic direct summands lift modulo every coatomic submodule of M , then M is coatomic refinable.

2. Coatomic Refinable Modules

In this section, we give basic properties of (strongly) coatomic refinable modules. In particular, we show that every direct summand of a coatomic refinable module is coatomic refinable.

Definition 2.1. Let M be a module. M is called *coatomic refinable* if, for any coatomic submodule N of M and any submodule K of M with $M = N + K$, there exists a direct summand N' of M with $N' \subseteq N$ and $M = N' + K$, and M is called *strongly coatomic refinable* if, for any coatomic submodule N of M and any submodule K of M with $M = N + K$, there exist submodules N' and K' of M with $N' \subseteq N$, $K' \subseteq K$, $M = N' + K$ and $M = N' \oplus K'$.

Clearly, (strongly) refinable modules are (strongly) coatomic refinable. The following example shows that a strongly coatomic refinable module need not be refinable. The following well-known fact is given for completeness.

Lemma 2.2. Let R be a Dedekind domain and M be an R -module. Assume that a submodule N of M is contained in $\text{Rad}(M)$. Then, N is a small submodule of M if and only if N is coatomic.

Example 2.3. Consider the \mathbb{Z} -module $M = {}_{\mathbb{Z}}\mathbb{Q}$. Therefore, we have that $M = \text{Rad}(M)$. If N is a coatomic submodule of M , it follows from Lemma 2.2 that N is a small submodule of M . So we can write $M = N + M = 0 \oplus M$ and $0 \subset N$. It means that M is a strongly coatomic refinable module. On the other hand, M is not refinable.

Proposition 2.4. Let R be a commutative noetherian ring and M be a (strongly) coatomic refinable R -module. If M is coatomic, then it is (strongly) refinable.

Proof. Let M be a coatomic R -module. Since R is commutative noetherian, every submodule of M is coatomic. By the assumption, M is (strongly) refinable.

Proposition 2.5. Let R be a left max ring and M be an R -module. Then, M is (strongly) coatomic refinable if and only if it is (strongly) refinable.

Proof. (\Rightarrow) Let M be a (strongly) coatomic refinable module. Since R is a left max ring, it follows that every submodule of M is coatomic. Hence M is (strongly) refinable.

(\Leftarrow) It is clear.

Recall from [4, 41.13] that M is called π -projective if for every submodule N, K of M and identity homomorphism $I_M : M \rightarrow M$ with $M = N + K$ there exists $y \in \text{End}(M)$ with $I_M(y) \subseteq N$ and $Im(I_M - y) \subseteq K$.

Proposition 2.6. Let M be a coatomic refinable module and π -projective. Then M is strongly coatomic refinable.

Proof. Let N be any coatomic submodule of M . Suppose that there exists a submodule K of M such that $M = N + K$. By the hypothesis, there exist submodules N', L of M such that $N' \subseteq N$, $M = N' \oplus L$ and $M = N' + K$. By Clark et al. (2006, 4.14), there exists a submodule K' of M such that $K' \subseteq K$ and $M = N \oplus K'$. Thus M is strongly coatomic refinable.

Now we prove that the property coatomic refinable is inherited by direct summands.

Theorem 2.7. Let M be a module. If M is a coatomic refinable, then every direct summand of M is coatomic refinable.

Proof. Let N be any direct summand of M . Then there exists a submodule K of M such that $M = N \oplus K$. Let U be any coatomic submodule of N . Suppose that $N = U + V$ for some submodule V of N . Then $M = U + (V + K)$. Since U is coatomic submodule of M and M is a coatomic refinable module, then there exist submodules U', T of M such that $U' \subseteq U$, $M = U' \oplus T$ and $M = U' + (V + K)$.

By the modularity, we have $N = N \cap [U' + (V + K)] = U' + [V + (N \cap K)] = U' + V$. Thus U is a coatomic refinable module.

Recall from Wisbauer (1991) that A submodule N of M is called *fully invariant* if $f(N)$ is contained in N for every $f \in \text{End}(M)$. A module M is called a *duo module* provided every submodule of M is fully invariant (Ozcan et al., 2006).

Theorem 2.8. *Let $\{M_i\}_{i \in I}$ be a family of coatomic refinable modules and $M = \bigoplus_{i \in I} M_i$. If M is a duo module, then M is coatomic refinable.*

Proof. Let U be any coatomic submodule of M . Suppose that there exists a submodule V of M such that $M = U + V$. By the hypothesis, $U = \bigoplus_{i \in I} (U \cap M_i)$. Since, for every $i \in I$, $U \cap M_i$ is homomorphic image of U , $U \cap M_i$ is a coatomic submodule of U . So, for every $i \in I$, $U \cap M_i$ is a coatomic submodule of M_i . Now we can write

$$\begin{aligned} M_i &= M_i \cap (U + V) \\ &= M_i \cap [\bigoplus_{i \in I} (U \cap M_i) + V] \\ &= (U \cap M_i) + M_i \cap [\bigoplus_{i \neq j \in I} (U \cap M_j) + V] \end{aligned}$$

for every $i \in I$. Since M_i is coatomic refinable for every $i \in I$, we obtain that

$$M_i = U_i + [\bigoplus_{i \neq j \in I} (U \cap M_j) + V] \quad \text{and} \quad M_i = U_i \oplus V_i$$

Where U_i and V_i are submodules of M_i with $U_i \subseteq U \cap M_i$. So, $\bigoplus_{i \in I} U_i \subseteq U$ and

$$M = (\bigoplus_{i \in I} U_i) + (\bigoplus_{i \in I} V_i) \text{ Note that}$$

$$\begin{aligned} M &= \bigoplus_{i \in I} M_i \\ &= \bigoplus_{i \in I} \{U_i + [(\bigoplus_{i \neq j \in I} (U \cap M_j) + V)]\} \\ &= \bigoplus_{i \in I} U_i + \bigoplus_{i \in I} [\bigoplus_{i \neq j \in I} (U \cap M_j) + V] \\ &= \bigoplus_{i \in I} U_i + \bigoplus_{i \in I} (V \cap M_i) \end{aligned}$$

It follows from the hypothesis that $V = \bigoplus_{i \in I} (V \cap M_i)$. Therefore $M = \bigoplus_{i \in I} U_i + V$. Thus M is coatomic refinable.

Recall from Clark et al. (2006, 11.26) that a module M is called *direct summands lift modulo a submodule N of M* if, under the canonical projection $p : M \rightarrow M/N$, every direct summand of M/N is an image of a direct summand of M . Similarly, a module M is called

(finite) decompositions lift modulo N if, whenever M/N is expressed as a (finite) direct sum of submodules M_i/N , then $M = \bigoplus_{i \in I} K_i$, where $p(K_i) = K_i/N$ for each $i \in I$.

Proposition 2.9. Let M be a module. If coatomic direct summands lift modulo every coatomic submodule of M , then M is coatomic refinable.

Proof. Let N be any coatomic submodule of M . Suppose that a submodule K of M with $M = N + K$ and the canonical projection $p : M \rightarrow M/K$. Since N is coatomic submodule of M , then $N+K/K$ is coatomic direct summand of M/K . By the hypothesis, N is a coatomic direct summand of M . So M is coatomic refinable.

Proposition 2.10. Let M be a module. If coatomic decompositions lift modulo every coatomic submodule of M , then M is strongly coatomic refinable.

Proof. Let N be any coatomic submodule of M . Suppose that a submodule K of M with $M=N+K$. Say $L=N \cap K$. Then $M/L=N/L \oplus K/L$. Suppose the canonical projection $g:M \rightarrow M/L$. Since $g(N)=N/L$ and $g(K)=K/L$, we have $M=N \oplus K$ by the hypothesis. Therefore M is strongly coatomic refinable.

References

- Clark, J., Lomp, C., Vanaja, N. Wisbauer, R. 2006. Lifting modules supplements and projectivity in module theory. *Frontiers in Mathematics*, Birkhauser Basel.
- Mohamed, S.H., Müller, B.J. 1990. Continuous and discrete modules, *London Mathematical Society Lecture Note Series 147*. Cambridge University Press, New York.
- Ozcan, A.C, Harmanci, A. Smith, P.F. 2006. Duo modules, *Glasgow Mathematical Journal*, 48: 533-545.
- Wisbauer, R. 1991. *Foundations of modules and rings*. Gordon and Breach.
- Wisbauer, R. 1996. *Modules and algebras: bimodule structure and group actions on algebras*, Pitman Monographs *and Surveys in Pure and Applied Mathematics* 81. Longman.
- Zöschinger, H. 1974. Komplementierte moduln uber Dedekindringen, *Journal of Algebra*, 29: 42-56.