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Certain Generalized Fractional Integral Inequalities

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Abstract

By employing the Saigo k-fractional integral operators, some new inequalities for the Chebyshev functional are formulated for two synchronous functions in this article. Further generalisations of these inequalities, including three monotonous functions, are also mentioned. In addition, as special cases of our key results, inequalities for the Chebyshev functional about Saigo fractional integrals are obtained. The main results are of a general nature and, as a special case, give rise to integral inequalities describing the Saigo's, Riemann-Liouville and Erdélyi-Kober fractional integral operators referred to the literature.

Keywords: Chebyshev functional; Integral inequalities; Saigo k-fractional integral operator.
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1. Introduction and Preliminaries

Fractional integral inequalities become important in determining the validity of solutions for certain partial differential fractional equations. They often describe upper and lower limits for the solutions to problems with fractional boundary value. These implications, involving fractional calculus operators, have directed numerous studies in the context of integral inequalities to investigate other extensions and generalizations [12, 18, 19, 20, 22, 26]. For a comprehensive overview of the different applications of fractional integral

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inequalities, attention can be made to [1, 2, 3, 5, 6, 10, 14, 15, 16, 17, 28] as well as the references listed in it.

Chebyshev [8] introduced and defined a functional for two functions $\mathcal{F}$ and $\mathcal{G}$, which are synchronous and integrable on the interval $[m, n]$ as follows:

$$\mathrm{T}(\mathcal{F}, \mathcal{G}) = \frac{1}{n-m}\int_m^n \mathcal{F}(\omega)\mathcal{G}(\omega)d\omega - \left(\frac{1}{n-m}\int_m^n \mathcal{F}(\omega)d\omega\right)\left(\frac{1}{n-m}\int_m^n \mathcal{G}(\omega)d\omega\right), \tag{1}$$

where, functions $\mathcal{F}$ and $\mathcal{G}$ are synchronous, if for any $\zeta, \eta \in [0, \infty)$ the following inequality holds:

$$\{(\mathcal{F}(\zeta) - \mathcal{F}(\eta))(\mathcal{G}(\zeta) - \mathcal{G}(\eta))\} \geq 0. \tag{2}$$

The Chebyshev functional (1) has been widely used in the theory of fractional integral inequalities, science and engineering fields. Therefore, it has attracted the attention of many researchers [7, 13, 23]. Before describing the main results, we recollect a few definitions and results of fractional calculus available in the literature.

Recently, Gupta and Parihar [11] introduced and defined Saigo k-fractional integral involving k-hypergeometric function for $x \in \mathbb{R}^+$, $\omega, \xi, \gamma \in \mathbb{C}$, $\Re(\omega) > 0$ and $k > 0$ as:

$$\left(I_{0+,k}^{\omega,\xi,\gamma}\mathcal{F}\right)x = \frac{x^{\frac{-\omega-\xi}{k}}}{k\Gamma_k(\omega)}\int_0^x (x-t)^{\frac{\omega}{k}-1} \times {}_2F_{1,k}\left((\omega+\xi, k), (-\gamma, k); (\omega, k); \left(1-\frac{t}{x}\right)\right)\mathcal{F}(t)\, dt, \tag{3}$$

and

$$\left(I_{-,k}^{\omega,\xi,\gamma}\mathcal{F}\right)x = \frac{1}{k\Gamma_k(\omega)}\int_x^\infty (t-x)^{\frac{\omega}{k}-1}\, t^{\frac{-\omega-\xi}{k}} \times {}_2F_{1,k}\left((\omega+\xi, k), (-\gamma, k); (\omega, k); \left(1-\frac{x}{t}\right)\right)\mathcal{F}(t)\, dt, \tag{4}$$

where, the k-hypergeometric function ${}_2F_{1,k}\left((\omega, k), (\xi, k); (\gamma, k); x\right)$ is defined in series as well as in integral form under the same set of conditions setout in [21] as follows:

$${}_2F_{1,k}\left((\omega, k), (\xi, k); (\gamma, k); x\right) = \sum_{l=0}^{\infty} \frac{(\omega)_{l,k}(\xi)_{l,k}}{(\gamma)_{l,k}}\frac{x^l}{l!} \tag{5}$$

$$= \frac{\Gamma_k(\gamma)}{k\Gamma_k(\xi)\Gamma_k(\gamma-\xi)}\int_0^1 t^{\frac{\xi}{k}-1}(1-t)^{\frac{\gamma-\xi}{k}-1}(1-kxt)^{-\frac{\omega}{k}}\, dt, \tag{6}$$

and k-gamma function [9], $\Gamma_k(z)$, is given as:

$$\Gamma_k(z) = \int_0^\infty t^{z-1}e^{-\frac{t^k}{k}}dt, \quad z \in \mathbb{C}. \tag{7}$$

The generalized k-fractional calculus operators have benefit that they generalizes Saigo's fractional integral and derivative operators, therefore number authors labeled this as a general operator. For recent work, one can see [27]. Further, it is interesting to note that for $k = 1$, equations (3) to (4) reduce into Saigo's fractional order integral operators [25], equations (5) to (6) reduce into hypergeometric function and equation (7) reduces into gamma function.

The below stated Lemma is a well known result [11], it will be utilized in the consequent theorems.

Lemma 1.1. *Let $\omega, \xi, \gamma \in \mathbb{C}$ and $\Re(\omega) > 0$, $k \in \mathbb{R}^+(0, \infty)$ such that $\Re(\vartheta) > max\,[0, \Re(\xi - \gamma)]$, then*

$$\left(I_{0+,k}^{\omega,\xi,\gamma} t^{\frac{\vartheta}{k}-1}\right) = \sum_{r=0}^{\infty} \frac{k^r\, \Gamma_k(\vartheta - \xi + \gamma)\, x^{\frac{\vartheta-\xi}{k}-1}}{\Gamma_k(\vartheta - \xi)\Gamma_k(\vartheta + \omega + \gamma)}. \tag{8}$$

The purpose of this analysis is to obtain some integral inequalities of the Chebyshev form associated with the Saigo k-fractional integral operators.

2. Main Results

Here, the Chebyshev type inequalities associated with the Saigo k-fractional integral operator are defined. Throughout the paper we considered $\mathcal{F}$ and $\mathcal{G}$ as synchronous functions defined on the interval $[0, \infty)$.

Theorem 2.1. *If $x > 0$, $\omega > max\,\{0, -\xi\}$, $\xi < k$ and $\xi - k < \gamma < 0$, then*

$$\left(I_{0+,k}^{\omega,\xi,\gamma} \mathcal{F}\,\mathcal{G}\right)(x) A_k(x) \quad \geq \quad \left(I_{0+,k}^{\omega,\xi,\gamma} \mathcal{F}\right)(x) \left(I_{0+,k}^{\omega,\xi,\gamma} \mathcal{G}\right)(x), \tag{9}$$

where, $A_k(x) = \sum_{r=0}^{\infty} \frac{k^r\, \Gamma_k(k - \xi + \gamma)\, x^{-\frac{\xi}{k}}}{\Gamma_k(k - \xi)\Gamma_k(k + \omega + \gamma)}$.

Proof. The inequality (2) yields that

$$\mathcal{F}(\zeta)\mathcal{G}(\zeta) + \mathcal{F}(\eta)\mathcal{G}(\eta) \quad \geq \quad \mathcal{F}(\zeta)\mathcal{G}(\eta) + \mathcal{F}(\eta)\mathcal{G}(\zeta). \tag{10}$$

Consider

$$\begin{aligned} \mathrm{F}(x,\zeta) &= \frac{x^{\frac{-\omega-\xi}{k}} (x-\zeta)^{\frac{\omega}{k}-1}}{k\Gamma_k(\omega)} \times {}_2F_{1,k}\left((\omega+\xi, k), (-\gamma, k); (\omega, k); \left(1 - \frac{\zeta}{x}\right)\right) \\ &= \frac{1}{k\Gamma_k(\omega)} \frac{(x-\zeta)^{\frac{\omega}{k}-1}}{x^{\frac{\omega+\xi}{k}}} + \frac{(\omega+\xi)(-\gamma)}{k\Gamma_k(\omega+k)} \frac{(x-\zeta)^{\frac{\omega}{k}}}{x^{\frac{\omega+\xi}{k}+1}} \\ &+ \frac{(\omega+\xi)(\omega+\xi+k)(-\gamma)(-\gamma+k)}{k\Gamma_k(\omega+2k)} \frac{(x-\zeta)^{\frac{\omega}{k}+1}}{2!\, x^{\frac{\omega+\xi}{k}+2}} + \cdots. \end{aligned} \tag{11}$$

Note that the function $\mathrm{F}(x, \zeta)$ stays positive, for all $\zeta \in (0, x)$ $(x > 0)$ considering the conditions stated in Theorem 2.1.

Now, multiply (10) by $\mathrm{F}(x, \zeta)$ (given in (11)) and integrating from 0 to x with respect to ζ, with the help of (3), we get

$$\begin{aligned} &\left(I_{0+,k}^{\omega,\xi,\gamma} \mathcal{F}\,\mathcal{G}\right)(x) + \mathcal{F}(\eta)\mathcal{G}(\eta) \left(I_{0+,k}^{\omega,\xi,\gamma} 1\right)(x) \geq \\ &\qquad \mathcal{G}(\eta) \left(I_{0+,k}^{\omega,\xi,\gamma} \mathcal{F}\right)(x) + \mathcal{F}(\eta) \left(I_{0+,k}^{\omega,\xi,\gamma} \mathcal{G}\right)(x). \end{aligned} \tag{12}$$

Next, multiply (12) by $\mathrm{F}(x, \eta)$ $(\eta \in (0, x),\ x > 0)$ and integrating with respect to η from 0 to x, and using formula (8)(when $\vartheta = k$), we obtain the result (9). □

Theorem 2.2. *If $x > 0$, $\alpha > max\,\{0, -\beta\}$, $\omega > max\,\{0, -\xi\}$, $\beta, \xi < k$, $\beta - k < \eta < 0$ and $\xi - k < \gamma < 0$, then*

$$\begin{aligned} &\left(I_{0+,k}^{\omega,\xi,\gamma} \mathcal{F}\,\mathcal{G}\right)(x) B_k(x) + \left(I_{0+,k}^{\alpha,\beta,\eta} \mathcal{F}\,\mathcal{G}\right)(x) A_k(x) \geq \\ &\qquad \left(I_{0+,k}^{\alpha,\beta,\eta} \mathcal{F}\right)(x) \left(I_{0+,k}^{\omega,\xi,\gamma} \mathcal{G}\right)(x) + \left(I_{0+,k}^{\omega,\xi,\gamma} \mathcal{F}\right)(x) \left(I_{0+,k}^{\alpha,\beta,\eta} \mathcal{G}\right)(x), \end{aligned} \tag{13}$$

where, $B_k(x) = \sum_{r=0}^{\infty} \frac{k^r\, \Gamma_k(k - \beta + \eta)\, x^{-\frac{\beta}{k}}}{\Gamma_k(k - \beta)\Gamma_k(k + \alpha + \eta)}$ *and $A_k(x)$ is defined in Theorem 2.1.*

Proof. Multiply both sides of (12) by

$$\frac{x^{\frac{-\alpha-\beta}{k}}}{k\Gamma_k(\alpha)}(x-\eta)^{\frac{\alpha}{k}-1} \times {}_2F_{1,k}\left((\alpha+\beta,k),(-\eta,k);(\alpha,k);\left(1-\frac{\eta}{x}\right)\right). \tag{14}$$

Function given by above equation (14) stays positive in the light of the above claims in the proof of Theorem 2.1 and stated conditions in Theorem 2.2. Next, integrate the resulting inequality from 0 to x with respect to η, we get

$$\begin{aligned}\left(I^{\alpha,\beta,\eta}_{0+,k}1\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{G}\right)(x)+\left(I^{\omega,\xi,\gamma}_{0+,k}1\right)(x)\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\,\mathcal{G}\right)(x)\geq\\ \left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\right)(x)\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{G}\right)(x)+\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{G}\right)(x),\end{aligned} \tag{15}$$

which on using the result (8)(when $\vartheta = k$) gives the result (14). □

Remark 2.1. *By substituting $\alpha = \omega$, $\beta = \xi$ and $\eta = \gamma$ in Theorem 2.2, it reduces into Theorem 2.1.*

Theorem 2.3. *If $x > 0$, $\omega > max\,\{0, -\xi\}$, $\xi < k$ and $\xi - k < \gamma < 0$, then*

$$\left(I^{\omega,\xi,\gamma}_{0+,k}\prod_{i=1}^{n}\mathcal{F}_i(t)\right)(x)\,(A_k(x))^{n-1}\geq\prod_{i=1}^{n}\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}_i(t)\right)(x), \tag{16}$$

where, $A_k(x)$ is defined in Theorem 2.1.

Proof. We will prove this with the help of induction. Obviously, we have for $n = 1$ in (16)

$$\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}_1(t)\right)(x)\geq\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}_1(t)\right)(x)\ \ (x>0,\omega>0).$$

Next, for $n = 2$, in (16), we get

$$\begin{aligned}\sum_{r=0}^{\infty}k^r\frac{\Gamma_k(k-\xi+\gamma)}{\Gamma_k(k-\xi)\Gamma_k(k+\omega+\gamma)}x^{-\frac{\xi}{k}}\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}_1(t)\mathcal{F}_2(t)\right)(x)\\ \geq\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}_1(t)\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}_2(t)\right)(x),\end{aligned}$$

that holds the Theorem 2.1 in view of (9).

By the theory of induction we assume the inequality

$$\begin{aligned}\left[\sum_{r=0}^{\infty}k^r\frac{\Gamma_k(k-\xi+\gamma)}{\Gamma_k(k-\xi)\Gamma_k(k+\omega+\gamma)}x^{-\frac{\xi}{k}}\right]^{n-2}\left(I^{\omega,\xi,\gamma}_{0+,k}\prod_{i=1}^{n-1}\mathcal{F}_i(t)\right)(x)\\ \geq\prod_{i=1}^{n-1}\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}_i(t)\right)(x),\end{aligned} \tag{17}$$

holds true for $n \geq 2$.

Now, $\prod_{i=1}^{n}\mathcal{F}_i(t)$ is an increasing function as $\mathcal{F}_i$ $(i = 1, \cdots, n)$ are increasing functions. We then add the Theorem 2.1's inequality (9) to the $\prod_{i=1}^{n-1}\mathcal{F}_i(t) = g$ and $\mathcal{F}_n = f$ functions in order to get

$$\begin{aligned}\left[\sum_{r=0}^{\infty}k^r\frac{\Gamma_k(k-\xi+\gamma)}{\Gamma_k(k-\xi)\Gamma_k(k+\omega+\gamma)}x^{-\frac{\xi}{k}}\right]^{n-1}\left(I^{\omega,\xi,\gamma}_{0+,k}\prod_{i=1}^{n}\mathcal{F}_i(t)\right)(x)\\ \geq\prod_{i=1}^{n-1}\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}_i(t)\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}_n(t)\right)(x),\end{aligned} \tag{18}$$

provided that $x>0$, $\omega>\max\{0,-\xi\}$, $\xi<k$, $\xi-k<\gamma<0$.

Using (17) in (18), we land the inequality (16). □

Remark 2.2. *If we substitute $k=1$ into Theorems 2.1 to 2.3, then theorems reduced into the Chebyshev type inequalities involving Saigo fractional integral operator (see, Purohit and Raina [23]).*

Theorem 2.4. *If $x>0$, $\omega>\max\{0,-\xi\}$, $\xi<k$, $\xi-k<\gamma<0$ and $\mathcal{H}>0$, then*

$$\begin{aligned}\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{G}\,\mathcal{H}\right)(x)A_k(x)\geq&\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{G}\,\mathcal{H}\right)(x)\\&+\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{G}\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{H}\right)(x)-\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{H}\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{G}\right)(x),\end{aligned}\tag{19}$$

where, $A_k(x)$ defined in Theorem 2.1.

Proof. Given $\mathcal{H}>0$, for all $\zeta,\eta\geq0$, using (2), we obtain

$$\{(\mathcal{F}(\zeta)-\mathcal{F}(\eta))\,(\mathcal{G}(\zeta)-\mathcal{G}(\eta))\,(\mathcal{H}(\zeta)+\mathcal{H}(\eta))\}\geq0,$$

which refer that

$$\begin{aligned}&\mathcal{F}(\zeta)\mathcal{G}(\zeta)\mathcal{H}(\zeta)+\mathcal{F}(\eta)\mathcal{G}(\eta)\mathcal{H}(\eta)\geq\mathcal{F}(\zeta)\mathcal{G}(\eta)\mathcal{H}(\eta)+\mathcal{F}(\eta)\mathcal{G}(\zeta)\mathcal{H}(\zeta)\\&+\mathcal{G}(\zeta)\mathcal{F}(\eta)\mathcal{H}(\eta)+\mathcal{G}(\eta)\mathcal{F}(\zeta)\mathcal{H}(\zeta)-\mathcal{H}(\zeta)\mathcal{F}(\eta)\mathcal{G}(\eta)-\mathcal{H}(\eta)\mathcal{F}(\zeta)\mathcal{G}(\zeta).\end{aligned}\tag{20}$$

Now, multiply(20) by $\mathrm{F}(x,\zeta)$ (given in (11)) and integrating from 0 to x with respect to ζ, and using (3), we get

$$\begin{aligned}&\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{G}\,\mathcal{H}\right)(x)+\mathcal{F}(\eta)\mathcal{G}(\eta)\mathcal{H}(\eta)\left(I^{\omega,\xi,\gamma}_{0+,k}1\right)(x)\geq\\&\quad\mathcal{G}(\eta)\mathcal{H}(\eta)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\right)(x)+\mathcal{F}(\eta)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{G}\,\mathcal{H}\right)(x)\\&\quad+\mathcal{F}(\eta)\mathcal{H}(\eta)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{G}\right)(x)+\mathcal{G}(\eta)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{H}\right)(x)\\&\quad-\mathcal{F}(\eta)\mathcal{G}(\eta)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{H}\right)(x)-\mathcal{H}(\eta)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{G}\right)(x).\end{aligned}\tag{21}$$

Next, multiply (21) by $\mathrm{F}(x,\eta)$ ($\eta\in(0,x)$, $x>0$) (defined in (11)) and then integrate with respect to η from 0 to x, and using formula (8)(when $\vartheta=k$), we get the desired result (19). □

Theorem 2.5. *If $x>0$, $\alpha>\max\{0,-\beta\}$, $\omega>\max\{0,-\xi\}$, $\beta,\xi<k$, $\beta-k<\eta<0$, $\xi-k<\gamma<0$ and $\mathcal{H}>0$, then*

$$\begin{aligned}&\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{G}\,\mathcal{H}\right)(x)B_k(x)+\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\,\mathcal{G}\,\mathcal{H}\right)(x)A_k(x)\geq\\&\quad\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{G}\,\mathcal{H}\right)(x)+\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\right)(x)\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{G}\,\mathcal{H}\right)(x)\\&\quad+\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{G}\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{H}\right)(x)+\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{G}\right)(x)\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\,\mathcal{H}\right)(x)\\&\quad-\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{H}\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{G}\right)(x)-\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{H}\right)(x)\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\,\mathcal{G}\right)(x),\end{aligned}\tag{22}$$

where, $A_k(x)$ and $B_k(x)$ are defined in Theorems 2.1 and 2.2, respectively.

Proof. Multiply (21) by (14) which stays positive under the conditions setout in Theorem 2.5. Next, integrate the derived inequality with respect to η from 0 to x, we obtain

$$\begin{aligned}&\left(I^{\alpha,\beta,\eta}_{0+,k}1\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{G}\,\mathcal{H}\right)(x)+\left(I^{\omega,\xi,\gamma}_{0+,k}1\right)(x)\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\,\mathcal{G}\,\mathcal{H}\right)(x)\geq\\&\quad\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\right)(x)\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{G}\,\mathcal{H}\right)(x)+\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{G}\,\mathcal{H}\right)(x)\\&\quad+\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{G}\right)(x)\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\,\mathcal{H}\right)(x)+\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{G}\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{H}\right)(x)\\&\quad-\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{H}\right)(x)\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\,\mathcal{G}\right)(x)-\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{H}\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{G}\right)(x),\end{aligned}\tag{23}$$

which on using the result (8)(when $\vartheta = k$) yields the desired result (22). □

Theorem 2.6. *Let monotonic functions $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ defined on $[0,\infty)$ satisfy the inequality*

$$\{(\mathcal{F}(\zeta)-\mathcal{F}(\eta))\,(\mathcal{G}(\zeta)-\mathcal{G}(\eta))\,(\mathcal{H}(\zeta)-\mathcal{H}(\eta))\}\geq 0, \tag{24}$$

and if $x>0$, $\alpha > max\,\{0,-\beta\}$, $\omega > max\,\{0,-\xi\}$, $\beta,\xi<k$, $\beta-k<\eta<0$, and $\xi-k<\gamma<0$, then

$$\begin{aligned}&\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{G}\,\mathcal{H}\right)(x)B_k(x)-\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\,\mathcal{G}\,\mathcal{H}\right)(x)A_k(x)\geq\\ &\quad\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{G}\,\mathcal{H}\right)(x)-\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\right)(x)\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{G}\,\mathcal{H}\right)(x)\\ &\quad+\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{G}\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{H}\right)(x)-\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{G}\right)(x)\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\,\mathcal{H}\right)(x)\\ &\quad+\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{H}\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{G}\right)(x)-\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{H}\right)(x)\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\,\mathcal{G}\right)(x),\end{aligned} \tag{25}$$

where, $A_k(x)$ and $B_k(x)$ are defined in Theorems 2.1 and 2.2, respectively.

Proof. Equation (24) implies that

$$\begin{aligned}&\mathcal{F}(\zeta)\mathcal{G}(\zeta)\mathcal{H}(\zeta)-\mathcal{F}(\eta)\mathcal{G}(\eta)\mathcal{H}(\eta)\geq\mathcal{F}(\eta)\mathcal{G}(\zeta)\mathcal{H}(\zeta)-\mathcal{F}(\zeta)\mathcal{G}(\eta)\mathcal{H}(\eta)\\ &+\mathcal{G}(\eta)\mathcal{F}(\zeta)\mathcal{H}(\zeta)-\mathcal{G}(\zeta)\mathcal{F}(\eta)\mathcal{H}(\eta)+\mathcal{H}(\eta)\mathcal{F}(\zeta)\mathcal{G}(\zeta)-\mathcal{H}(\zeta)\mathcal{F}(\eta)\mathcal{G}(\eta).\end{aligned} \tag{26}$$

Now, Multiply (26) by $\mathrm{F}(x,\zeta)$ (defined in (11)) and then integrate with respect to ζ from 0 to x, and using (3), we obtain

$$\begin{aligned}&\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{G}\,\mathcal{H}\right)(x)-\mathcal{F}(\eta)\mathcal{G}(\eta)\mathcal{H}(\eta)\left(I^{\omega,\xi,\gamma}_{0+,k}\,1\right)(x)\geq\\ &\quad\mathcal{F}(\eta)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{G}\,\mathcal{H}\right)(x)-\mathcal{G}(\eta)\mathcal{H}(\eta)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\right)(x)\\ &\quad+\mathcal{G}(\eta)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{H}\right)(x)-\mathcal{F}(\eta)\mathcal{H}(\eta)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{G}\right)(x)\\ &\quad+\mathcal{H}(\eta)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{G}\right)(x)-\mathcal{F}(\eta)g(\eta)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{H}\right)(x).\end{aligned} \tag{27}$$

Next, Multiply (27) by (14) which stays positive by virtue of the conditions setout in Theorem 2.6. Integrate the resulting inequality so obtained with respect to η from 0 to x, we obtain

$$\begin{aligned}&\left(I^{\alpha,\beta,\eta}_{0+,k}1\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{G}\,\mathcal{H}\right)(x)-\left(I^{\omega,\xi,\gamma}_{0+,k}1\right)(x)\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\,\mathcal{G}\,\mathcal{H}\right)(x)\geq\\ &\quad\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{G}\,\mathcal{H}\right)(x)-\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\right)(x)\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{G}\,\mathcal{H}\right)(x)\\ &\quad+\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{G}\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{H}\right)(x)-\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{G}\right)(x)\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\,\mathcal{H}\right)(x)\\ &\quad+\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{H}\right)(x)\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{F}\,\mathcal{G}\right)(x)-\left(I^{\omega,\xi,\gamma}_{0+,k}\mathcal{H}\right)(x)\left(I^{\alpha,\beta,\eta}_{0+,k}\mathcal{F}\,\mathcal{G}\right)(x),\end{aligned} \tag{28}$$

which on using the result (8)(when $\vartheta = k$) yields the desired result (25). □

Remark 2.3. *Again, if we substitute $k=1$ into Theorems 2.4 to 2.6, then theorems reduced into the inequalities involving Saigo fractional integral operators (see, Purohit, Ucar and Yadav [24]).*

3. Concluding Remark

Here, we have described some inequalities involving Saigo k-fractional integrals. The inequalities have been derived by taking account of the Chebyshev functional. Further generalizations of these inequalities are often mentioned, containing three monotonic functions. We have also find the results associated with the inequalities involve Saigo fractional integral as particular cases. The generalized operators of k-fractional calculus have the benefit of generalizing Saigo's, Riemann-Liouville and Erdélyi-Kober fractional integral and derivative operators, so several researchers term this a general operator. We summarize this study by stressing that several other fascinating integral inequalities can be obtained from our leading findings as the particular cases.

Declarations

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

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Author's contributions

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