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Evolute-Involute Partner Curves According to Darboux Frame in the Euclidean 3-space E^3

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Abstract: In this study, evolute-involute curves are researched. Characterization of evolute-involute curves lying on the surface are examined according to Darboux frame and some curves are obtained.

Keywords: Curve, surface, geodesic, curvature, frame.

1. Introduction

The interest of special curves has increased recently. Some of these are associated curves. They are curves where one of the Frenet vectors at opposite points is linearly dependent to the other curve. One of the best examples of these curves is the evolute-involute partner curves.

An involute thought known to have been used in his optical work came up in 1658 by C. Huygens. C. Huygens discovered involute curves while trying to make more accurate measurement studies [5].

Many researches have been conducted about evolute-involute partner curves. Some of them conducted recently are Bilici and Çalışkan [4], Özyılmaz and Yılmaz [9], As and Sarıoğlugil [2]. Bektaş and Yüce consider the notion of the involute-evolute curves lying on the surfaces for a special situation. They determine the special involute-evolute partner $D-$curves in E^3. By using the Darboux frame of the curves they obtain the necessary and sufficient conditions between κ_g, τ_g, κ_n and κ_n^* for a curve to be the special involute partner $D-$curve. κ_g^* and τ_g^* of this special involute partner $D-$curve are found, here κ_g, τ_g, κ_n are the coefficients in the derivative changes of the Darboux frame [3]. Almaz and Külahçi consider the notion of the involute-evolute curves in Minkowski $3-$space E_1^3. They give the representation formulae for spacelike curves in Minkowski $3-$space E_1^3. By using the Darboux frame of the curves they obtain the necessary and sufficient

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conditions between the coefficients in the derivative changes of the Darboux frame, [1].

In this study, we review the basic notions and information in Section 2. In Section 3, we analyze evolute-involute partner curves by using Darboux frame, next we research the states of geodesic curve, asymptotic curve and principle line of them.

2. Preliminaries

Let the curve of $\alpha: \ I \subset R \longrightarrow E^3$ be a regular curve specified by arc parameter s. The triadic $\{T, N, B\}$ obtained as indicated below by means of this curve is called as Frenet vector field at the $\alpha(s)$ point of the curve α,

$$T(s) = \alpha'(s), \qquad N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \qquad B(s) = T(s) \times N(s).$$

In this point, "$\times$" indicates vector product on E^3 and $\alpha'(s) = \dfrac{\mathrm{d}\alpha}{\mathrm{d}s}$.

These vectors form is an orthogonal vector system. In that point, whereas T is called as tangent vector field, N and B are called as principles normal vector field and binormal vector field, respectively. Frenet formulas could be given such as

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Here κ and τ are called as curvature and torsion of the curve α, respectively.

If the curve α is over a surface S, another frame which is called as Darboux frame and showed with $\{T, g, n\}$, could be found. In this frame, T is tangent vector of the curve α, n is unit normal vector field of the surface S and finally g is another vector field written with $g = n \times T$. Relation between the Frenet and the Darboux frames could be given such as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ g \\ n \end{bmatrix}. \tag{1}$$

Here θ is the angle between g and N. Derivative change of Darboux frames is as shown below:

$$\begin{bmatrix} T' \\ g' \\ n' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ g \\ n \end{bmatrix}.$$

Here, κ_g , κ_n and τ_g in the equation above are called as geodesic curvature, normal curvature and geodesic torsion of the curve α, respectively. The relationship between geodesic curvature, normal curvature, geodesic torsion and κ, τ are given such as

$$\kappa_g = \kappa\cos\theta, \qquad \kappa_n = \kappa\sin\theta, \qquad \tau_g = \tau + \theta'.$$

For the curve α in surface S, we can say the followings [8]:

(i) α is a geodesic curve if and only if $\kappa_g = 0$,

(ii) α is an asymptotic line if and only if $\kappa_n = 0$,

(iii) α is a principle curvature line if and only if $\tau_g = 0$.

Definition 2.1 [7] *Let E^3 be the 3-dimensional Euclidean space with the standard inner product. Let two regular and unite speed curves be (α, α^*) in E^3 and also the Frenet vectors of the curves α and α^* be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$, respectively. If*

$$< T, T^* >= 0,$$

in that case the curve α is called the evolute curve of the curve α^* and the curve α^* is called the involute curve of the curve α (See Figure 1.).

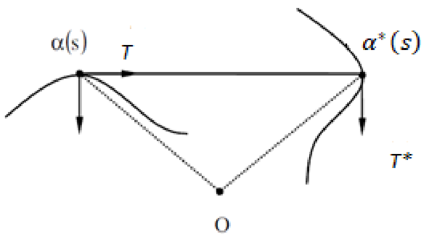

Figure 1: Evolute-involute partner curves

In this study, we are going to use $\{T, N, B, \kappa, \tau, s\}$ as Frenet elements of the curve α and $\{T^*, N^*, B^*, \kappa^*, \tau^*, s^*\}$ as Frenet elements of the curve α^*. In this point, s and s^* are arc parameter of the curve α and arc parameter of the curve α^*, respectively.

Theorem 2.2 [10] *Let (α, α^*) be evolute-involute partner curves. There is equilibrium indicated below between those partner curves*

$$\alpha^*(s^*) = \alpha(s) + \lambda(s)T(s). \tag{2}$$

Here $\lambda(s) = c - s$ and c is a constant in the equation.

Theorem 2.3 [10] *Let (α, α^*) be evolute-involute partner curves. The distance between $\alpha(s)$ and $\alpha^*(s^*)$ is*

$$d\left(\alpha(s), \alpha^*(s^*)\right) = |c - s|\,.$$

Here $\dfrac{ds^*}{ds} = (c - s)\,\kappa\,(s)\,.$

Theorem 2.4 [6] *Let (α, α^*) be evolute-involute partner curves. We could write the followings:*

(i) $T^* = \pm N$.

(ii) *Let two points which is corresponding to the point P of two different evolute curves of the curve α be P_1 and P_2. The angle $\measuredangle(P_1 P P_2)$ is constant. Here P on α.*

(iii) *If the curve α is a circular helix, then all involute curves of its are planar curves and the planes of these curves are perpendicular to the axis of the cylinder.*

(iv) $(\kappa^*)^2 = \dfrac{\kappa^2 + \tau^2}{\kappa^2 (c-s)^2}$ *and* $\tau^* = \dfrac{\kappa\tau' - \kappa'\tau}{\kappa(\kappa^2+\tau^2)(c-s)}$. *Here* $\tau' = \dfrac{d\tau}{ds}$.

(v) *Let the Frenet vectors of curve α be $\{T, N, B\}$ and its Darboux vector be*

$$W = \tau T + \kappa B.$$

Tangent indicatrix, binormal indicatrix and Darboux indicatrix of the curve α be α_T, α_B and α_W, respectively. α_T, α_B and α_W are involute curves.

Theorem 2.5 [10] *Let (α, α^*) be evolute-involute partner curves. The following equations exist:*

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \cos\psi & 0 & \sin\psi \\ -\sin\psi & 0 & \cos\psi \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

In this equation, ψ is angle between vectors T and N^*.

3. Evolute-Involute Curves According to Darboux Frame in E^3

In this part, we are going to present the characterization of evolute-involute partner curves by paying into consideration of Darboux frame.

Definition 3.1 *Let S and S^* be directed two surface in E^3 and also the curves of α and α^* be given on the surface of S and S^*, respectively. If the curves α and α^* are evolute-involute curves, these two curves are called as evolute-involute pair curves in accordance with the surfaces of S and S^*. They are indicated as $(S_\alpha, S^*_{\alpha^*})$. We are going to show Darboux elements of the curve α as $\{T, g, n, \kappa_g, \kappa_n, \tau_g, s\}$ and Darboux elements of the curve α^* as $\{T^*, g^*, n^*, \kappa^*_g, \kappa^*_n, \tau^*_g, s^*\}$ for $(S_\alpha, S^*_{\alpha^*})$ evolute-involute pair. Here κ_g, κ_n, τ_g and $\kappa^*_g, \kappa^*_n, \tau^*_g$ are geodesic curvature, normal curvature, torsion curvature of the curves α and α^*, respectively. At the same time s and s^* are arc elements of the curves α and α^*, respectively.*

Taking the derivative of the equation (2) with respect to s and use the Darboux frame, we find the equation below:

$$T^{*}\frac{\mathrm{d}s^{*}}{\mathrm{d}s}=\lambda\kappa_{g}g+\lambda\kappa_{n}n. \tag{3}$$

If we take the norm of the equation (3), we obtain

$$\frac{\mathrm{d}s^{*}}{\mathrm{d}s}=|\lambda|\sqrt{\kappa_{g}^{2}+\kappa_{n}^{2}}=|\lambda|\,\kappa. \tag{4}$$

Theorem 3.2 *Evolute-involute partner curves* $(S_{\alpha},S^{*}_{\alpha^{*}})$ *are given. Relation between Darboux frames are as like stated below:*

$$\begin{bmatrix} T \\ g \\ n \end{bmatrix}=\begin{bmatrix} 0 & \cos\gamma & -\sin\gamma \\ \cos\theta & \sin\theta\sin\gamma & \sin\theta\cos\gamma \\ 0 & \cos\theta\sin\gamma & \cos\theta\cos\gamma \end{bmatrix}\begin{bmatrix} T^{*} \\ g^{*} \\ n^{*} \end{bmatrix}. \tag{5}$$

Here the angle γ *is* $\psi+\theta^{*}$*.* ψ *is the angle between the vectors* T *and* N*. Angles* θ *and* θ^{*} *are the angle between the vectors* g *with* N *and the angle between the vectors* g^{*} *with* N^{*}*, respectively.*

Proof We have given the equation (1) that the relation of the curve α between the Frenet frame and the Darboux frame is

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} T \\ g \\ n \end{bmatrix}.$$

The relation of the curve α^{*} between the Frenet frame and the Darboux frame is

$$\begin{bmatrix} T^{*} \\ N^{*} \\ B^{*} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta^{*} & -\sin\theta^{*} \\ 0 & \sin\theta^{*} & \cos\theta^{*} \end{bmatrix}\begin{bmatrix} T^{*} \\ g^{*} \\ n^{*} \end{bmatrix}. \tag{6}$$

On the other side, the relation of the curves α and α^{*} between their Frenet frames is

$$\begin{bmatrix} T^{*} \\ N^{*} \\ B^{*} \end{bmatrix}=\begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{7}$$

If we use the equation (7) in the equation (6), we obtain the equation below:

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}=\begin{bmatrix} 0 & \cos(\psi+\theta^{*}) & -\sin(\psi+\theta^{*}) \\ 1 & 0 & 0 \\ 0 & \sin(\psi+\theta^{*}) & \cos(\psi+\theta^{*}) \end{bmatrix}\begin{bmatrix} T^{*} \\ g^{*} \\ n^{*} \end{bmatrix}. \tag{8}$$

Finally, taking $\gamma=\psi+\theta^{*}$ and if we use the equation (6) in the equation (8), we could finish the proof of the theorem. □

Theorem 3.3 *The relationship of evolute-involute partner curves* $(S_\alpha, S^*_{\alpha^*})$ *between their curvatures are*

$$\kappa^*_g = -\frac{1}{\lambda}\cos\gamma + \frac{\tau}{\kappa\lambda}\sin\gamma, \tag{9}$$

$$\kappa^*_n = \frac{1}{\lambda}\sin\gamma + \frac{\tau}{\kappa\lambda}\cos\gamma. \tag{10}$$

Proof We have given Teorem 2.4 that

$$T^* = \pm N. \tag{11}$$

If we take the derivative of the equation (11) with respect to s and use the equation (4), we find the equation below:

$$\kappa^*_g g^* + \kappa^*_n n^* = \frac{1}{\lambda}T + \frac{\tau}{\kappa\lambda}B. \tag{12}$$

If we multiply both sides of the equation (12) by g^* and use in the equation (5), we obtain the equation (9), and If we multiply both sides of the equation (12) by n^* and use in the equation (5), we obtain the equation (10). □

When we consider Theorem 3.3, we could find the corollary below:

Corollary 3.4 *Let evolute-involute partner curves* $(S_\alpha, S^*_{\alpha^*})$ *be specified. Then*

$$\left(\kappa^*_g\right)^2 + \left(\kappa^*_n\right)^2 = \frac{1}{\lambda^2} + \frac{\tau^2}{\kappa^2\lambda^2}.$$

Theorem 3.5 *Let evolute-involute partner curves* $(S_\alpha, S^*_{\alpha^*})$ *be specified. There is equation below:*

$$\kappa^*_g\cos\gamma - \kappa^*_n\sin\gamma = \frac{-\kappa_g\cos\theta + \kappa_n\sin\theta}{\lambda\sqrt{\kappa_g^2+\kappa_n^2}}. \tag{13}$$

Proof We know the equation (11) that $T^* = \pm N$. Then we can write

$$N = \cos\theta g - \sin\theta n$$

and

$$T^* = \pm\left(\cos\theta g - \sin\theta n\right). \tag{14}$$

If we take the derivative of the equation (14) with respect to s and use the Darboux frame, we find the equation below:

$$\begin{aligned}\left(\kappa^*_g g^* + \kappa^*_n n^*\right)\frac{\mathrm{d}s^*}{\mathrm{d}s} &= \pm\left(-\kappa_g\cos\theta + \kappa_n\sin\theta\right)T \\ &\quad \pm\sin\theta\left(\tau_g - \theta'\right)g \pm \cos\theta\left(\tau_g - \theta'\right)n. \end{aligned} \tag{15}$$

If we multiply both sides of the equation (15) by T and use in the equation (5), we obtain

$$\left(\kappa_g^* \cos\gamma - \kappa_n^* \sin\gamma\right) \frac{\mathrm{d}s^*}{\mathrm{d}s} = \pm\left(-\kappa_g \cos\theta + \kappa_n \sin\theta\right). \tag{16}$$

If we use the equation (4) in the equation (16), we could finish the proof of the theorem. □

If the Theorem 3.5 is paid attention, we could find the corollaries below:

Corollary 3.6 *Let evolute-involute partner curves* $(S_\alpha, S^*_{\alpha^*})$ *be specified. Then*

(i) If both of the curves α and α^* are geodesic,

$$\kappa_n^* = \pm\frac{\sin\theta}{\lambda\sin\gamma}.$$

(ii) If both of the curves α and α^* are asymptotic curves,

$$\kappa_g^* = \pm\frac{\cos\theta}{\lambda\cos\gamma}.$$

(iii) If the curve α is an asymptotic curve and the curve α^* is a geodesic curve,

$$\kappa_n^* = \pm\frac{\cos\theta}{\lambda\sin\gamma}.$$

(iv) f the curve α is a geodesic curve and the curve α^* is an asymptotic curve,

$$\kappa_g^* = \pm\frac{\sin\theta}{\lambda\cos\gamma}.$$

Let us remind again; $\lambda(s) = c - s$ and c is a constant in the above equations.

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