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Some Semisymmetry Curvature Conditions on Paracontact Metric (k,μ)-Manifolds

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Abstract: The aim of the article is to study paracontact metric (k,μ)-manifolds satisfying some semisymmetry curvature conditions. Also, we show that if a paracontact metric (k,μ)-manifold is Ricci pseudo-symmetric then it is an Einstein manifold provided $k \neq 1$.

Keywords: Paracontact metric (k,μ)-manifolds, h-projectively semisymmetry, ϕ-projectively semisymmetry, Ricci pseudo-symmetry.

1. Introduction

Paracontact metric structures have been examined in [5], as a natural odd-dimensional counterpart to para-Hermitian structures, like contact metric structures correspond to the Hermitian ones. Paracontact metric manifolds have been characterized by many authors, particularly since the appearance of [13]. An important class among this manifolds is that of the paracontact (k,μ)-manifolds, which satisfy [1]

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{1}$$

for all X,Y vector fields on M, where k and μ are constants and $h = \frac{1}{2}\mathcal{L}_\xi\phi$. This class includes the para-Sasakian manifolds [5, 13], the paracontact metric manifolds satisfying $R(X,Y)\xi = 0$ for all X,Y [14].

Among the geometric properties of manifolds symmetry is an important one. From the local point view it was introduced by Shirokov as a Riemannian manifold with covariant constant curvature tensor R, that is, with $\nabla R = 0$, where ∇ is the Levi-Civita connection [7]. A wide theory of symmetric Riemannian manifolds was introduced by Cartan [3]. A manifold is called *semisymmetric* if the curvature tensor R satisfies $R(X,Y)\cdot R = 0$, where $R(X,Y)$ is considered to

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be a derivation of the tensor algebra for the tangent vectors X, Y. Semisymmetric manifolds were locally introduced by Szabó [9]. A manifold is said to be *Ricci semisymmetric* if $R(X,Y)\cdot S=0$, where S denotes the Ricci tensor. Also, in [11] Yıldız and De studied h-Weyl semisymmetric, ϕ-Weyl semisymmetric, h-projectively semisymmetric and ϕ-projectively semisymmetric non-Sasakian (k,μ)-contact metric manifolds. Recently, Mandal and De studied certain curvature conditions on paracontact (k,μ)-spaces [6].

In [1], the authors have studied a new type of paracontact manifold, so-called paracontact metric (k,μ)-spaces, where that the values of k and μ in (1) remains unchanged under $\mathcal{D}$-homothetic deformation. Namely, unlike in the contact Riemannian status, a paracontact (k,μ)-manifold with $k=-1$ in general is not para-Sasakian. In fact, there are paracontact (k,μ)-manifolds such that $h^2=0$ (which is equal to take $k=-1$) but with $h\neq 0$. Montano and Terlizzi gave the first example of paracontact metric $(-1,2)$-space $(M^{2n+1},\phi,\xi,\eta,g)$ with $h^2=0$ but $h\neq 0$ for 5-dimensional in [2] and then Montano et al. gave the first paracontact metric structures defined on the tangent sphere bundle and constructed an example with arbitrary n in [1]. Later, in [4] for 3-dimensional, the first numerical example was given. Another important difference with the contact Riemannian status, due to the non-positive definiteness of the metric, is that while for contact metric (k,μ)-spaces the constant k can not be greater than 1, paracontact metric (k,μ)-space has no limitation for k and μ. Also, in [12] Yıldız and De studied some curvature conditions paracontact metric (k,μ)-manifolds provided $k\neq 1$.

The projective curvature tensor is a significant tensor from the differential geometric point of view. Let M be a $(2n+1)$-dimensional semi-Riemannian manifold with the metric g. The Ricci operator Q is defined by $g(QX,Y)=S(X,Y)$. For $n\geq 1$, M is locally projectively flat if and only if projective curvature tensor P vanishes which is defined by [8]

$$P(X,Y)Z=R(X,Y)Z-\frac{1}{2n}\{S(Y,Z)X-S(X,Z)Y\} \tag{2}$$

for all $X,Y,Z\in T(M)$.

In fact M is projectively flat if and only if it is of constant curvature [10]. Thus the projective curvature tensor is the measure of the failure of a semi-Riemannian manifold to be of constant curvature.

A paracontact metric (k,μ)-manifold is called to be an *Einstein manifold* if satisfies $S=\lambda_1 g$, and η-*Einstein manifold* if satisfies $S=\lambda_1 g+\lambda_2\eta\otimes\eta$, where λ_1 and λ_2 are constants.

In this paper, we study some curvature properties of a paracontact metric (k,μ)-space. The outline of the paper goes as follows: After introduction, in Section 2, we give basic facts which we will use throughout the paper. Section 3 deals with some basic results of paracontact metric manifolds with characteristic vector field ξ belonging to the (k,μ)-nullity distribution. In section 4, we introduce h-projectively semisymmetric and ϕ-projectively semisymmetric paracontact metric (k,μ)-manifolds provided $k \neq 1$. In the last section, we show that if a paracontact metric (k,μ)-manifold is Ricci pseudo-symmetric, then it is an Einstein manifold provided $k \neq 1$.

2. Preliminaries

For more information about paracontact metric geometry, we may refer to [5], [13] and references therein.

A $(2n+1)$-dimensional manifold M is said to have an *almost paracontact structure* if it admits a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying the following equations:

(i) $\eta(\xi) = 1$, $\phi^2 = I - \eta \otimes \xi$,

(ii) The tensor field ϕ induces an almost paracomplex structure on each fibre of $\mathcal{D} = \ker(\eta)$, i.e., the ±1-eigendistributions, $\mathcal{D}^{\pm} = \mathcal{D}_{\phi}(\pm 1)$ of ϕ have equal dimension n.

From the definition, we have $\phi\xi = 0$, $\eta \circ \phi = 0$ and the endomorphism ϕ has rank $2n$. The Nijenhius torsion tensor field $[\phi,\phi]$ is given by

$$[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

When the tensor field $N_{\phi} = [\phi,\phi] - 2d\eta \otimes \xi$ vanishes identically the almost paracontact manifold is said to be *normal*. If an almost paracontact manifold admits a pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y) \tag{3}$$

for all $X, Y \in \Gamma(TM)$, then we say that (M,ϕ,ξ,η,g) is an *almost paracontact metric manifold*. Such a pseudo-Riemannian metric is necessarily of signature $(n+1,n)$. For an almost paracontact metric manifold, there exists an orthogonal basis $\{X_1,\ldots,X_n,Y_1,\ldots,Y_n,\xi\}$, such that $g(X_i,X_j) = \delta_{ij}$, $g(Y_i,Y_j) = -\delta_{ij}$, $g(X_i,Y_j) = 0$, $g(\xi,X_i) = g(\xi,Y_j) = 0$ and $Y_i = \phi X_i$ for any $i,j \in \{1,\ldots,n\}$, which is called a ϕ-*basis*.

We can now define the *fundamental form* of the almost paracontact metric manifold by $\Phi(X,Y) = g(X,\phi Y)$. If $d\eta(X,Y) = g(X,\phi Y)$, then (M,ϕ,ξ,η,g) is said to be *paracontact metric manifold*. In a paracontact metric manifold one defines a symmetric, trace-free operator $h = \frac{1}{2}\mathcal{L}_{\xi}\phi$, where $\mathcal{L}_{\xi}$, denotes the Lie derivative. It is known [13] that h anti-commutes with ϕ and satisfies $h\xi = 0$, $\operatorname{tr} h = \operatorname{tr} h\phi = 0$ and

$$\nabla\xi = -\phi + \phi h, \tag{4}$$

$$\phi h + h\phi = 0. \tag{5}$$

Also, $h=0$ if and only if ξ is Killing vector field. Then (M,ϕ,ξ,η,g) is said to be a K-*paracontact manifold*. A normal paracontact metric manifold is called a *para-Sasakian manifold*. Also, the para-Sasakian case implies the K-paracontact case and the converse holds only in dimension 3. Moreover, any para-Sasakian manifold satisfies

$$R(X,Y)\xi=\eta(X)Y-\eta(Y)X.$$

3. Paracontact Metric (k,μ)-Manifolds

Let (M,ϕ,ξ,η,g) be a paracontact manifold. The (k,μ)-nullity distribution of a (M,ϕ,ξ,η,g) for the pair (k,μ) is a distribution

$$N(k,\mu):p\to N_p(k,\mu)=\left\{\begin{array}{c} Z\in T_pM \mid R(X,Y)Z=k(g(Y,Z)X-g(X,Z)Y) \\ +\mu(g(Y,Z)hX-g(X,Z)hY) \end{array}\right\} \tag{6}$$

for some real constants k and μ. If the characteristic vector field ξ belongs to the (k,μ)-nullity distribution we have (6). [1] is a complete study of paracontact metric manifolds for which the Reeb vector field of the underlying contact structure satisfies a nullity condition (namely the condition (6) for some real numbers k and μ).

In a $(2n+1)$-dimensional paracontact metric (k,μ)-manifold for $k\neq -1$, the following relations hold [1]:

$$h^2=(k+1)\phi^2 \tag{7}$$

and

$$(\nabla_X\phi)Y=-g(X-hX,Y)\xi+\eta(Y)(X-hX). \tag{8}$$

Lemma 3.1 *[1] Let (M,ϕ,ξ,η,g) be a paracontact metric (k,μ)-manifold of dimension $2n+1$. Then*

$$(\nabla_X h)Y-(\nabla_Y h)X=-(1+k)(2g(X,\phi Y)\xi+\eta(X)\phi Y-\eta(Y)\phi X)$$
$$+(1-\mu)(\eta(X)\phi hY-\eta(Y)\phi hX)$$

for any vector fields X,Y on M.

Lemma 3.2 *[1] In any $(2n+1)$-dimensional paracontact metric (k,μ)-manifold (M,ϕ,ξ,η,g) with $k\neq -1$, the Ricci operator Q is given by*

$$Q=(2(n-1)+\mu)h+(2(1-n)+n\mu)I+(2(n-1)+n(2k-\mu))\eta\otimes\xi. \tag{9}$$

From (9), we have

$$S(X,\xi)=2nk\eta(X), \tag{10}$$

$$Q\xi=2nk\xi. \tag{11}$$

4. Main Results on Paracontact Metric (k,μ)-Manifolds

Definition 4.1 *A semi-Riemannian manifold (M^{2n+1},g), $n>1$, is said to be h-projectively semisymmetric if*

$$(P(X,Y)\cdot h)Z=0$$

holds on M.

However if we consider three-dimensional paracontact metric (k,μ)-manifold, then the manifold is either a paracontact metric $N(k)$-manifold or a para-Sasakian manifold.

Now let M be a h-projectively semisymmetric paracontact metric (k,μ)-manifold with $k\neq -1$. Then above equation is equivalent to

$$P(X,Y)hZ-hP(X,Y)Z=0$$

for $k\neq -1$. Firstly, we get

$$\begin{aligned} R(X,Y)hZ-hR(X,Y)Z \;=\; & \mu(k+1)\{g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi \\ & +\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\} \\ & +k\{g(hY,Z)\eta(X)\xi-g(hX,Z)\eta(Y)\xi \\ & +\eta(X)\eta(Z)hY-\eta(Y)\eta(Z)hX \\ & +g(\phi Y,Z)\phi hX-g(\phi X,Z)\phi hY\} \\ & +(\mu+k)\{g(\phi hX,Z)\phi Y-g(\phi hY,Z)\phi X\} \\ & +2\mu g(\phi X,Y)\phi hZ. \end{aligned} \tag{12}$$

Then we can write

$$\begin{aligned} P(X,Y)hZ-hP(X,Y)Z \;=\; & R(X,Y)hZ-hR(X,Y)Z \\ & -\frac{1}{2n}\{S(Y,hZ)X-S(X,hZ)Y \\ & -S(Y,Z)hX+S(X,Z)hY\}=0. \end{aligned} \tag{13}$$

Using (12) in (13), we get

$$\begin{aligned} & \mu(k+1)\{g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi+\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\} \\ & +k\{g(hY,Z)\eta(X)\xi-g(hX,Z)\eta(Y)\xi+\eta(X)\eta(Z)hY-\eta(Y)\eta(Z)hX \\ & +g(\phi Y,Z)\phi hX-g(\phi X,Z)\phi hY\}+2\mu g(\phi X,Y)\phi hZ \\ & +(\mu+k)\{g(\phi hX,Z)\phi Y-g(\phi hY,Z)\phi X\} \\ & -\frac{1}{2n}\{S(Y,hZ)X-S(X,hZ)Y \\ & \quad -S(Y,Z)hX+S(X,Z)hY\}=0. \end{aligned} \tag{14}$$

Putting $Y = hY$ in (14), we have

$$\begin{aligned}
&\mu(k+1)\{g(hY,Z)\eta(X)\xi + \eta(X)\eta(Z)hY\} \\
&+k\{g(h^2Y,Z)\eta(X)\xi + \eta(X)\eta(Z)h^2Y \\
&+g(\phi hY,Z)\phi hX - g(\phi X,Z)\phi h^2Y\} \\
&+(\mu+k)\{g(\phi hX,Z)\phi hY - g(\phi h^2Y,Z)\phi X\} \\
&+2\mu g(\phi X,Y)\phi h^2 Z \\
&-\frac{1}{2n}\{S(hY,hZ)X - S(X,hZ)hY \qquad (15)\\
&\qquad - S(hY,Z)hX + S(X,Z)h^2Y\} = 0.
\end{aligned}$$

Multiplying both sides of (15) with ξ, we obtain

$$(k+1)\mu g(hY,Z) + kg(Y,Z) - \frac{1}{2n}S(Y,Z) + k\eta(Y)\eta(Z)\eta(X) = 0.$$

Then we have

$$\mu g(hY,Z) + kg(Y,Z) + k\eta(Y)\eta(Z) - \frac{1}{2n}S(Y,Z) = 0.$$

From Lemma 3.2, we can write

$$\begin{aligned}
S(X,Y) &= (2(1-n)+n\mu)g(X,Y) + (2(n+1)+\mu)g(hX,Y) \\
&\quad +(2(n-1)+n(2k-\mu))\eta(X)\eta(Y).
\end{aligned}$$

Thus we have

$$\begin{aligned}
g(hX,Y) &= \frac{1}{2(n+1)+\mu}S(X,Y) - \frac{2(1-n)+n\mu}{2(n+1)+\mu}g(X,Y) \\
&\quad -\frac{2(n-1)+n(2k-\mu)}{2(n+1)+\mu}\eta(X)\eta(Y). \qquad (16)
\end{aligned}$$

Thus from (16), we have

$$\begin{aligned}
&\frac{\mu}{2(n+1)+\mu}S(Y,Z) - \frac{\mu(2(1-n)+n\mu)}{2(n+1)+\mu}g(Y,Z) \\
&-\frac{\mu(2(n-1)+n(2k-\mu))}{2(n+1)+\mu}\eta(Y)\eta(Z) \\
&+kg(Y,Z) + k\eta(Y)\eta(Z) - \frac{1}{2n}S(Y,Z) = 0,
\end{aligned}$$

i.e.,

$$\left(\frac{\mu}{2(n+1)+\mu}-\frac{1}{2n}\right)S(Y,Z)$$
$$-\left(\frac{\mu(2(1-n)+n\mu)}{2(n+1)+\mu}-k\right)g(Y,Z)$$
$$-\left(\frac{\mu(2(n-1)+n(2k-\mu))}{2(n+1)+\mu}-k\right)\eta(Y)\eta(Z)=0,$$

which turns to

$$S(Y,Z)=\frac{\lambda_2}{\lambda_1}g(Y,Z)+\frac{\lambda_3}{\lambda_1}\eta(Y)\eta(Z),$$

where

$$\begin{aligned}\lambda_1 &= \frac{\mu}{(2(n+1)+\mu)}-\frac{1}{2n},\\ \lambda_2 &= \frac{\mu(2(1-n)+n\mu)}{2(n+1)+\mu}-k,\\ \lambda_3 &= \frac{\mu(2(n-1)+n(2k-\mu))}{2(n+1)+\mu}-k.\end{aligned}$$

So the manifold M is an η-Einstein manifold. Hence, we have the following:

Theorem 4.2 *Let (M,ϕ,ξ,η,g) be a $(2n+1)$-dimensional paracontact (k,μ)-manifold with $k\neq -1$. If M is an h-projectively semisymmetric manifold, then M is an η-Einstein manifold provided $\mu\neq 2(1-n)$.*

Definition 4.3 *A semi-Riemannian manifold (M^{2n+1},g), $n>1$, is said to be ϕ-projectively semisymmetric if*

$$(P(X,Y)\cdot\phi)Z=0$$

holds on M.

Let M be a ϕ-projectively semisymmetric paracontact metric (k,μ)-manifold with $k\neq -1$. Then above equation is equivalent to

$$P(X,Y)\phi Z-\phi P(X,Y)Z=0$$

for $k \neq -1$. Firstly, we get

$$\begin{aligned}
R(X,Y)\phi Z - \phi R(X,Y)Z &= g(X,\phi Z)Y - g(Y,\phi Z)X + g(Y,Z)\phi X \\
&-g(X,Z)\phi Y - g(X,\phi Z)hY + g(Y,\phi Z)hX \\
&+g(hY,\phi Z)X - g(hX,\phi Z)Y - g(Y,Z)\phi hX \\
&+g(X,Z)\phi hY - g(hY,Z)\phi X + g(hX,Z)\phi Y \qquad (17) \\
&+\frac{-1-\frac{\mu}{2}}{k+1}\{g(hY,\phi Z)hX - g(hX,\phi Z)hY - g(hY,Z)\phi hX \\
&+g(hX,Z)\phi hY\} - \frac{-k+\frac{\mu}{2}}{k+1}\{g(hX,\phi Z)\phi hY - g(hY,\phi Z)\phi hX \\
&-g(\phi hY,Z)hX + g(\phi hX,Z)hY\} \\
&+(k+1)\{g(\phi X,Z)\eta(Y)\xi - g(\phi Y,Z)\eta(X)\xi \\
&+\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X\} \\
&+(\mu-1)\{g(\phi hX,Z)\eta(Y)\xi - g(\phi hY,Z)\eta(X)\xi \\
&+\eta(X)\eta(Z)\phi hY - \eta(Y)\eta(Z)\phi hX\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
P(X,Y)\phi Z - \phi P(X,Y)Z &= R(X,Y)\phi Z - \phi R(X,Y)Z \\
&-\frac{1}{2n}\{S(Y,\phi Z)X - S(X,\phi Z)Y \\
&-S(Y,Z)\phi X + S(X,Z)\phi Y\} = 0. \qquad (18)
\end{aligned}$$

Using (17), putting $X = \phi X$ and multiplying with W in (18), we obtain

$$\begin{aligned}
&g(\phi X,\phi Z)g(Y,W) - g(Y,\phi Z)g(\phi X,W) - g(Y,Z)g(\phi X,\phi W)\\
&-g(\phi X,Z)g(\phi Y,W) - g(\phi X,\phi Z)g(hY,W) + g(Y,\phi Z)g(h\phi X,W)\\
&+g(hY,\phi Z)g(\phi X,W) - g(h\phi X,\phi Z)g(Y,W) - g(Y,Z)g(\phi h\phi X,W)\\
&+g(\phi X,Z)g(\phi hY,W) + g(hY,Z)g(\phi X,\phi W) + g(h\phi X,Z)g(\phi Y,W)\\
&+\frac{-1-\frac{\mu}{2}}{k+1}\{g(hY,\phi Z)g(h\phi X,W) - g(h\phi X,\phi Z)g(hY,W)\\
&-g(hY,Z)g(\phi h\phi X,W) + g(h\phi X,Z)g(\phi hY,W)\}\\
&-\frac{-k+\frac{\mu}{2}}{k+1}\{g(h\phi X,\phi Z)g(\phi hY,W) - g(hY,\phi Z)g(\phi h\phi X,W)\\
&-g(\phi hY,Z)g(h\phi X,W) + g(\phi h\phi X,Z)g(hY,W)\}\\
&-(k+1)\{g(\phi X,\phi Z)\eta(Y)\eta(W) - \eta(Y)\eta(Z)g(\phi X,\phi W)\} \qquad (19)\\
&-(\mu-1)\{g(h\phi X,\phi Z)\eta(Y)\eta(W) - \eta(Y)\eta(Z)g(\phi h\phi X,W)\}\\
&-\frac{1}{2n}\{S(Y,\phi Z)g(\phi X,W) - S(\phi X,\phi Z)g(Y,W)\\
&-S(Y,Z)g(\phi^2 X,W) + S(\phi X,Z)g(\phi Y,W)\} = 0.
\end{aligned}$$

Putting $Y = W = \xi$ in (19), we get

$$S(\phi X,\phi Z) + 2n\mu g(\phi hX,\phi Z) + k\eta(X)\eta(Z) = 0. \qquad (20)$$

Using (3) in (20), we have

$$S(X,Z) - (2n-1)k\eta(X)\eta(Z) - [2n\mu + 4n - 4 + 2\mu]g(hX,Z) = 0,$$

i.e.,

$$\begin{aligned}
&S(X,Z) - (2n-1)k\eta(X)\eta(Z)\\
&-[2n\mu + 4n - 4 + 2\mu]\{\frac{1}{2(n+1)+\mu}S(X,Z)\\
&-\frac{2(1-n)+n\mu}{2(n+1)+\mu}g(X,Z) - \frac{2(n-1)+n(2k-\mu)}{2(n+1)+\mu}\eta(X)\eta(Z)\} = 0.
\end{aligned}$$

Hence

$$\begin{aligned}
&[1-\frac{2n\mu+4n-4+2\mu}{2(n+1)+\mu}]S(X,Z)\\
=\;&\frac{2(1-n)+n\mu}{2(n+1)+\mu}g(X,Z) + [(2n-1)k + \frac{2(n-1)+n(2k-\mu)}{2(n+1)+\mu}]\eta(X)\eta(Z).
\end{aligned}$$

Thus we have

$$S(X,Z) = \frac{\lambda_2'}{\lambda_1'} g(X,Z) + \frac{\lambda_3'}{\lambda_1'} \eta(X)\eta(Z),$$

where

$$\begin{aligned} \lambda_1' &= 1 - \frac{2n\mu + 4n - 4 + 2\mu}{2(n+1) + \mu}, \\ \lambda_2' &= \frac{2(1-n) + n\mu}{2(n+1) + \mu}, \\ \lambda_3' &= (2n-1)k + \frac{2(n-1) + n(2k-\mu)}{2(n+1) + \mu}. \end{aligned}$$

So the manifold M is an η-Einstein manifold. Hence, we have the following:

Theorem 4.4 *Let (M,ϕ,ξ,η,g) be a $(2n+1)$-dimensional paracontact (k,μ)-manifold with $k \neq -1$. If M is a ϕ-projectively semisymmetric manifold, then M is an η-Einstein manifold provided $\mu \neq 2(1-n)$.*

5. Ricci Pseudo-Symmetric Paracontact Metric (k,μ)-Manifolds

Definition 5.1 *A semi-Riemannian manifold (M^{2n+1},g), $n > 1$, is said to be Ricci pseudo-symmetric if*

$$(R(X,Y)\cdot S)(Z,W) = fQ(g,S)(X,Y;Z,W),$$

where

$$\begin{aligned} (R(X,Y)\cdot S)(Z,W) &= R(X,Y)S(Z,W) \\ &\quad -S(R(X,Y)Z,W) - S(Z,R(X,Y)W) \end{aligned}$$

and

$$fQ(g,S)(X,Y;Z,W) = f\{S((X\wedge Y)Z,W) + S(Z,(X\wedge Y)W)\}$$

hold on M, f is some function and $(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y$, for all $X,Y,Z,W \in \chi(M)$.

Let M be a Ricci pseudo-symmetric paracontact metric (k,μ)-manifold with $k \neq -1$. Then we can write

$$\begin{aligned} & -S(R(X,Y)Z,W) - S(Z,R(X,Y)W) \\ = \; & f\{g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ & +g(Y,W)S(X,Z) - g(X,W)S(Y,Z)\}. \end{aligned} \tag{21}$$

Putting $X = \xi$ in (21), we get

$$\begin{aligned} & S(R(\xi,Y)Z,W) + S(Z,R(\xi,Y)W) \\ = \; & f\{\eta(Z)S(Y,W) - g(Y,Z)S(\xi,W) \\ & +\eta(W)S(Y,Z) - g(Y,W)S(\xi,Z)\}. \end{aligned} \tag{22}$$

On the other hand from (6), we obtain

$$R(\xi,Y)Z = k\{g(Y,Z)\xi - \eta(Z)Y\} + \mu\{g(hY,Z)\xi - \eta(Z)hY\}.$$

Using this fact in (22), we get

$$\begin{aligned} & k\{g(Y,Z)S(\xi,W) - \eta(Z)S(Y,W) \\ & +g(Y,W)S(\xi,Z) - \eta(W)S(Y,Z)\} \\ & +\mu\{g(hY,Z)S(\xi,W) - \eta(Z)S(hY,W) \\ & +g(hY,W)S(\xi,Z) - \eta(W)\alpha(SY,Z)\} \\ = \; & f\{\eta(Z)S(Y,W) - g(Y,Z)S(\xi,W) \\ & +\eta(W)S(Y,Z) - g(Y,W)S(\xi,Z)\}, \end{aligned} \tag{23}$$

which turns to

$$\begin{aligned} & (k+f)\{g(Y,Z)S(\xi,W) - \eta(Z)S(Y,W) \\ & +g(Y,W)S(\xi,Z) - \eta(W)S(Y,Z)\} \\ & +\mu\{g(hY,Z)S(\xi,W) - \eta(Z)S(hY,W) \\ & +g(hY,W)S(\xi,Z) - \eta(W)S(hY,Z)\} = 0. \end{aligned} \tag{24}$$

Now we have three cases:

(i) $k + f \neq 0$, $\mu = 0$,

(ii) $k + f = 0$, $\mu \neq 0$,

(iii) $k + f \neq 0$, $\mu \neq 0$.

For proof of (i), putting $W = \xi$ in (24), we get

$$g(Y,Z)S(\xi,\xi) - S(Y,Z) = 0, \tag{25}$$

which turns to

$$S(Y,Z) = 2nkg(Y,Z).$$

For proof of (ii), putting $W = \xi$ in (24),

$$g(hY,Z)S(\xi,\xi) - S(hY,Z) = 0. \tag{26}$$

Taking $Y = hY$ in (26), we obtain

$$(k+1)\{g(Y,Z)S(\xi,\xi) - S(Y,Z)\} = 0.$$

Since $k \neq -1$, then we get

$$S(Y,Z) = 2nkg(Y,Z).$$

For proof of (iii), putting $W = \xi$ in (24), we have

$$\begin{aligned}&(k+f)\{g(Y,Z)S(\xi,\xi) - \eta(Z)S(Y,\xi) + g(Y,\xi)S(\xi,Z) - S(Y,Z)\} \\ &+\mu\{g(hY,Z)S(\xi,\xi) - S(hY,Z)\} = 0.\end{aligned} \tag{27}$$

Taking $Y = hY$ in (27), we obtain

$$(k+f)\{g(hY,Z)S(\xi,\xi) - S(hY,Z)\} - \mu\{g(h^2Y,Z)S(\xi,\xi) - S(h^2Y,Z)\} = 0,$$

which turns to

$$\begin{aligned}&(k+f)\{g(hY,Z)S(\xi,\xi) - S(hY,Z)\} \\ &-\mu(k+1)\{g(Y,Z)S(\xi,\xi) - \eta(Y)\eta(Z)S(\xi,\xi) \\ &\quad - S(Y,Z) + \eta(Y)S(Z,\xi)\} = 0.\end{aligned} \tag{28}$$

Then using (26) in (28), we have

$$(k+1)\mu\{g(Y,Z)S(\xi,\xi) - S(Y,Z)\} = 0.$$

Since $k \neq -1$, then we get

$$S(Y,Z) - g(Y,Z)S(\xi,\xi) = 0,$$

which turns to

$$S(Y,Z) = 2nkg(Y,Z).$$

Considering above facts, we state the following:

Theorem 5.2 *Let (M,ϕ,ξ,η,g) be a $(2n+1)$-dimensional paracontact (k,μ)-manifold with $k \neq -1$. If M is a Ricci pseudo-symmetric manifold, then the manifold is an Einstein manifold.*

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