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On the Numerical Solution of a Semilinear Sobolev Equation Subject to Nonlocal Dirichlet Boundary Condition

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Abstract: A semilinear Sobolev equation $\partial_t(u - \Delta u) - \Delta u = f(\nabla u)$ with a Dirichlet-type integral boundary condition is investigated in this contribution. Using the Rothe method which is based on a semi-discretization of the problem under consideration with respect to the time variable, we prove the existence and uniqueness of a weak solution. Moreover, a suitable approach for the numerical solution based on Legendre spectral-method is presented.

Keywords: Nonlocal boundary condition, Rothe method, Sobolev equation, Spectral method.

1 Introduction

Various phenomena in diverse areas of engineering, physics, and biological systems lead to nonlocal boundary value problem for partial differential equations. This kind of problems have been investigated extensively by many researchers (see e.g [3]-[4]).

In this contribution we aim to study the solvability of the following problem:

$$\begin{aligned} &\partial_t u - \partial_t \Delta u - \Delta u = f(\nabla u) \quad , (x,t) \in Q_T := \Omega \times [0,T], \\ &u(x,t) = \int_\Omega K(x,z)u(z,t)\mathrm{d}z \quad , (x,t) \in \Gamma \times [0,T], \\ &u(x,0) = u_0(x) \quad , x \in \Omega, \end{aligned} \tag{1}$$

where Ω with is a bounded sub-domain of $\mathbb{R}^d, d \geq 1$ with smooth and regular boundary Γ.

As mentioned previously, partial differential equations have been extensively studied in the literature. The main reason for such widespread interest in this class of problems is that some problems in physics, chemistry and many other fields of sciences can be modelled using nonlocal problems [2]-[3]-[4]-[5].

Plenty of papers have been developed to examine the solvability of initial boundary value problems of PDEs with nonlocal Dirichlet boundary condition. While much literature basically focuses on parabolic and hyperbolic equations [1]-[9]-[10]-[11], there are very few works concerning pseudo-parabolic equations, in particular Sobolev-type equations, with nonlocal boundary conditions. Motivated by the work of A. Guezane-Lakoud and D. Belakroum [7] , in which, an integrodifferential Sobolev-type equation subject to purely integral conditions is considered. Following the same approach as in [12], we study the problem (1) in two aspects, first, the existence and uniqueness of solution is proved using Rothe method [8], and second, the problem under consideration is studied form numerical point of view.

The outline of this paper is as follows: In the next section, we transform the problem (1) to its weak formulation and give some conditions on the kernel K to ensures the uniqueness of the solution, in the same section, we discretize the problem in time direction using an explicit schema and derive some a priori estimates, which helps us to establish the convergence of the method and the existence of a unique solution. In the third section, Legendre pseudo-spectral method is employed for the space discretization that leads to fully-discretization. Finally, numerical tests are presented in the last section to demonstrate the effectiveness and accuracy of the proposed approach.

2 Solvability of the problem

In this section we establish the existence and uniqueness of the weak solution to problem; first we reduce the nonlocal problem (1) into an appropriate weak problem. For this purpose, multiplying both sides of equation (1) by a test function $\phi \in H_0^1(\Omega)$ and integrate over Ω, then applying Green formula leads to the following weak problem, Find $u \in C(0,T;L_2(\Omega)) \cap L_\infty(0,T;H^1(\Omega))$ such that:

$$
\begin{aligned}
&(\partial_t u, \phi) + (\partial_t \nabla u, \nabla \phi) + (\nabla u, \nabla \phi) = (f(\nabla u), \phi) \quad \forall \phi \in H_0^1(\Omega), \\
&u(x,t) = \mathcal{K}_u(x,t) := \int_\Omega K(x,z) u(z,t) \mathrm{d}z \text{ on } \partial\Omega \times [0,T]\,.
\end{aligned}
\tag{2}
$$

Now, we derive two auxiliary inequalities, which we will needed later. Their derivation is based on the Cauchy inequality.

$$
\begin{aligned}
|(v(t), \mathcal{K}_u(t))| &\leq \left| \int_\Omega v(x,t) \int_\Omega K(x,z) u(z,t) \mathrm{d}z \right| \\
&\leq \int_\Omega |v(x,t)| \left(\int_\Omega |K(x,z)|^2 \mathrm{d}z \right)^{\frac{1}{2}} \mathrm{d}x \left(\int_\Omega |u(y,t)|^2 \mathrm{d}y \right)^{\frac{1}{2}} \\
&\leq \alpha_K \|v(t)\|.\|u(t)\|
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
|(\nabla v(t), \nabla \mathcal{K}_u(t))| &\leq \left| \int_\Omega \nabla v(x,t) \int_\Omega \nabla_x K(x,z) u(z,t) \mathrm{d}z \right| \\
&\leq \int_\Omega \nabla |v(x,t)| \left(\int_\Omega \nabla_x |K(x,z)|^2 \mathrm{d}z \right)^{\frac{1}{2}} \mathrm{d}x \left(\int_\Omega |u(y,t)|^2 \mathrm{d}y \right)^{\frac{1}{2}} \\
&\leq \beta_K \|\nabla v(t)\|.\|u(t)\|
\end{aligned}
\tag{4}
$$

with $\alpha_K = \left(\int_\Omega \int_\Omega |K(x,z)|^2 \mathrm{d}z \mathrm{d}x \right)^{\frac{1}{2}}$ and $\beta_K = \left(\int_\Omega \int_\Omega \nabla_x |K(x,z)|^2 \mathrm{d}z \mathrm{d}x \right)^{\frac{1}{2}}$.

Throughout this paper we denote by C, C_ε and ε generic positive constants where $C_\varepsilon = C(\frac{1}{\varepsilon})$ and ε is sufficiently small.

2.1 Uniqueness

We establish the uniqueness of a solution to problem (2) by stating the following theorem.

Theorem 2.1. *Let f be Lipschitz continuous function, and the kernel K belongs to $L^2(\Omega \times \Omega)$, moreover, we assume that: $\alpha_K < \frac{1}{2}$ and $\beta_K < C$. Then the variational problem* (2) *admits at most one solution.*

Proof: Let u and v be two solutions of (1), we set $w = u - v$. By subtracting the corresponding variational formulations for both solutions from each other and put $\phi = \partial_t(z - \mathcal{K}_z)$ for any $t \in [0,T]$, one can obtain,

$$
\|\partial_t z(t)\|^2 + \|\partial_t \nabla z(t)\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla z(t)\|^2 = (f(\nabla u - \nabla v), \partial_t z - \partial_t \mathcal{K}_z) + (\partial_t z, \partial_t \mathcal{K}_z) + (\partial_t \nabla z, \partial_t \nabla \mathcal{K}_z) + (\nabla z, \partial_t \nabla \mathcal{K}_z) \tag{5}
$$

Using Cauchy and ε-Young inequalities and basic estimates (3)-(4), we can bound the right hand-side of the above identity as the following,

$$
|R.H.S| \leq (\frac{1}{2} + \alpha_K + \varepsilon)\|\partial_t z(t)\|^2 + \frac{\alpha_K^2}{2}\|\partial_t \nabla z(t)\|^2 + C_\varepsilon \|\nabla z(t)\|^2. \tag{6}
$$

Putting things together and fixing ε sufficiently, we obtain,

$$
\|\partial_t z(t)\|^2 + \|\partial_t \nabla z(t)\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla z(t)\|^2 \leq C\|\nabla z(t)\|^2. \tag{7}
$$

Integrating (7) over $[0,t]$ and applying Gronwall inequality gives,

$$
\int_0^t \|\partial_t z(s)\|^2 \mathrm{d}s + \int_0^t \|\partial_t \nabla z(s)\|^2 \mathrm{d}s + \|\nabla z(t)\|^2 \leq 0. \tag{8}
$$

This yields that $u(x,t) = v(x,t) = z(x,t) = 0$ almost everywhere in $\Omega \times (0,T)$, which concludes the proof. □

2.2 Existence

To prove the existence of a solution to weak problem (2) we will use the Rothe method of time discretization. To this end, we divide the time interval $[0,T]$ into $n \in \mathbb{N}$ equidistant subintervals $[t_i, t_{i-1}]$ with $ti = i\tau$ where $\tau = \frac{T}{n}$. We introduce the following notation,

$$
v_i = v(t_i), \qquad \delta v_i = \frac{v_i - v_{i-1}}{\tau}.
$$

We suggest the following time-discrete variational formulation,

$$
\begin{aligned}
&(\delta u_i, \phi) + (\delta \nabla u_i, \nabla \phi) + (\nabla u_i, \nabla \phi) = (f(\nabla u_i), \phi) \quad , \forall \phi \in H_0^1(\Omega), \\
&u_i = \mathcal{K}_{u_{i-1}} \quad \text{on} \quad \partial\Omega.
\end{aligned}
\tag{9}
$$

Therefore, the weak problem (2) is approximated by a sequence of linear elliptic boundary value problem that have to be solved. The well-posedness of (9) is addressed in the following lemma.

Lemma 2.1. *Let the assumptions of Theorem* (2.1) *be satisfied and* $u_0 \in H^2(\Omega)$*. Then for any* $i = 1, \cdots, n$*, the variational problem* (9) *admits unique solution* $u_i \in H^1(\Omega)$*.*

Proof: We set $v_i = u_i - \mathcal{K}_{u_{i-1}}$. Then the variational problem (9) can be read as: Find $v_i \in H_0^1(\Omega)$ such that,

$$
\begin{aligned}
&\mathcal{A}(v_i, \phi) = \mathcal{F}(\phi) \quad, \forall \phi \in H_0^1(\Omega), \\
&\mathcal{A}(v_i, \phi) = (v_i, \phi) + (\delta \nabla v_i, \nabla \phi) + (\nabla v_i, \nabla \phi), \\
&\mathcal{F}(\phi) = (f(\nabla u_{i-1}), \phi) + (\delta \mathcal{K}_{u_{i-1}}, \phi) + (\delta \nabla \mathcal{K}_{u_{i-1}}, \phi) + (\nabla \mathcal{K}_{u_{i-1}}, \phi).
\end{aligned} \tag{10}
$$

Obviously, $\mathcal{A}(\cdot, \cdot)$ is a continuous coercive bilinear functional on $H_0^1(\Omega) \times H_0^1(\Omega)$ and $\mathcal{F}(\cdot)$ is a bounded linear functional on $H_0^1(\Omega)$. By the aid of Lax–Milgram lemma, we obtain the existence of a unique solution v_i for (10), therefore there exists a unique solution u_i to (9). □

Next, we state some stability results of u_i.

Lemma 2.2. *Let the assumptions of Lemma* (2.1) *be fulfilled. Then there exists* $C > 0$ *such that,*

$$
\tau \sum_{i=1}^{n} \|\delta u_i\|^2 + \tau \sum_{i=1}^{n} \|\delta \nabla u_i\|^2 + \max_{0 \le j \le n} \|\nabla u_i\|^2 + \sum_{i=1}^{n} \|\nabla u_i - \nabla u_{i-1}\|^2 \le C. \tag{11}
$$

Proof: Put $\phi = \tau(\delta u_i - \delta \mathcal{K}_{u_{i-1}})$ into (2) and sum it up for $i = 1, \cdots, j$. We obtain,

$$
\tau \sum_{i=1}^{j} \|\delta u_i\|^2 + \tau \sum_{i=1}^{j} \|\delta \nabla u_i\|^2 + \frac{1}{2}\|\nabla u_j\|^2 + \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2 - \frac{1}{2}\|\nabla u_0\|^2 = \sum_{k=1}^{4} I_k, \tag{12}
$$

where

$$
\begin{aligned}
I_1 &= \tau \sum_{i=1}^{j} (f(\nabla u_{i-1}), \delta u_i - \nabla \mathcal{K}_{u_{i-1}}), \\
I_2 &= \tau \sum_{i=1}^{j} (\delta u_i, \delta \mathcal{K}_{u_{i-1}}), \\
I_3 &= \tau \sum_{i=1}^{j} (\delta \nabla u_i, \delta \nabla \mathcal{K}_{u_{i-1}}), \\
I_4 &= \tau \sum_{i=1}^{j} (\nabla u_i, \delta \nabla \mathcal{K}_{u_{i-1}}).
\end{aligned} \tag{13}
$$

By the use of Cauchy and ε-Young inequalities besides basic estimates (3) and (4), we estimate each terms I_i separately,

$$
\begin{aligned}
|I_1| &\le C_\varepsilon \sum_{i=1}^{j} \tau \|\nabla u_{i-1}\|^2 + \varepsilon \sum_{i=1}^{j} \tau \|\delta u_i\|^2 + \varepsilon \sum_{i=1}^{j} \tau \|\delta u_{i-1}\|^2, \\
|I_2| &\le \frac{1}{2} \sum_{i=1}^{j} \tau \|\delta u_i\|^2 + \frac{\alpha_K^2}{2} \sum_{i=1}^{j} \tau \|\delta u_{i-1}\|^2, \\
|I_3| &\le \frac{1}{2} \sum_{i=1}^{j} \tau \|\delta \nabla u_i\|^2 + \frac{\beta_K^2}{2} \sum_{i=1}^{j} \tau \|\delta u_{i-1}\|^2, \\
|I_4| &\le C_\varepsilon \sum_{i=1}^{j} \tau \|\nabla u_i\|^2 + \varepsilon \sum_{i=1}^{j} \tau \|\delta u_{i-1}\|^2.
\end{aligned} \tag{14}
$$

Combining (14) with (12), we can obtain:

$$
\begin{aligned}
&\tau \sum_{i=1}^{j} \|\delta u_i\|^2 + \tau \sum_{i=1}^{j} \|\delta \nabla u_i\|^2 + \frac{1}{2}\|\nabla u_j\|^2 + \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2 \le \\
&\qquad C_\varepsilon \sum_{i=1}^{j} \tau \|\nabla u_i\|^2 + \frac{1}{2} \sum_{i=1}^{j} \tau \|\delta u_i\|^2 + (\frac{\alpha_K^2 + \beta_K^2}{2} + \varepsilon) \sum_{i=1}^{j} \tau \|\delta u_{i-1}\|^2 + \frac{1}{2} \sum_{i=1}^{j} \tau \|\delta \nabla u_i\|^2.
\end{aligned} \tag{15}
$$

Fixing $\varepsilon < \frac{1}{2} - \frac{\alpha_K^2 + \beta_K^2}{2}$ and applying Gronwall's lemma, we conclude the proof. □

Now, let us introduce the following piecewise linear in time functions $u_n(t)$ and $\bar{u}_n(t)$,

$$u_n = t \mapsto \begin{cases} u_0 & , t = 0 \\ u_{i-1} + (t - t_{i-1})\delta u_i & , t \in (t_{i-1}, t_i], 1 \leq i \leq n \end{cases}$$

$$\bar{u}_n = t \mapsto \begin{cases} u_0 & , t = 0 \\ u_i & , t \in (t_{i-1}, t_i], 1 \leq i \leq n \end{cases}$$

Using this notation, we can rewrite (2) as follows:

$$\begin{aligned} &(\partial_t u_n(t), \phi) + (\partial_t \nabla u_n(t), \nabla\phi) + (\nabla\bar{u}_n(t), \nabla\phi) = (f(\nabla\bar{u}_n(t-\tau)), \phi) \quad , \forall\phi \in H_0^1(\Omega) \\ &\bar{u}_n = \mathcal{K}_{\bar{u}(t-\tau)} \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{16}$$

Lemma 2.3. *Let the assumptions of Lemma* (2.1) *be fulfilled. There exists a subsequence of* (u_n)*, which is a Cauchy sequence in* $L_2(0, T; H^1(\Omega))$.

Proof: Lemma (2.2) together with [8, Lemma 1.3.13] imply the existence of a subsequence of (u_n) (denoted, by the same symbol again) which is a Cauchy sequence in $C(0, T; L^2(\Omega))$. Moreover, $\bar{u}_n(t) \rightharpoonup u(t)$ in $H^1(\Omega)$ for all $t \in [0, T]$ and $\partial u_n \rightharpoonup \partial_t u$ in $L^2(0, T; L^2(\Omega))$ Take the difference of (2) for $n = p$ and (2) for $n = q$, then put $\phi = \bar{u}_p - \bar{u}_q - \mathcal{K}_{\bar{u}_p(t-\tau)} + \mathcal{K}_{\bar{u}_q(t-\tau)}$, and integrate over $(0, T)$. One obtains:

$$\begin{aligned} &\int_0^T \|\nabla u_p(t) - \nabla u_q(t)\|^2 \mathrm{d}t = J_1 + J_2 + J_3 + J_4, \\ &J_1 = \int_0^T \left(\|\nabla u_p(t) - \nabla u_q(t)\|^2 - \|\nabla\bar{u}_p(t) - \nabla\bar{u}_q(t)\|^2\right) \mathrm{d}t, \\ &J_2 = \int_0^T (\partial_t u_p(t) - \partial_t u_q(t), \phi)\mathrm{d}t, \\ &J_3 = \int_0^T (\partial_t \nabla u_p(t) - \partial_t \nabla u_q(t), \nabla\phi)\mathrm{d}t, \\ &J_4 = \int_0^T (\nabla\bar{u}_p(t) - \nabla\bar{u}_q(t), \nabla\mathcal{K}_{\bar{u}_p(t-\tau)} - \nabla\mathcal{K}_{\bar{u}_q(t-\tau)}), \\ &J_5 = \int_0^T (f(\nabla\bar{u}_p(t) - \nabla\bar{u}_q(t)), \phi). \end{aligned} \tag{17}$$

Now, we estimate the terms $J_i, i = 1, \cdots, 5$, seperatly . The procedure is standard. We employ the Cauchy inequality, (3), (4) and Lemma (2.2). We arrive at:

$$\begin{aligned} &|J_1| \leq C_1(\tau_p + \tau_q), \\ &|J_2| \leq C_2\sqrt{\tau_p^2 + \tau_q^2 + \int_0^T \|u_p(t) - u_q(t)\|^2 \mathrm{d}t}, \\ &|J_3| \leq C_3\sqrt{\tau_p^2 + \tau_q^2 + \int_0^T \|u_p(t) - u_q(t)\|^2 \mathrm{d}t}, \\ &|J_4| \leq C_4(\tau_p + \tau_q + \int_0^T \|u_p(t) - u_q(t)\|\mathrm{d}t), \\ &|J_5| \leq C_5\sqrt{\tau_p^2 + \tau_q^2 + \int_0^T \|u_p(t) - u_q(t)\|^2 \mathrm{d}t}. \end{aligned} \tag{18}$$

Since (u_n) is a Cauchy sequence in $C(0, T; L^2(\Omega))$ so $\|u_p(t) - u_q(t)\| \to 0$ as $p, q \to \infty$, which implies that (∇u_n) is a Cauchy sequence in $C(0, T; L^2(\Omega))$. □

Now, we are in a position to prove the solvability of (1)

Theorem 2.2. *Let the assumptions of Lemma* (2.1) *be fulfilled. Then the direct problem* (1) *admits a unique solution* $u \in C(0, T; L_2(\Omega) \cap L_\infty((0, T), H^1(\Omega))$ *obeying* $\partial_t u \in L_2(0, T; L_2(\Omega))$.

Proof: Integrate (16) over $(0,t)$ to get:

$$(u_n(t)-u_0,\phi)+(\nabla u_n(t)-\nabla u_0,\nabla\phi)\int_0^t(\nabla\nabla\bar{u}_n(s)-\nabla u_0,\nabla\phi)\mathrm{d}s=\int_0^t(f(\nabla\bar{u}(s-\tau)),\phi)\mathrm{d}s. \tag{19}$$

Use:

$$\int_0^t(f(\nabla\bar{u}(s-\tau)),\phi)\mathrm{d}s=\int_0^t(f(\nabla\bar{u}(s-\tau))-f(\nabla\bar{u}(s)),\phi)\mathrm{d}s+\int_0^t(f(\nabla\bar{u}(s))-f(\nabla u(s)),\phi)\mathrm{d}s+\int_0^t(f(\nabla u(s-\tau)),\phi)\mathrm{d}s.$$

Making $n\to+\infty$ in (19) and differentiating with respect to t implies that u is a weak solution to (1). □

Following the same arguments as in [12] to check the behaviour of $u_n(t)$ on the boundary. We may have:

$$\lim_{\tau\to 0}\max_{t\in[0,T]}\|u_n(t)-\mathcal{K}_{u(t)}\|=0.$$

Having this, we see that the u satisfies the boundary condition.

3 Full-discretization

This section is devoted to the space discretization of problem (2), which completes the full-discretization of the nonlocal problem (1). For this purpose, we adopt Legendre pseudo-spectral method. Based on the weak formulation (2) we define the conventional Legendre-Galerkin semi-discrete approximation reads as:
Find $u_i^N\in\mathbb{P}_N(\Lambda)$ such that, for any $\forall\phi\in\mathbb{P}_N^0(\Lambda)$

$$\begin{aligned}
&(\delta u_i^N,\phi)+(\partial_x\delta u_i^N,\partial_x\phi)+(\partial_x u_i^N,\partial_x\phi)=(f(u_{i-1}^N),\phi),\\
&u_i^N(-1)=\int_{-1}^1 u_{i-1}^N(x)K_1(x)\mathrm{d}x,\\
&u_i^N(1)=\int_{-1}^1 u_{i-1}^N(x)K_2(x)\mathrm{d}x,\\
&u_0^N=\mathcal{I}_N^C u_0.
\end{aligned} \tag{20}$$

Where $\mathcal{I}_N^C$ stands for the interpolation operator at the Chebyshev-Gauss-Lobatto points $\xi_i=\cos\left(\frac{i\pi}{N}\right)$. Let us denote $L_k(x)$ the k-th degree Legendre polynomial. The set of Legendre polynomials $\{L_k\}_{k=0}^\infty$ forms an orthogonal basis for the space $L^2(\Lambda)$. Let N be a positive integer, we define

$$\begin{aligned}
&\phi_k(x)=L_k(x)-L_{k+2}(x),\quad 0\le k\le N-2,\\
&\phi_{N-1}(x)=\frac{1}{2}\left(L_0(x)+L_1(x)\right),\\
&\phi_N(x)=\frac{1}{2}\left(L_0(x)-L_1(x).\right)
\end{aligned} \tag{21}$$

Obviously, the set $\{\phi_k\}_{k=0}^N$ consists of $N+1$ linearly independent elements, therefore form a basis function for $\mathbb{P}_N(\Lambda)$. We set

$$u_i^N(x)=\sum_{k=0}^N\tilde{u}_k^i\phi_k(x). \tag{22}$$

Inserting (22) into the variational equation in (2) and taking $v=\phi_j, j=0,\cdots,N-2$ yields

$$\sum_{k=1}^N(m_{jk}+p_{jk}+\tau p_{jk})\hat{u}_k^i=\sum_{k=1}^N(m_{jk}+p_{jk})\hat{u}_k^{i-1}+F_j^{i-1}\quad ,i\ge 1, \tag{23}$$

where

$$m_{jk}=\int_{-1}^1\varphi_j(x)\varphi_k(x)\mathrm{d}x\quad ,p_{jk}=\int_{-1}^1\varphi_j'(x)\varphi_k'(x)\mathrm{d}x\quad ,F_j^i=\int_{-1}^1 f(u_{i-1}^N(x))\varphi_j(x)\mathrm{d}x.$$

The boundary conditions give us two supplementary algebraic equations

$$\left.\begin{aligned}
&\sum_{k=1}^N\hat{u}_k^i\varphi_k(-1)=\int_{-1}^1 u_{i-1}^N(x)K_1(x)\mathrm{d}x,\\
&\sum_{k=1}^N\hat{u}_k^i\varphi_k(1)=\int_{-1}^1 u_{i-1}^N(x)K_2(x)\mathrm{d}x.
\end{aligned}\right.\quad ,i\ge 1 \tag{24}$$

The coefficients m_{jk} and p_{jk} can be determined by using the orthogonal properties of Legendre polynomials as in [6, 13], and the term F_j^i can be calculated approximately. The resulting system (23)-(24) can be easily solved using a direct or either iterative method to solve it.

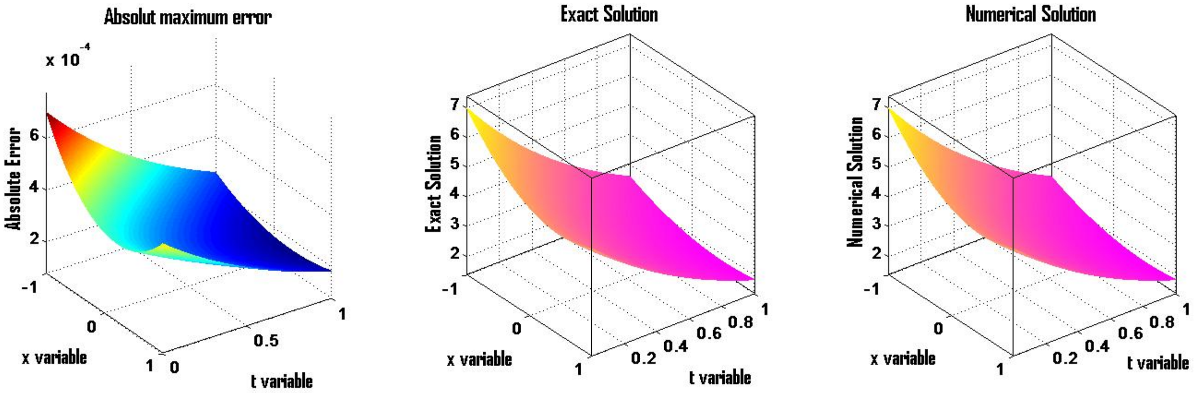

Fig. 1: Profiles of exact solution, numerical solution and absolute error of problem test of Example (4.1) for the parameters $N = 8$ and $\Delta t = 10^{-3}$

4 Numerical tests

In this section, the method presented in previous sections is applied to solve two problems of the form (1).

Example 4.1. *Let us first consider problem*

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial t \partial x^2} - \frac{\partial^2 u}{\partial x^2} = -(x^2 - x + 5)e^{-t},$$
$$u(-1,t) = \int_{-1}^{1} \sin(\pi x) u(x,t) \mathrm{d}x,$$
$$u(1,t) = \int_{-1}^{1} \cos(\pi x) u(x,t) \mathrm{d}x,$$

with the exact solution

$$u(x,t) = \left(x^2 - x + 5\right) e^{-t}.$$

Without loss of generality, we consider that $T = 2$ in our numerical experiments. The comparisons between the exact and the numerical solution are illustrated in Figure (1). From the computational results illustrated in Figure (1), one can observe that the numerical and exact solutions are in good agreement, that confirms the accuracy and effectiveness of the proposed method.

Moreover, according to results listed in Table (1) the rate of the temporal convergence is almost $\mathcal{O}(\tau)$, which seems reasonable since the employed method of discretization in time is of first-order. From the computational results illustrated in Figure 1, one can observe that the numerical and exact solutions are in good agreement. Moreover, according to results listed in Table 1 the rate of the temporal convergence is almost $\mathcal{O}(\tau)$, which seems reasonable since the employed method of discretization in time is of first-order.

Table 1 Temporal convergence rates at $t = 1$ and $t = T$ for Example 4.1

	t = 1		t = T	
Δt	L^2-error	order	L^2-error	order
10^{-2}	2.6148e-002	-	1.0325e-002	-
10^{-3}	2.6046e-003	1.0000	1.0287e-003	1.0016
10^{-4}	2.6036e-004	1.0002	1.0284e-004	1.0001
10^{-5}	2.6035e-005	1.0017	1.0283e-005	1.0000

Example 4.2. *Now, let us consider*

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial t \partial x^2} - \frac{\partial^2 u}{\partial x^2} = f\left(\frac{\partial u}{\partial x}\right) + q(x,t),$$
$$u(-1,t) = \int_{-1}^{1} \sin(\pi x) u(x,t) \mathrm{d}x,$$
$$u(1,t) = \int_{-1}^{1} \cos(\pi x) u(x,t) \mathrm{d}x,$$

where: $f(v) = v(v-1)$ *and* $q(x,t) = e^{2x}(1+\cos(2\pi t))$. *The exact solution is given by,*

$$u^*(x,t) = e^x \cos(\pi t).$$

As in the previous example, the proposed method is applied for the problem above and the numerical results are showed in Figure (2) *for* $N = 16$ *and* $\tau = 10^{-4}$. *Tables* (2) *and* (3) *represent some comparisons between the exact and numerical solutions. As a conclusion, the computational results demonstrate that the numerical solution accurately approaches the exact solution.*

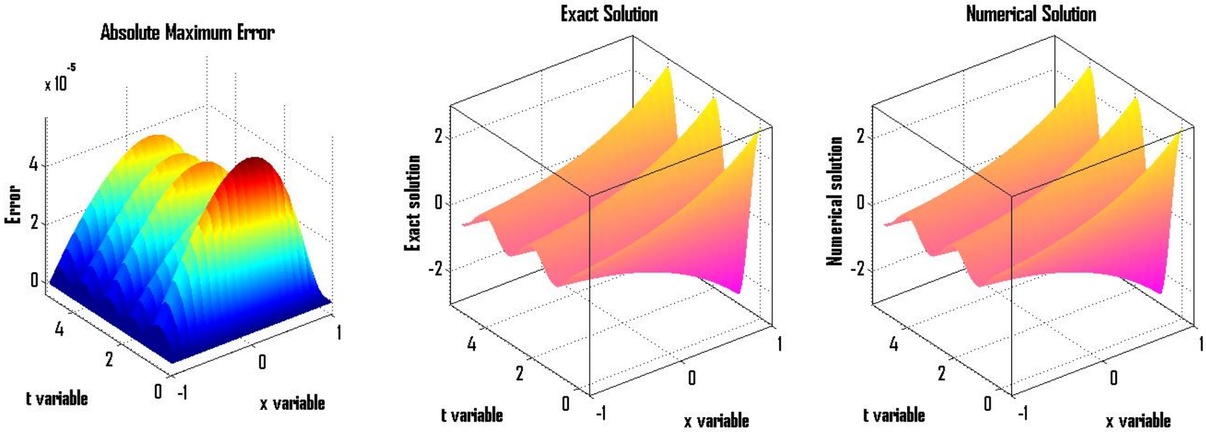

Fig. 2: Profiles of exact solution, numerical solution and absolute error of problem test of Example (4.2) for the parameters $N = 16$ and $\Delta t = 10^{-4}$

Table 2 Companions between exact and numerical solutions at $t = 1$ with: $N = 16$ and $\tau = 10^{-4}$ for Example 4.2

	$u^*(x,t)$	$u_N(x,t)$	$\lvert u^*(x,t) - u_N(x,t)\rvert$
$x = 0.0$	-1.00004	-1.00000	4.72958e-005
$x = 0.4$	-1.49187	-1.49182	4.75656e-005
$x = 0.8$	-2.22556	-2.22554	2.51051e-005
$x = 1.0$	-2.71828	-2.71827	3.71056e-006

Table 3 Companions between exact and numerical solutions at $t = 1$ with: $N = 16$ and $\tau = 10^{-4}$ for Example 4.2

	$u^*(x,t)$	$u_N(x,t)$	$\lvert u^*(x,t) - u_N(x,t)\rvert$
$x = 0.1$	1.10521	1.10517	3.50484e-005
$x = 0.3$	1.34989	1.34985	3.52534e-005
$x = 0.7$	2.01377	2.01375	2.41711e-005
$x = 0.9$	2.45961	2.45960	1.00561e-005

5 Conclusion

Within this paper, we studied the well-posedness of a class of nonlocal boundary value problems of pseudo-parabolic equations with Dirichlet-type integral condition. We showed the uniqueness and the existence of a solution in appropriate function spaces. Besides this, we also designed a numerical scheme based on backward Euler difference schema for the temporal discretization and Legendre pseudo-spectral method for the space discretization. The contribution of this contribution lies in extend and improvement of some existing results concerning the solvability of similar nonlocal boundary value problems.

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