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Öğrencilerin Örüntülere İlişkin Matematiksel Anlamalarının Pirie-Kieren Modeli ile İncelenmesi

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ÖZET

Bu çalışmanın amacı öğrencilerin örüntülere ilişkin matematiksel anlamalarını araştırmaktır. Bir devlet okulunda öğrenim gören üç yedinci sınıf öğrencisi çalışmaya katılmış ve örüntülerle ilgili soruları çözmüştür. Bu öğrencilerle çözümlerine yönelik yarı yapılandırılmış görüşmeler gerçekleştirilmiştir. Veriler Pirie-Kieren teorisi kullanılarak analiz edilmiştir. Elde edilen bulgular, öğrencilerin matematiksel anlamalarının ön bilgiden gözlem yapmaya kadar ilk altı düzey arasında çeşitlilik gösterdiğini ve genellikle imaj oluşturma ile formülleştirme aşamalarında gerçekleştiğini ortaya koymaktadır. Teoriye göre, öğrencilerin ilk ve ikinci ihtiyaç duyulmayan sınırların ilerisine gidebildikleri fakat üçüncü sınırı geçemedikleri görülmüştür. Sonuçlar öğrencilerin örüntülere ilişkin bilgilerinin olduğunu ve örüntünün genel formülünü bulmak için çoğunlukla bir kural belirlemeye ve bunun doğrulunu ilk üç terim için kontrol etmeye çalıştıklarını göstermektedir.

Anahtar Sözcükler: matematiksel anlama, Pirie-Kieren model, örüntüler, ortaokul öğrencileri

Examining Students' Mathematical Understanding of Patterns by Pirie-Kieren Model

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ABSTRACT

The purpose of this study is to investigate students' mathematical understanding of patterns. Three 7th grade students who enrolled in a public school solved the questions regarding the patterns. Semi-structured interviews were conducted with them about their solutions. The data were analyzed by using the Pirie-Kieren theory of mathematical understanding. The findings of this study revealed that students' mathematical understanding varied between first six levels from primitive knowing to observing and their mathematical understanding mostly occurred between Image Making and Formalising layers. In terms of theory, students were able to pass the first and second "Don't Need" boundaries but they could not progress their understanding over the third "Don't Need" boundary. The results also illustrated that all of the students had knowledge about the patterns. In order to find the general rule of the pattern, they mostly endeavored to determine a formula and check its correctness by writing initial three steps.

Keywords: mathematical understanding, Pirie-Kieren model, patterns, middle school students

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1. INTRODUCTION

Currently, interest in mathematical understanding and the need for teaching mathematics with understanding is on the increase (Barmby, Harries, Higgins, and Suggate, 2007). In recent, curriculum emphasizes the necessity of mathematical thinking, interaction and operating deep understanding in students for effective education in many countries (Muir, Beswick, and Williamson, 2008). Based on the suggestions of National Council of Teachers of Mathematics ([NCTM], 2000), "Students must learn mathematics by understanding, actively building new knowledge from experience and prior knowledge." (p. 20).

As to Pirie and Kieren (1994) mathematical understanding is a dynamic, nonlinear, self-replicating continuum and goes through different phases. It is an essential theory in mathematics education since it provides deep insights about the meaning of understanding something (Lester, 2005) and mathematical understanding (Towers and Martin, 2014). In addition, since prior

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knowledge and perception of the students are different, their mathematical understanding might be different from each other (George, 2017). Describing students` ways of understanding and thinking provides to realize their lacks of mathematical concepts and errors in their solutions and to understand how they construct the knowledge. It also helps curriculum developers to revise and organize the curriculum and teachers to design their teaching according to the growth of mathematical understanding (MacCullough, 2007). At this point, Pirie-Kieren theory has been considered as well-structured perspective explaining the nature of mathematical understanding (Towers and Martin, 2006; Martin, 2008).

In literature, a good number of empirical studies have investigated the growth of mathematical understanding through Pirie-Kieren theory using it as a theoretical and analytic lens (Codes, González, Delgado and Monterrubio, 2013). Besides, this theory have also been used as a tool in some fields such as teacher preparation (Borgen, 2006; Nillas, 2010), developing teaching models (Higgins and Parsons, 2009; Wright, 2014), teacher actions (Warner, 2008), the nature of mathematical understanding (Martin, Towers, and Pirie, 2006; Towers and Martin, 2006), the development of teachers' mathematical understanding (Borgen, 2006; Cavey and Berenson, 2005) and collective mathematical understanding (Martin and Towers, 2016; Towers and Martin, 2014). Various studies have used folding back component of the theory to search the growth of mathematical understanding (Martin, 2008; Lawan, 2011; Valcarce et al. 2012; Wright, 2014). On the other hand, the number of research related to this theory is limited in Turkey (Argat, 2012; Arslan, 2013; Gülkılık et al. 2015; Gökalp, 2012). Moreover, it is seen that mathematical understanding in some topics such as fractions (Arslan, 2013; George, 2017; Gökalp, 2012), rational number (Lawan, 2011), frequencies as proportions (Wright, 2014), permutation and factorial (Argat, 2012), numerical series (Valcarce et al., 2012), geometry (Gülkılık, Uğurlu and Yürük, 2015; Mabotja, 2017) have been focused on with the use of Pirie-Kieren theory in literature.

Differently from the above topics, the focus of this study is on patterns. It is one of important topics in mathematics since the nature of mathematics is based on searching patterns (Hargreaves, Threlfall, Frobisher, and Shorrocks-Taylor, 1999). It provides the transition from arithmetic to algebra through generalizations (English and Warren, 1998) and facilitates to understand the other mathematical concepts and the relationships among them (Burns, 2000). It develops essential skills such as reasoning, problem solving and calculation (Reys, Suydam, Lindquist, and Smith, 1998) and makes mathematical knowledge more meaningful (Fox, 2005).

Identifying, extending and reasoning on patterns helps the students develop algebraic and functional thinking (Van de Walle 2004). NCTM (2000) emphasizes that learning to notice relationships and recognizing patterns enables to take a beginning step towards algebra and to structure new and important mathematical knowledge related to other concepts (Phillips, 1995). On the other hand, most of the studies in literature have focused on generalization and generalization strategies in patterns (Alajmi, 2016; Hallagan et al. 2009; Lannin, 2005; Lannin, Barker and Townsend, 2006). There are a few research on mathematical understanding of patterns (Manu, 2005; Wilson and Stein, 2007). According to Manu (2005) mathematical understanding is dependent on ideas and images rather than the words. Wilson and Stein (2007) finds that there is a relationship between students' mathematical understanding and their representations for patterns. However, these studies mostly reveals the important role of representations in understanding of mathematical ideas. It is felt the need of attention to how mathematical understanding occurs in patterns. Therefore, it is aimed to investigate students' mathematical understanding of patterns and describe their mathematical understanding profiles in the current study. In this respect, the answer of the following research question is investigated in the current study:

1. How can the students' mathematical understanding of the patterns be classified by Pirie Kieren model?

1.1. Theoretical Framework

Pirie and Kieren (1989) developed a theory for the dynamic growth of mathematical understanding offering a mean for operation of acquired knowledge, and the learners' thinking related to and building their understanding (Martin, 2008). The Pirie-Kieren theory considers students' understanding in the framework of a whole dynamic, layered, nonlinear, recursive process of their knowledge structure (Pirie and Kieren, 1992).

The theory comprises of eight different layers of actions describing one`s development of understanding and represented by eight nested circles (see Fig. 1). Each layer includes all previous subsequent layers and development moves outward (Martin, 2008). Outer layers in the model represent deeper understanding levels. However, students go back and forth within these layers while generalizing mathematical knowledge or remembering previous knowledge to interrelate new concepts (Thom and Pirie, 2006). Pirie and Kieren (1991) name this dynamic process as "Folding Back" emphasizing on its important effects on the development. Folding back provides "reconstruction and elaboration of inner level understanding to support and lead to new outer level understanding". It makes understanding deeper when inner levels are revisited (p.172).

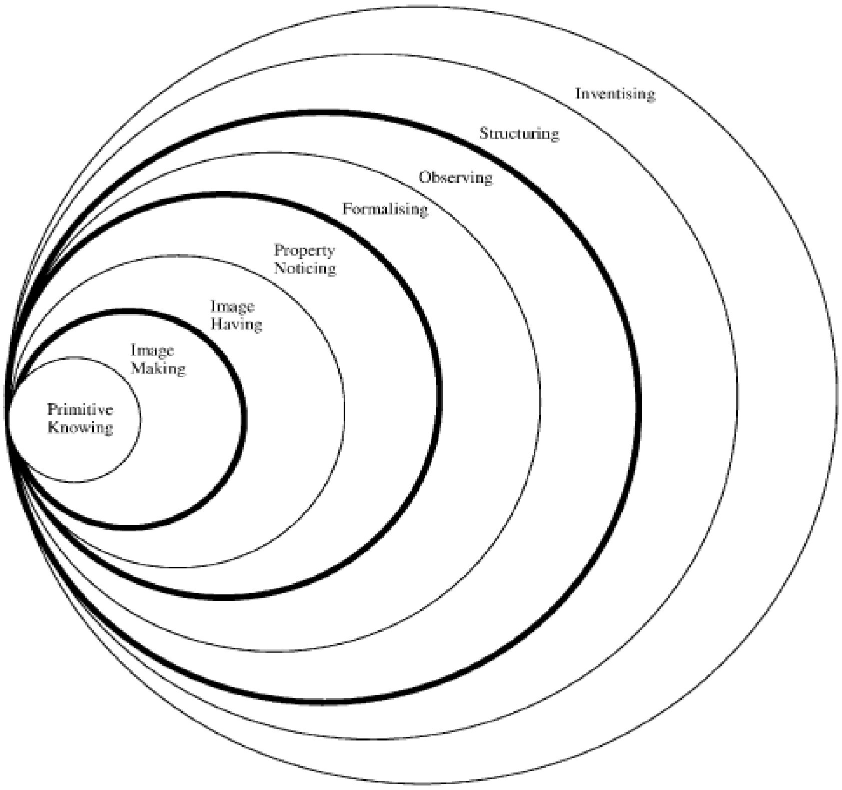

Figure 1. Pirie-Kieren model of mathematical understanding (Pirie and Kieren, 1994)

Primitive Knowing is the essential cognitive knowledge in order to structure new concepts learned. For example, in the case of percentages, it includes knowing that 50% are one half of something (Dole, 2000). Image Making is related to the activities which students attend to get an idea of what concept is about. For example, in the case of fractions, it involves making different combinations of a kit containing halves, thirds, fourths, sixths, eighths, twelfths, and twenty-fourths for the fraction amount of 3/4 (Pirie and Kieren, 1994). At Image Having stage, the learner has a mental approach to apply without the need for engaging in particular activities. One does not need to rely on inner understanding. This situation is called "Don't Need" boundaries which occur between three different layers of understanding (the darker lines in the model). According to Thom and Pirie (2006) having an image is to know "some piece of mathematics as a "matter of fact" (right or wrong!)" (p. 190). Property Noticing is the stage which the learner is able to realize connections and distinctions between images s/he has constituted and the properties of the concept learned. For instance, regarding percentages, it includes noticing that "a given fraction can be represented as a percentage, such as one fifth of the whole grid is 20% because 5×20=100" (Wright, 2014). At Formalising stage, the learner generalizes the properties s/he has realized depending on previous displays without specific reference to a particular action or images, and can explain how s/he makes generalizations. For example, it involves calculating 20% of $120 as one fifth of 120 (Wright, 2014). At Observing stage, the learner is able to organize personal thinking process by recognizing branching. Observing requires to reflect on and coordinate one`s formal mathematical activities. For instance, it includes to understand that there cannot be a quantity such as the smallest fraction (Pirie and Kieren, 1994). Structuring is a stage which learner is able to explain considerations and experiences in sensible mathematical structure. For example, in the case of fractions, a student can see a rational number in the form of a set of ordered pairs (Pirie and Kieren, 1994). At Inventising stage, the learner structures understanding completely, so the learner reaches new understanding level and is able to produce new questions by extending the existing mental constructions (Thom and Pirie, 2006; Martin, 2008). According to Borgen (2006) in this level, a question such as "How could the fifth or sixth dimension be?" can be produced (p. 34-35).

In this study, the focus is on the solutions as a whole or a process rather than the correct or wrong answers of the students for the questions regarding patterns. Pirie-Kieren theory is used in analysis to reveal mathematical understanding of patterns embedded in the students` solutions and dialogues recorded by video camera.

2. METHODOLOGY

Case study is a kind of qualitative research design providing the opportunities of exploring a case or cases in detail with the help of various data collection process including multiple resources. In this study, case study design was employed since it supports a depth understanding of the cases (Stake, 1995).

2.1. Participants and Data Collection

The participants of this study are three 7th grade students, two females and one male student, attending a public school in the northwest of Turkey. According to mathematics curriculum, it is aimed to make students engage in number and geometric patterns from the first grade to the seventh grade. They are expected to recognize the number patterns whose difference is constant, find the relationships between items in patterns, identify missing items, form different patterns with the same relationship through numbers or images and further the patterns a few near steps. At seventh grade, they are also expected to express the rule of the patterns with letters and further the patterns far steps. Accordingly, it was paid attention to that the students have achieved all the gains related to the patterns. Thus, three 7th grade students were determined by typical sampling technical to represent the class. They were the class members familiar with the concept of the patterns. They also had average achievement scores in mathematics.

The data were collected through the interviews and a test including seven questions regarding patterns which was prepared by the researchers (Figure 2). Students' understanding might arise in different ways depending on the questions asked during the interview. Thus, different kind of questions such as furthering number and figure patterns, defining patterns, finding general formula of patterns and generating patterns were prepared. Because the students were at 7th grade, the test included linear patterns that the difference between the numbers was constant and the questions that enabled to reveal the property of folding back were preferred. Students individually solved these questions in an hour. The processes of the students' engagement in these questions were recorded by video camera. Then, the interviews were performed. The questions such as "How do you solve the question?", "Why do you think in this way?" were asked to encourage the students to share their ideas and to obtain detailed information about their solution processes and mathematical understanding. The video recordings of the interviews approximately took thirty minutes for each student's work. The interviews were used to reveal mathematical understanding map of each student.

Question 1: What is the meaning of pattern? Can you write an example for a pattern?

Question 2: In the pattern of 3, 9, ..., 21, 27 can you write the appropriate number into the blank?

Question 3: What is the algebraic expression corresponding to 1, 5, 9, 13,... number pattern? Find the number at the 13. step of pattern with the help of algebraic expression.

Question 4: What is the rule of pattern given at the table?

Sequence Number in Pattern	Stick Number
1	4
2	7
3	10
...	...
n	

Question 5: Fill the table with the pattern which you determine the rule of it.

Sequence Number in Pattern	Stick Number
1	
2	
...	
n	

Question 6:

a) Construct the triangles for the 5th and 6th steps.
b) Find the number of sticks needed to construct the triangle of the 9th step.
c) Find the number of sticks needed to construct the triangle of 20th step without drawing figures.
d) Write algebraic expression for the pattern.

Question 7: By using the information of 1x4, 2x4, 3x4,...., form a pattern using any figure.

Figure 2. The questions related to patterns

2.2. Data Analysis

In data analysis, content analysis technique was used to analyze the qualitative data. In this study, considering the students' solutions and the video recordings including the actions and statements of the students, their solutions were categorized by using the Pirie–Kieren model to illustrate the mathematical understanding of them. The map of each student show in which layers they have gone forward and backward.

Table 1.
The codes for mathematical understanding of patterns

Layers	Codes
Primitive Knowing	Recognition of patterns, knowing what patterns are, creating patterns
Image Making	Trying to find the relationship between items of patterns using various numbers or images, trying to further patterns by extending with some numbers
Image Having	Furthering patterns without the need for engaging in particular activities such as drawing images or trying some numbers
Property Noticing	Expressing some relationship between items of patterns
Formalising	Making generalization of the rule of patterns and writing general formula representing all relationships between items of patterns
Observing	Generating general formula for extending patterns to far steps, connecting patterns with different mathematical topics such as algebraic expressions, equations, geometry
Structuring	Trying to generate general comprehension about how to write algebraic expressions of patterns, making logical comments on the meaning of patterns in the form of a.n+b
Inventising	Creating new questions on patterns

With the aim of performing the reliability and validity studies for the findings, direct quotations belonged to the interview processes were given for the detailed information and description of the students` solutions, thinking and understanding related to patterns. In addition, two researchers (the authors conducting this study and having Ph.D. degree in mathematics education) separately analyzed the transcripts of the video recordings and made their own coding by using Pirie–Kieren model. In consequence of comparing coding, the consistency between them was found as 90%. They discussed about the remaining 10% of the coding list in order to reach consensus. For example, when a case expressed the properties of the patterns researchers remained between the layers of property noticing and formalizing. At this situation, the examples from literature were examined again and it was decided to code it as property noticing when all relationships in the patterns were not indicated as a general formula.

3. FINDINGS

The questions regarding patterns were respectively asked to the participants and the dialogs for the questions were given in this section.

3.1. Case 1: Selin

The following dialogs report the answers of Selin to each question and her mathematical understanding layers.

Dialog 1 for first question:

R: *What is the meaning of pattern? Can you write an example for a pattern?*
Selin: *Recurring…the numbers or objects which continue depending on each other, backwards.*
R: *Explain recurring. Can you give an example?*
Selin: *For example, two ımmm… later… two times two is four… they are ordered in a way that each step has a number twice the number of previous step. Four times two is eight, there is two times between them as well. Eight times two is sixteen… it goes like this.*
R: *What is your example of patterns?*
Selin: *1, 2, 3, 5, 8, 13, 21*
R: *Can you explain the relationship?*
Selin: *One plus two is three, two plus three is five, three plus five is eight… it continues in this form*.

Primitive knowing is the base which mathematical knowledge can be built upon. It is seen that Selin had appropriate knowledge about patterns. Because of the concept “depending” that she used, it could be claimed that she knew there was relationships between the elements of patterns and the elements might be number or object. This was her primitive knowledge related to patterns and presented that she worked at **Primitive Knowing** layer.

Dialog 2 for second question:

R: *In the pattern of 3, 9, …., 21, 27 can you write the appropriate number into the blank?*
Selin: *First, I think three times three is nine and I can continue by multiplying but I realize it does not become.*
R: *Why do you think in this way?*
Selin: *If I say nine times three, twenty-seven is here [shows the twenty seven at the end of the pattern]. It is impossible, pattern cannot be like this.*
R: *What do you do then?*
Selin: *It may be fifteen odds.*
R: *Why?*
Selin: *There are six between numbers, it continues by increasing six by six.*

Selin initially tried to take forward the pattern by extending the previous numbers with three but she understood that her approach was wrong and continued to seek another relationship. Since she tried to comprehend relationships, we could say that she was at the **Image Making** layer. At her second approach, she understood the rule at the pattern. Thus, her explanation of "increasing six by six" was the evidence of her moving out to **Property Noticing** layer by combining her approaches.

Dialog 3 for third question:

R: *What is the algebraic expression corresponding to 1, 5, 9, 13,... number pattern? Find the number at the 13. step of pattern with the help of algebraic expression.*
Selin: *I try n+4 however it is not appropriate.*
R: *Ok, how can we express the connection among the numbers?*
Selin: *A term is four more than the next term.*
R: *How can we express this connection with parameters?*
Selin: *It can be 4n-3 [written on her piece of paper]. ... I think when we replace one into the place of n, four times one is four, the result becomes one. Then, if I replace by two, the result becomes five. Hence, I can justify the pattern.*
R: *How do you get the relationship?*
Selin: *First, I consider five because one is more general... because of increasing four by four, I think it should be related to four. Four times n but when we check by two it is too much so I subtract three and it match with all.*
R: *How do you decide to subtract three? Why do not you subtract two?*
Selin: *Four times two eight. In order to get five, I decrease three.*
R: *The question also requires finding the number at the 13th step.*
Selin: *Forty-nine.*
R: *How do you find?*
Selin: *We wrote thirteen instead of n. Fifty-two minus three is forty nine.*

Selin's examination of the idea whether n+4 was suitable to be the general formula showed that she was working at the **Image Making** layer. Although she was not able to get the rule, while she was working at this layer, she realized the relationship between numbers. Her statement of "increasing four by four" supported that she passed the **Property Noticing** layer. Then, she checked whether 4n-3 was the formula of pattern or not by writing one, two, three instead of n respectively. Thus, she went back **Image Making** layer from property noticing layer. After she had controlled 4n-3 with numbers and recognized outcomes were the numbers of pattern, she decided 4n-3 was the general term of the pattern. Because of that, Selin indicated the rule of pattern with the help of a parameter like n, she was observed to move out the **Formalising** layer. In the second part of the question, Selin wrote 13 instead of n in the formula and said the number at the 13th step in a short time without counting or writing the next numbers. Therefore, it showed that she was at the **Image Having** layer. Image having occurred just outside of the first "Don't Need" Boundaries in the Pirie-Kieren Model. Due to her action that Selin did not need to engage in the activities such as counting or extending pattern by writing the latter numbers, namely, because of not relying on the more specific inner understanding, working out of the first "**Don`t Need**" boundary was also observed here.

Dialog 4 for fourth question:

R: *What is the rule of pattern given at the table?*
Selin: *[she thinks on the question and writes something on paper]. I find.*
R: *How do you find?*
Selin: *ıımmm... n.3+1 [written on her piece of paper]. If I replace 1 into the place of n, the result is four and if I replace 2, the result becomes seven and so on. Hence, I can justify the pattern in this way.*
R: *Why do you use the expression of 3.n?*
Selin: *Because a term is three more than the next term.*
R: *Why do you add one?*
Selin: *Since, it seems appropriate.*
R: *Why?*
Selin: According to me, the numbers on the right hand side of the table are one more than multiplies of three.

After she thought on the question for a while, she directly formed the correct formula of the pattern. Due to the idea that she made generalization of the rule in the pattern and wrote the expression including all situations, we could say that she worked at **Formalising** layer in this question. Formalising occurs just outside of the second "**Don't Need**" boundaries according to the Pirie-Keiren Model. It could be stated that the example of the second "**Don't Need**" boundary was observed.

Dialog 5 for fifth question:

R: *How do you start to solve the question?*
Selin: *The answer is up to us, I primarily determine the rule, the algebraic expression as 2n+1 [she shows her paper]...ıımmm. Then, as an example for the first one, one times two plus one is three; for the second five comes; and for the third seven comes. I form two-column table including the number of steps and its value based on the formula to check the formula of the pattern.*

It was observed that she first determined a formula and then obtained the numbers of the pattern by writing one, two, three instead of n. Hence, we could say that she had already experienced this situation. She determined the rule independent from activities and knew something about the pattern as evidence of her work. Thus, it could be said that she was at the **Image Having** layer. The example of working out of the second "**Don't Need**" boundary was also observed.

Dialog 6 for sixth question:

a) Construct the triangles for the 5th and 6th steps.
Selin: *There exists one triangle in the initial stage, two in the second stage… the number of triangles is the same as the number of stage of the pattern. Hence, there are three triangles at third stage, four triangles at fourth, five triangles at fifth and six triangles at sixth stage [then, she completes the stages by drawing the figures].*

We saw that she worked at the **Property Noticing** layer. Her statement indicated that she formed connections between the number of steps and the number of triangles.

b) Find the number of sticks needed to construct the triangle of the 9th step.
Selin: *Twenty-seven.*
R: *How did you solve this question?*
Selin: *There are nine triangles at the 9th step. Triangle is formed via three sticks. Hence, I acquire twenty-seven when I multiply nine and three.*

She said the answer immediately and she did not need to draw figures. We could say that this situation occurred because of working at **Image Having** layer and passing the first **"Don't Need"** boundary. She constituted the links between the number of edges of triangle and the step number. Therefore, she was at the **Property Noticing** layer.

c) Find the number of sticks needed to construct the triangle of 20th step without drawing figures.
Selin: *Sixty…I wrote twenty instead of n as in the previous question.*
R: *how can we solve by a different strategy?*
Selin: *Different way… immm we can find. We can multiply another number with n or add another number so we can change the formula.*

As in the previous question, she did not draw figures and find the outcome in a short time as an evidence for the observation that she continued working at the **Property Noticing** layer.

d) Write algebraic expression for the pattern.
Selin: *3.n…the number of step times the number of sticks of a triangle.*

Because of that, Selin indicated the general formula of pattern with the help of a parameter like n, she was observed to move out to the **Formalising** layer. We also saw the example of working out of the second **"Don't Need"** boundary.

Dialog 7 for seventh question:

R: *By using the information of 1x4, 2x4, 3x4,…., form a pattern using any figure.*
Selin: *I make like this [she shows the figure pattern composed of squares on her paper]. One square, two square, three square… I made the figure according to the previous sticks.*

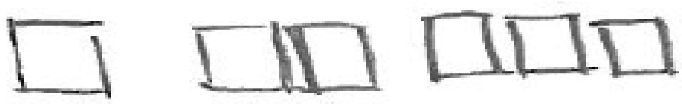

Figure 3. Selin's representation of pattern

She formed the figure pattern by using square in the similar way as in the previous question related to the triangular shapes. Therefore, it could be claimed that the previous problem-solving process influenced her understanding. She did not engage in any activities to solve the question and draw figures so that we could say that she worked at **Image Having** layer. In addition, it was observed that she worked out of the first "**Don't Need**" boundary.

The map of Selin's mathematical understanding during the solution of pattern problems is as follows:

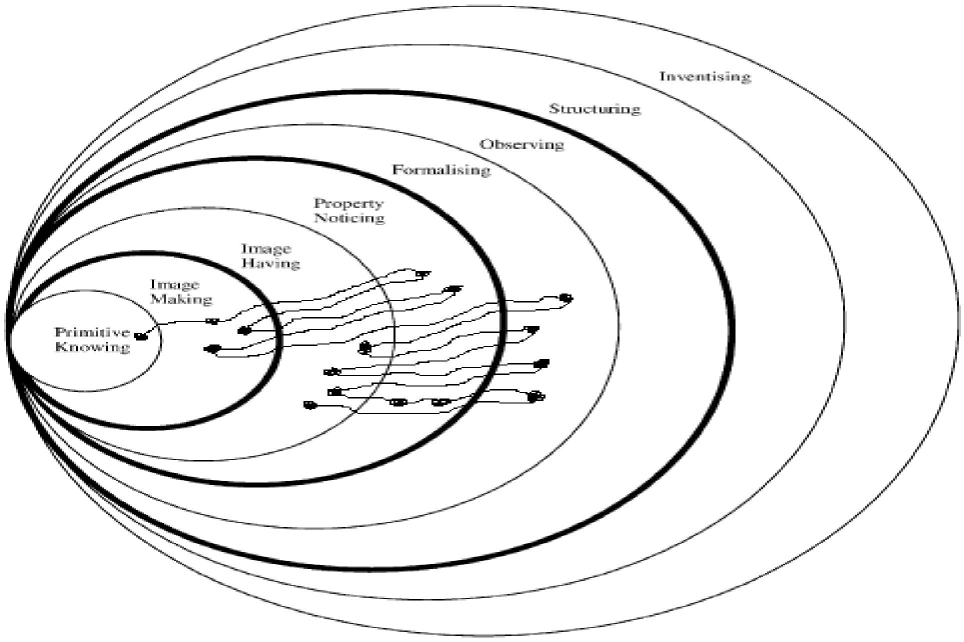

Figure 4. The Map of Selin`s Mathematical Understanding

3.2. Case 2: Ufuk

The following dialogs report the answers of Ufuk to each question and his mathematical understanding layers.

Dialog 1 for first question:

R: *What is the meaning of pattern? Can you write an example for a pattern?*
Ufuk: *The numbers continue in the form of product of the previous number, namely, the numbers are which proceed in a specific rhythm.*
R: *Can you give an example?*
Ufuk: *5, 10, 15, 20*
R: *What is the relationship in this example?*
Ufuk: *It goes on increasing five by five*.

It could be stated that Ufuk was aware of the idea that pattern included relationship among numbers and he thought only in terms of numbers, but he did not focus on the relationships among objects. Although his explanation had deficiencies, his answer was the evidence for his work on **Primitive Knowing** layer.

Dialog 2 for second question:

R: *In the pattern of 3, 9, …., 21,27 can you write the appropriate number into the blank?*
Ufuk: *It continues increasing six by six. 3, 9, 15, 21, 27.*
R: *At the beginning, how do you think?*
Ufuk: *First I find the odds between these two [he shows 3 and 9]… there are six odds between them. Then, I find fifteen.*

In this dialog, he found the answer easily and explained the relationship among the numbers. Therefore, his statements proved that he was at **Property Noticing** layer.

Dialog 3 for third question:

R: *What is the algebraic expression corresponding to 1, 5, 9, 13,… number pattern? Find the number at the 13. step of pattern with the help of algebraic expression.*
Ufuk: *Here, 1, 5, 9, 13… it increases four by four?*
R: *Can you state the general formula?*
Ufuk: *…*
R: *You say that it increases four by four, try to do something considering this [orients student to how to think].*
Ufuk: *2n+3, when I replace 1 into n, I acquire 5. It is justified.*
R: *Actually, what is first term that you should obtain?*
Ufuk: *One… ın the expression of 4n+1, when I replace 1 into n, it gets 5.*
R: *But you said that you had to acquire the first term as one.*
Ufuk: *[he tries to obtain formula]*
R: *Ok, you are expected to reach the number at the 13. step. What is 13th term?*

> ***Ufuk:*** *[he writes something on his paper]… forty nine.*
> ***R:*** *Why?*
> ***Ufuk:*** *I do it when it is increased four by four.*
> ***R:*** *What about the 120th term?*
> ***Ufuk:*** *I can use the expression of this pattern for this term…*

In his first statement, it was observed that he realized the relationship in the pattern and worked at **Property Noticing** layer. Later, he tried to obtain the general formula but he was not able to find it appropriately. Hence, he was directed to how to think and promoted returning an inner layer in order to extend his inadequate mathematical understanding. Then, he wrote algebraic expression and checked whether it proved the pattern or not. After controlling, he sought another formula but he was not able to reach correct algebraic expression. Here, he engaged in the activities to obtain formula as an evidence for his work on **Images Making** layer. It was observed that he removed to Image Making layer from Property Noticing layer. He needed to revisit inner level for going deeper his understanding about general formula. This situation presented that there was **Folding Back** here. Then, in the second part of the question, because he calculated the answer by counting and writing on his paper the pattern`s number in the form of extended until the 13th step, it could be claimed that he proceeded to work on **Image Making** layer.

Dialog 4 for fourth question:

> ***R:****What is the rule of pattern given at the table?*
> ***Ufuk:*** *It goes on three by three.*
> ***R:*** *Can you find the formula of this pattern?*
> ***Ufuk:*** *With n?*
> ***R:*** *Yes*
> ***Ufuk:*** *I find 3n +1.*
> ***R:*** *How do you think?*
> ***Ufuk:*** *First I think to multiply with three and then if I add one it, will be four [he shows 4 at the table]. Later I think for three and two, six comes and after adding one I find seven [he shows 7 at the table].*

It could be stated that he recognized the relationship among numbers with the help of his statement "It goes on three by three" as an evidence for his work on **Property Noticing** layer. Then, he was able to find the general formula of pattern and explained how he thought. Therefore, it indicated that he moved out to **Formalising** layer. The example of working out of the second "**Don't Need**" boundary was appeared in his explanations.

Dialog 5 for fifth question:

> ***R:*** *What is your pattern?*
> ***Ufuk:*** *5, 6, 7, 8 [he writes the numbers but leaves empty the cell of general term].*
> ***R:*** *Can you form the rule of this pattern?*
> ***Ufuk:*** *Is not it n?*
> ***R:*** *Try to check*
> ***Ufuk:*** *[he thinks for a while, and then he erases the numbers and tries to write another pattern]… This time, firstly, I wrote 3n+3 as the rule [he shows his paper]. Then, I find the numbers of the pattern. Three times one plus three is six. Three times two plus three is nine. Three times three plus three is twelve and… the next one is fifteen.*

Initially, he wrote the numbers and then, he sought the formula of this pattern. He controlled the suitability of the formula writing one, two, three instead of n but he recognized that it was not suitable. This effort showed that he worked at **Image Making** layer to get the general form of the rule in the pattern. Later, he decided to erase the numbers and then, he wrote a formula 3n+3 firstly and found the numbers in the pattern writing one, two, three instead of n. At the end, he indicated his pattern as 6, 9, 12, 15. By considering his writing of 3n+3, it could be claimed that he knew how to form a pattern formula and work on it. Thus, it was observed that he worked on **Image Having** layer and worked out of the first "**Don't Need**" boundary.

Dialog 6 for sixth question:

> a) Construct the triangles for the 5th and 6th steps.
> ***Ufuk:*** *There exists a triangle for the initial stage, two triangles for the second, three triangles for the third so there exist five triangles for the fifth and six triangles for the sixth stage [then, he completes the steps by drawing the figures].*

Due to the actions that he made connections between the steps and triangles, his working occurred in the **Property Noticing** layer.

> b) Find the number of sticks needed to construct the triangle of the 9th step.
> ***Ufuk:*** *Nine times three.*
> ***R:*** *Why?*
> ***Ufuk:*** *Because, there are three edges of triangle. With the aim of acquiring the number of sticks at the 9th step, I multiply nine with three.*

He completed the step by drawing triangles so he worked on **Image Having** layer. In addition, his statement of "there are three edges of triangle" presented that he had already had image so he worked on **Image Having** layer as well. Because of not needing to draw, it was observed that he worked out of the first "**Don't Need**" boundary. Then, he indicated that he multiplied 9 with three because of the number of edges of a triangle. He made connection between the number of triangle and step. Therefore, his explanation showed that he worked on the **Property Noticing** layer.

c) Find the number of sticks needed to construct the triangle of the 20th step without drawing figures.
Ufuk: *The sixty.*
R: *How do you find?*
Ufuk: *I multiply twenty with three by considering such in the previous one.*

He said that he solved question such in the previous one as an evidence that he continued working on **Property Noticing** layer.

d) Write algebraic expression for the pattern.
Ufuk: *n… is not it n? Because, there exists a triangle at the initial stage, two triangles at the second stage, three triangles at the third stage.*
R: *If you think in terms of the number of sticks?*
Ufuk: *3n.*

He formed both formulas; the first one indicates the relationship between the number of triangles and steps, and the other showed the relationship between the number of sticks and steps. Thus, it was observed that he moved out to the **Formalising** layer. We also saw the example of working out of the second "**Don't Need**" boundary.

Dialog 7 for seventh question:
R: *By using the information of 1x4, 2x4, 3x4,…., form a pattern using any figure.*
Ufuk: *I use circles. First 1x4 is four [shows the question] so I draw four circles, then 2x4 is eight so eight circles and then twelve circles.*

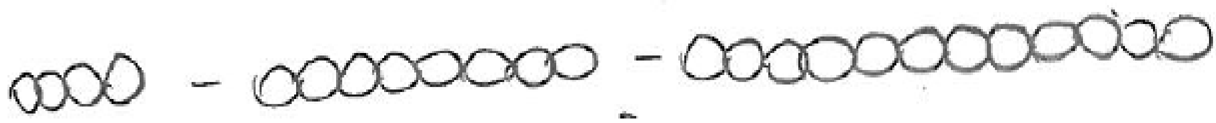

Figure 5. Ufuk's representation of pattern

First, he found the numbers of pattern as 4, 8, 12…. Then, he used four circles corresponding to first term of the pattern, eight circles corresponding to the second term and last twelve circles. Therefore, he indicated the same pattern using figure so he was at **Images Making** layer. The map of Ufuk's mathematical understanding during the solution of pattern problems is as follows:

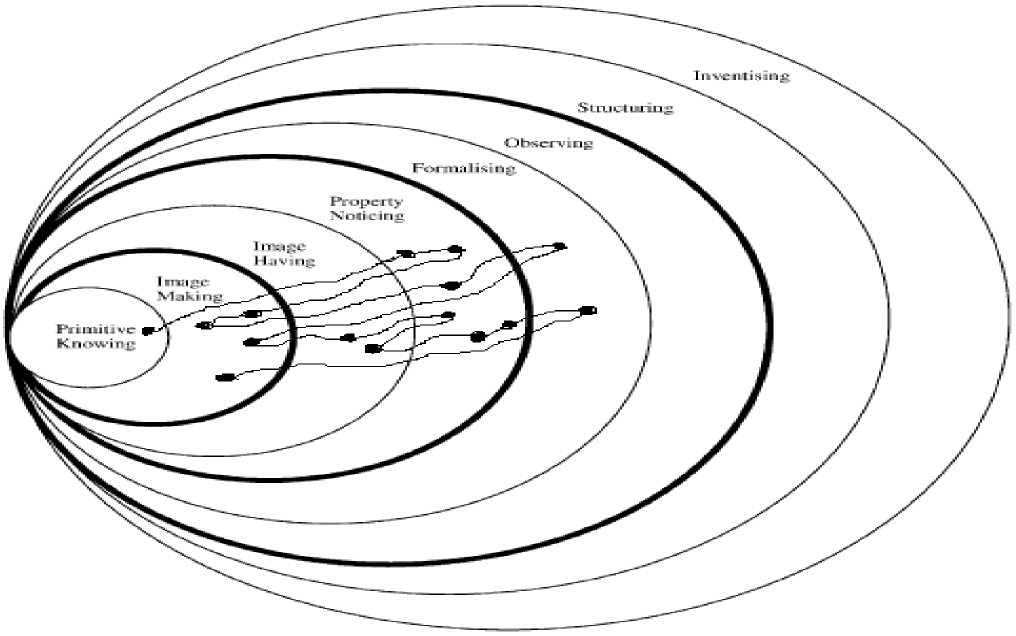

Figure 6. The Map of Ufuk`s Mathematical Understanding

3.3. Case 3: Irem

The following dialogs report the answers of Irem to each question and her mathematical understanding layers.

Dialog 1 for first question:

R: *What is the meaning of pattern? Can you write an example for a pattern?*
Irem: *How should I say?... when you look at the numbers, if there is a relationship between the previous step and the next step, we can say that there is a pattern.*
R: *What is your example of the pattern?*
Irem: *0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55*
R: *Can you explain the relationship?*
Irem: *Each number is the sum of previous two numbers… namely, zero plus one is one, one plus one is two here, one plus two is three… it goes on like this.*

She indicated that there should be connection between numbers or objects to be a pattern and had knowledge. This dialog showed her primitive knowledge related to patterns and she worked on **Primitive Knowing** layer.

Dialog 2 for second question:

R: *In the pattern of 3, 9, …., 21, 27 can you write the appropriate number into the blank?*
Irem: *Initially I supposed that the number on each step was three times the number on previous step. However, I realized that it was not accurate when I saw twenty-one. I considered initial two numbers of the pattern. If the numbers did not continue in this way, I thought that there might be the number in each step is six more of the number on the previous step of the pattern. I looked at two numbers on initial steps and then two numbers on the last steps…Then, I wrote 15 on the blank based on this rule.*

In this question, she looked for a rule to find the number in the blank. Because she could not obtain a result with her first opinion, she maintained to work on the question. Due to the fact that she engaged in checking numbers, it was observed that she worked at the **Image Making** layer. Then, she realized the relationships between numbers and found the number. Her statement of "there may be six odds among them" indicated that she moved out the **Property Noticing** layer from **Image Making.**

Dialog 3 for third question:

R: *What is the mathematical statement illustrating 1, 5, 9, 13,… number pattern? What is the number on the 13th stage of the pattern with the help of algebraic expression.*
Irem: *…n.2+(n-1)*
R: *What do you write in such a way? [n.2+(n-1)]*
Irem: *I focus on the second term. When I replace 2 into n, I get 5. Then, when I replace 3, I get 8. However, it does not fit the expression.. because I must get 9.*

Initially, she wrote a general formula to understand whether it proved the pattern or not. Then, she wrote two, three to control but she saw the outcomes were not same with the pattern`s number. She did not obtain the general formula of the pattern. In order to have an idea, owing to working on the question, she was at the **Images Making** layer.

Dialog 4 for fourth question:

R: *What is the rule of pattern given at the table?*
Irem: *I find n.4 - (n-1) [written on her piece of paper]. When I replace 1 into n, I get 4 and When I replace 2, I get 7 and so on. Hence, the expression is justified for the pattern.*
R: *Why do you use n.4 - (n-1)?*
Irem: *I look at the step numbers and values of these steps… I realize that I must decrease particular values on them. Hence, (n-1) for the second step. Then, I must to decrease two, for the third step and so on. It is appropriate for all steps.*

At the beginning, she wrote an expression and then, she controlled the accuracy of it by writing numbers instead of n. Afterwards, she obtained the general formula of the pattern. She generalized it to the pattern by indicating that her expression was true for the pattern`s all numbers as an evidence for her work on **Formalising** layer. The example of working out of the second "**Don't Need**" boundary was observed.

Dialog 5 for fifth question:

R: *What do you write?*
Irem: *3, 7, 11, 15 [initially, she writes the number of sticks but she does not write the rule of the pattern].*
R: *Can you write the general formula of the pattern?*
Irem: *ıımmm, now… it becomes similar to the above question. With this sense, we can write 3n-(n-1)… I guess. Three times two is six, ıımmm… two minus one is one… hmmm. One minute... now, we look at the first one, it comes one and it is true for one. For the second, three times two is six, two minus one is one and six minus one is five… hmm… it is not true for the second. What may it be? [she thinks for a while]…ıımmm. [she studies on the question for a while, she writes another expression of 2.n –(n+1)]*
R: *Could you find the rule?*

Irem: No, I could not. ıımmm… it may be 4n-1. Four times one minus one is three, four times two minus one is seven and four times three minus one is eleven, yes, it is true.
R: How do you decide this formula?
Irem: It increases four by four. So, I look at one and three [shows 1 and 3 in the table]. For one, it becomes three. Then, I try the others, because of that the same rule matches with all, I say 4n-1.

She began from **Images Making** layer in this question. First, she determined the numbers to form a pattern and then, she tried to find the rule of it writing some expressions and checking them. Afterwards, she found the correct algebraic expression of the pattern. Her statement of "increases four by four" showed that she noticed some relationships between numbers. Therefore, she moved out to the **Property Noticing** layer. Finally, she explained her formula. Her statement "the rule matches with pattern" showed that she worked on **Formalising** layer. We also saw an example of working out of the second "**Don't Need**" boundary.

Dialog 6 for sixth question:

a) Construct the triangles for the 5th and 6th steps.
Irem: It increases one by one so it is not difficult. In the first step, there is one triangle, for the second two, for the third three so for the fifth five and for the sixth there are six triangles [after explanation, she completes the steps by drawing the figures].

We observed that she realized the relationships between the steps and triangles. This dialog showed that she was at the **Property Noticing** layer.

b) Find the number of sticks needed to construct the triangle of the 9th step.
R: How many sticks are there on the 9th step?
Irem: Twenty-seven.
R: Why?
Irem: …three times of nine because of the number of the edges of a triangle.

She gave answer without drawing in a short time and indicated that triangle had three edges as an evidence for her work on **Images Having** layer. Therefore, an example of working out of the first "**Don't Need**" boundary was observed. Because making connection between steps and triangles, she removed to the **Property Noticing** layer.

c) Find the number of sticks needed to construct the triangle of the 20th step without drawing figures.
Irem: This time twenty times three is equal to sixty.
R: How can you solve by another strategy?
Irem: I can extend the figures. Drawing figure one by one until the ninth, it is easy. However, drawing figures for the far steps is not useful. For example, we cannot find until 18th step.

In a similar way to the prior question, she said the answer without need to draw figures with the help of property as an evidence for her work on the **Property Noticing** layer. In this dialog, she indicated that one preferred trying to obtain formula and using it rather than finding by drawing. Her idea showed that she observed some situations and made some inferences. Therefore, her interpretation and statement of "we cannot draw until eightieth" was an evidence for her working on the **Observing** layer.

d) Write algebraic expression for the pattern.
Irem: n.3… three comes from the edges of triangle and n is the number of sequence.

We see that she obtains the algebraic expression corresponding to the rule of pattern. Therefore, we can say she works at **Formalising** layer and passes the second "**Don`t Need**" boundary.

Dialog 7 for seventh question:

R: By using the information of 1x4, 2x4, 3x4,…., form a pattern using any figure.
Irem: It may be like this [shows the figure pattern composed of squares on her paper]. Because, in the previous question, we use triangle I think similarly which figure or geometric object may be. ıımmm… it may be square because square has four edges. If we draw square that the number of it is equal to the number of step, the pattern is formed.

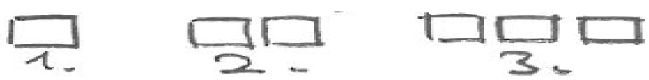

Figure 7. Irem's representation of pattern

She indicated that she thought such in the previous question. She formed pattern using her knowledge. Therefore, she worked on the **Image Having** layer and the example of working out of the first "**Don`t Need**" boundary was observed. In addition, she made connection between the edge of square and the pattern`s step and explained why she used square. Hence, it could be stated that she removed to the **Property Noticing** layer. The map of Irem's mathematical understanding during the solution of pattern problems is as follows:

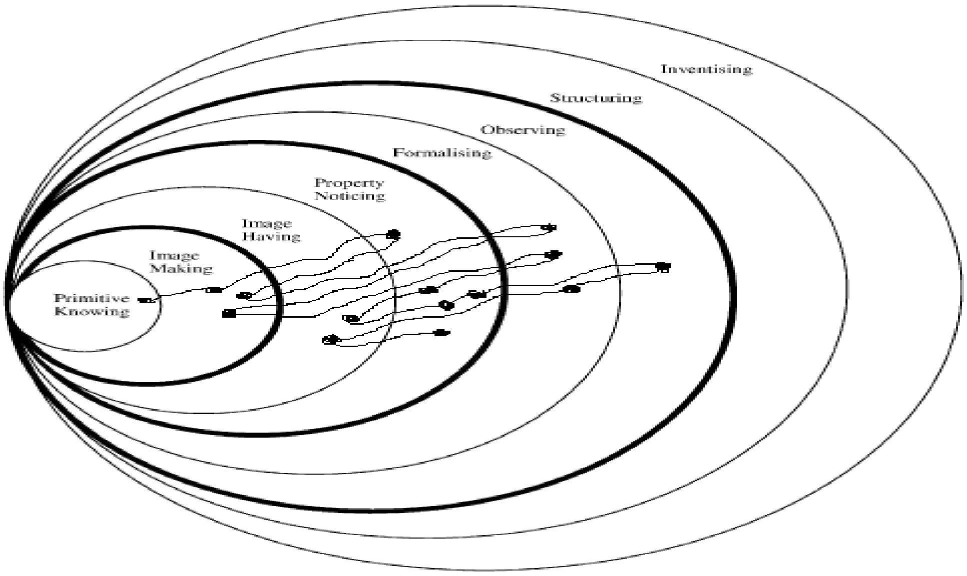

Figure 8. The Map of Irem`s Mathematical Understanding

4. RESULTS, DISCUSSION AND RECOMMENDATIONS

In consequence of the investigation students' understanding of patterns by using Pirie-Kieren theory, first six levels were observed from primitive knowing to observing and the students' mathematical understanding mostly occurred between Image Making and Formalising layers. In the study of Gülkılık et al. (2015) which investigated students' mathematical understanding of geometric transformations, similar to our study, it was found that students' mathematical understanding shaped among first six levels and they were active within the levels of Image Making and Property Noticing more than other levels. The studies of Nillas (2010) and Thom and Pirie (2006) regarding students' mathematical understanding also support these results. Argat (2012) and Arslan (2013) revealed that students' mathematical understanding was between image making and formalising. According to observed layers in this study, it can be said that the students' mathematical understanding levels were low. Likewise, the studies in literature also reveal that mathematical understanding of many students is not enough (Li et al. 2008; Torbeyns et al. 2009; TIMSS, 2007). After analyzing the cases, it was seen that only one student reached the observing layer. In addition, none of the students was able to move out to the Structuring and Inventising layers. Because of that these layers entail more complex skills of thinking and interpreting, the seventh-grade students in the study may not reach these layers.

The results showed that Selin and Irem worked more out of the second "Don't Need" boundary than Ufuk and "Folding Back" was mostly observed in the dialogs of Ufuk. It was found that the students went back and forth more than one layer in some questions. Although they passed higher levels, this progression was not constant and students needed to go back inner levels. It shows that the movements within layers are not linear and they are dynamic (Borgen, 2006; Nillas, 2010; Pirie and Kieren, 1994). Besides, this study showed that students were able to pass the first and second "Don't Need" boundaries but they were not able to move their understanding over the third "Don't Need" boundary. Gülkılık, Uğurlu and Yürük (2015) indicate that passing the second "Don't Need" boundary between the levels of Property Noticing and Formalising is not easy for students and takes time. The reason of not noticing a movement beyond the third Don't Need boundary may be inexperience of students in the levels of Observing, Structuring and Inventising since it is difficult for students to build a formal understanding (Gülkılık et al. 2015). Moreover, some students needed to go back the previous layer for solving the question and "Folding Back" occurred. Students need folding back movements when they cannot immediately solve a problem with the help of their current understanding and this movement provides them to extend their mathematical understanding (Pirie and Kieren, 1994; Martin, 2008; Lawan, 2011; Valcarce et al. 2012). Based on the example of "Folding Back", it can be said that noticing the challenges or errors of students and orienting them to how they think in this direction, making them realize their failure thinking and also moving them back inner layer is important to provide deeper understanding and removing challenges. In order to develop deeper mathematical understanding and move more outsider levels such as Formalising, Observing, Structuring and Inventising, more abstract and advanced mathematical activities may be provided students to engage in them and express their mathematical understanding (Pirie and Kieren, 1994).

The results revealed that all of the students had knowledge about the patterns. They generally indicated the meaning of this concept in terms of the relationship between the elements, interdependence or a particular rule. All of them were able to give an example of pattern and explain the rule of it. When they could not find the general rule of the pattern, they mostly endeavored to determine a formula and check its correctness by writing first three step. It was observed that the students did not randomly indicate a formula. Their ideas were mainly based on the relationship between numbers even if some of them were incorrect. They were able to define the patterns so that this knowledge helped them to focus on the key point and determine the rule. Therefore, it can be said that students' primitive knowledge influences their concept learning and subsequent mathematical understanding. Similarly, many studies reveal that mathematical understanding occurs based on previous mathematical concepts and knowledge and primitive knowing is critical for understanding (Grinevitch, 2004; Hollebrands, 2003; Pirie and Kieren, 1994). Besides, the students mostly checked their formulas and correct them after they noticed the errors. They did not have difficulty in finding general formula of the patterns in general. This may be due to the fact that the questions are based on linear patterns and the differences between the numbers of patterns were constant.

This study conducted with three seventh grade students and concentrated at patterns. The map of mathematical understanding should be examined at different topics with more and different grade students. The relationship between students' mathematical achievement and understanding also can be investigated. In this article, the focus was on how the students' understanding of patterns was classified with the Pirie-Kieren model while solving the problems, not how the mathematical understanding of the patterns developed. Future studies may focus on the development of mathematical understanding in patterns.

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The authors declared that research and publication ethics were followed in this study.

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Statement of Interest

The authors have no conflicts of interest to declare.

6. GENİŞ ÖZET

Son dönemde matematiksel anlamaya yönelik ilgi ve matematiği anlayarak öğretmeye ilişkin ihtiyaç artmaktadır (Barmby, Harries, Higgins ve Suggate, 2007). NCTM (2000), öğrencilerin matematiği anlayarak, yeni bilgileri deneyimlerinden ve önceki bilgilerinden aktif bir şekilde inşa ederek öğrenmeleri gerektiğini söylemektedir. Pirie ve Kieren'e (1994) göre matematiksel anlama dinamik, lineer olmayan, kendi kendini yineleyen bir süreçtir ve farklı düzeyleri içermektedir. Bir şeyi anlamanın ne demek olduğuyla ilgili derinlemesine bir bakış açısı sağladığı için eğitimde önemli bir teoridir (Lester, 2005). Bunun yanı sıra, matematiksel anlama öğrencinin önceki bilgilerine ve tecrübelerine dayandığı için her bir öğrenci için birbirinden farklı olabilir (George, 2017). Öğrencilerin anlama ve düşünme yollarını tanımlamak onların matematiksel kavramlarla ilgili eksikliklerini ve hatalarını tespit etmeyi ve bilgiyi nasıl yapılandırdıklarını anlamayı sağlar. Aynı zamanda, öğretim programlarını geliştiren kişilerin programları düzenlemesine ve revize etmesine, öğretmenlerin derslerini tasarlamasına da yardım eder (MacCullough, 2007). Bu noktada, Pirie-Kieren teorisinin öğrencilerin matematiksel anlamalarını açıklamada iyi yapılandırılmış bir perspektif sağlayacağı düşünülmektedir (Towers ve Martin, 2006; Martin, 2008).

Alan yazımda, çok sayıda deneysel çalışma bu teoriyi analitik bir bakış açısı olarak kullanarak matematiksel anlamanın gelişimini incelemiştir (Codes et al. 2013). Ayrıca, bu teori birçok farklı alanda bir araç olarak kullanılmıştır. Örneğin, öğretmen eğitimi (Borgen, 2006; Nillas, 2010), öğretim modeli geliştirme (Higgins ve Parsons, 2009; Wright, 2014), matematiksel anlamanın doğası (Martin, Towers ve Pirie, 2006; Towers ve Martin, 2006), öğretmen uygulamaları (Warner, 2008), öğretmenlerin matematiksel anlamalarının gelişimi (Borgen, 2006; Cavey ve Berenson, 2005), ortak matematiksel anlama (Martin ve Towers, 2015, 2016; Towers ve Martin, 2014). Bunların yanı sıra, matematiksel anlamanın kesirler (Arslan, 2013; George, 2017; Gökalp, 2012), rasyonel sayılar (Lawan, 2011), permütasyon ve faktöriyel (Argat, 2012), oranlar (Wright, 2014), sayısal seriler (Valcarce et al. 2012) ve geometri (Gülkılık, Uğurlu ve Yürük, 2015; Mabotja, 2017) gibi bazı konular için araştırıldığı görülmüştür.

Yukarıdaki konulardan farklı olarak bu çalışmada örüntüler konusu ele alınmıştır. Matematiğin doğası örüntüler aramaya dayandığı için bu kavram matematikteki önemli konulardan biridir (Hargreaves, Threlfall, Frobisher ve Shorrocks-Taylor, 1999). Genellemeye dayanan yapısıyla aritmetikten cebire geçişi sağlar (English ve Warren, 1998) ve matematiksel kavramaları ve bunlar arasındaki ilişkileri anlamayı kolaylaştırır (Burns, 2000). Akıl yürütme, problem çözme ve hesap yapma becerilerinin gelişiminde önemlidir (Reys, Suydam, Lindquist ve Smith, 1998). Diğer bir yandan, alan yazımdaki çalışmaların çoğu örüntülerde genelleme ve genelleme stratejileri üzerinde durmuştur (Alajmi, 2016; Hallagan et al. 2009; Lannin, 2005; Lannin, Barker ve Townsend, 2006). Bu nedenle, bu çalışmada örüntüler konusu matematiksel anlama boyutunda ele alınmış ve öğrencilerin örüntülere ilişkin matematiksel anlamalarını araştırmak amaçlanmıştır.

Pirie- Kieren teorisi sekiz düzeyden ve iç içe geçmiş sekiz halkadan oluşmaktadır. Her bir düzey önceki düzeylerin hepsini kapsamaktadır ve gelişim dış halkalara doğru gerçekleşmektedir (Martin, 2008). En dıştaki halka en derin anlamayı temsil etmektedir. Fakat öğrenciler matematiksel bilgiyi genellerken ya da yeni kavramlarla ilişkilendirmek için önceki bilgilerini hatırlarken bu düzeyler arasında ileri ya da geri gidebilirler (Thom ve Pirie, 2006). Ön Bilgi, yeni bilgileri öğrenmek için gerekli olan bilgidir. Örneğin, %50'nin yarım olduğunu bilmek (Dole, 2000). İmaj Oluşturma, öğrencilerin bir kavramı öğrenmek için yaptığı eylemlerle ilişkilidir. Örneğin, 3/4'ü göstermek için yarım, üçte bir, dörtte bir, altıda bir, on ikide bir, yirmi dörtte bir içeren kesir takımıyla farklı kombinler yapmak (Pirie ve Kieren, 1994). İmaja Sahip Olma, kişi herhangi bir eyleme ihtiyaç duymadan kullanabileceği bir zihinsel yaklaşıma sahiptir. Eğer bir kişi daha önceki düzeylerdeki eylemlere ihtiyaç duymuyorsa bu durum "İhtiyaç Duyulmayan Sınırlar" olarak adlandırılır. Özellikleri Fark Etme, kişi oluşturduğu imajlar ile öğrendiği kavramlar arasındaki ilişkileri ve farklılıkları keşfeder. Örneğin, bir bütünün beşte biri ile %20'nın aynı olduğunu fark etmek (Wright, 2014). Formülleştirme, keşfettiği özellikleri matematiksel durumlara göre geneller. Örneğin, 120'nin %20'sini beşte bir olarak hesaplamak. Gözlem Yapma, önceki düzeylerde öğrenmiş olduklarını kullanarak matematiksel eylemlerini düzenlemeyi içerir. Yapılandırma, kişi matematiksel bir yapıdaki ilişkileri ortaya koyabilir. Örneğin, bir rasyonel sayıyı sıralı ikililer seti olarak düşünmek. Keşfetme, kişi kendine yeni anlamalar kazandıracak sorular sorar. Örneğin, "Beşinci ya da altıncı boyut olabilir mi?" sorusunu sormak (Borgen, 2006).

Bu çalışmanın amacı öğrencilerin örüntülere ilişkin matematiksel anlamalarını Pirie- Kieren teorisini kullanarak ortaya koymaktır. Durum çalışması niteliğinde olan bu araştırma üç yedinci sınıf öğrencisiyle gerçekleştirilmiştir. Örüntülerle ilgili yedi soru sorulmuş ve öğrenciler bu soruları yaklaşık bir saat kadar sürede bireysel olarak çözmüştür. Ardından, her bir öğrenci ile çözümleri üzerine yarı yapılandırılmış mülakatlar yapılmış ve süreç video ile kayıt altına alınmıştır. Öğrencilerin matematiksel anlamalarını detaylandırmak için "Bu soruyu nasıl çözdün?", "Neden bu şekilde düşündün?" gibi sorular sorulmuştur. Mülakatlar her bir öğrenci için yaklaşık otuz dakika sürmüş ve veriler öğrencilerin matematiksel anlama haritasını oluşturmak için kullanılmıştır. İçerik analizi tekniği ile öğrencilerin çözümleri ve mülakat verileri analiz edilmiş ve Pirie-Kieren modeline göre yorumlanmıştır. Geçerlik ve güvenirliği sağlamak amacı ile öğrencilerin yanıtlarından doğrudan alıntılara ve detaylı çözümlerine yer verilmiştir. Ayrıca iki araştırmacı tarafından veriler ayrı ayrı kodlanmış ve uyum yüzdesi %90 olarak bulunmuştur.

Sonuçlar, öğrencilerin örüntülere ilişkin matematiksel anlamalarının ön bilgiden gözlem yapmaya kadar ilk altı düzey arasında çeşitlilik gösterdiğini ve genellikle imaj oluşturma ile formülleştirme aşamalarında gerçekleştiğini ortaya koymaktadır. Gülkılık, Uğurlu ve Yürük (2015), Nillas (2010), Thom ve Pirie (2006) çalışmalarında benzer sonuçlara ulaşmıştır. Bu sonuçlara göre öğrencilerin matematiksel anlamalarının düşük olduğu söylenebilir. Alan yazımdaki çalışmalar öğrencilerin çoğunun yeterli matematiksel anlamaya sahip olmadığını ifade etmektedir (Li et al. 2008; Torbeyns et al. 2009; TIMSS, 2007). Bu çalışmada, sadece bir öğrenci gözlem yapma seviyesine ulaşmıştır. Öğrencilerin hiç birinin yapılandırma ve keşfetme aşamasına ulaşamadığı görülmüştür. Bu düzeyler daha karmaşık düşünme ve yorumlama becerileri gerektirdiğinden yedinci sınıf öğrencileri bu düzeylere ulaşamamış olabilir. Bunların yanı sıra, öğrencilerin bazı sorularda birden fazla düzeye gidip geldikleri sonucuna ulaşılmıştır. Daha üst düzeylere çıkmalarına rağmen bu ilerleme sabit değildir ve önceki düzeylere geri gelmeye ihtiyaç duydukları görülmüştür. Bu da düzeyler arasındaki hareketin yani matematiksel anlamanın lineer olmadığını, dinamik olduğunu göstermektedir (Borgen, 2006; Nillas, 2010; Pirie ve Kieren, 1994). Teoriye göre, öğrencilerin ilk ve ikinci ihtiyaç duyulmayan sınırların ilerisine gidebildikleri fakat üçüncü sınırı geçemedikleri görülmüştür. Gülkılık, Uğurlu ve Yürük'e (2015) göre üçüncü ihtiyaç duyulmayan sınırın ilerisine geçilememesinin sebebi öğrencilerin gözlem yapma, yapılandırma ve keşfetme düzeylerinde tecrübesiz olmaları olabilir. Çünkü formel anlamayı inşa etmek öğrenciler için zordur. Bunların yanı sıra, öğrencilerin örüntülere ilişkin bilgilerinin olduğu ve örüntünün genel formülünü bulmak için çoğunlukla bir kural belirlemeye ve bunun doğrulunu ilk üç terim için kontrol etmeye çalıştıkları sonucuna ulaşılmıştır.