Some Applications of Free Group

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Abstract — *In this paper, we study many concepts as applications of free group, for example, presentation, rank of free group, and inverse of free group. We discussed some results about presentation concept and related it with free group. Our main result about free rank, is if G is a group, then G is free rank n if and only if* $G \cong Z^n$. *Also, we obtained a new fact about inverse semigroup which say there is no free inverse semigroup is finitely generated as a semigroup. Moreover, we studied some results of inverse of free semigroup.*

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1 Introduction

Several authors have investigated the free group. Sury in [1], defined a free group F on a set X by the universal property that abstract map from X to any group G can be extended uniquely as a homomorphism from F to G. Note that a group G is free abelian of rank n if f $G \cong Z^n$.

Derek Holt introduced the definition generate of subgroup by the following, for all $X \subseteq G$ with G is a group we say a subgroup $\langle X \rangle$ of G generated by X in two ways, ⟨X⟩ is intersection of all subgroup H of G that contains X, *i.e.*, $\langle X \rangle = \cap_{H \leq G, X \subseteq H} H$ and if $X^{-1} = \{x^{-1} \mid x \in X\}$, then $A = X \cup X^{-1}$. We say A^* to be the set of all words over A. Elements of A^* represent elements of G, it is closed under concatenation and inversion. So, it is a subgroup of G. So $\langle X \rangle = A^*$. In [1], the empty word represents 1_G introduced in details. Also, in [2], we found proof of the following, a finite group G is not free if $G \neq \{1\}$.

In [3], Jairo Gon, calves studied the existence of free subgroups in unit groups of matrix rings and group rings.

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Charles F. Miller In [4], proved that every group is a quotient group of a free group and then if G is a group there is a free group F and a normal subgroup N such that $G \cong F/N$. Let G be an abelian group and $S=\{x_1,\ldots, x_t\}$ be a subset of G. We said S linearly independent if for each $n_1,\ldots,n_t$ belong Z and $n_1x_1,\ldots,n_ix_i=0$, then for all $1\leq i\leq t$ [2]. Let G be an abelian group, then G is called a finitely generated free group if G has a finite basis (see [1]). Let S and S^{-1} be two sets such that $sns^{-1}=\phi$, $|s|=|s^{-1}|$. If $s^{-1}\in S$ is a unique elements of s in S for all $s\in S^{-1}$ and $t^{-1}\in S$ is a unique element of $t\in s^{-1}$, then $((s^{-1})+1=s)$, for all $s \in S$ [3]. Let $H\leq G$, then H is a free subgroup and $F(S) = Z$ is a free group.

In [5], Maltsev, studied the equation $zxyx^{-1} y^{-1} z^{-1} = aba^{-1} b^{-1}$ in a free group. Also, in [6] the authors studied the periodic automorphisms of free groups in details. Periodic automorphisms of the two-generator free group was investigated by Meskin in [7].

In this article, we introduced some applications of free group and some new results which explain the relationships between several concepts and free group.

2 Presentation of Free Group

Here we discuss some results about presentation concept and related it with free group. Let $G=\langle S\rangle$ and $F(S)=\langle S\rangle$ is a *F-group*. Let $R\subseteq F(S)$ and $N=\langle R\rangle$ be normal subgroups of free group F(S) which generated by set R in other words, $N=\langle\{a^{-1}ra:a\in F(S), r\in R\}\rangle$.
Let $\varphi:F(S)\rightarrow G$ be onto homomorphism. So $\langle S,R\rangle$ is presentation of G if Ker $\varphi=N$, i.e. $F(S)/N\cong G$. If S,R are finite sets, then $\langle S,R\rangle$ is a finite presentation of G.

Remark 2.1. Every element in F(S) is called realtor.

Note that the presentation of a group is a different from representation of free group. Therefore, a presentation of a group G is an expression of Gas a quotient of a free group.

Example 2.2. $G\neq\langle a, b|a^2=1, b^3=1\rangle$, then we say G is the quotient $F/N \ni F$ is the free group of rank 2 generated by $\{a, b\}$ and N is the smallest normal subgroup of F containing the element $a^2$and b^2.Also, we can describe the group Gabove as the free product $Z_2\times Z_3$. Then $a\neq\langle a,b|a^2=1, aba^{-1}b^{-1}=1\rangle\cong Z_2\times Z$. If we add a new relation, then we cannot change the group. See the following $G\neq\langle a,b|a^2=1, ab^2a^{-1}b^{-1}, ab^2a^{-2}b^{-2}=1\rangle$ and $G=\langle a_1, a_2,\ldots,a_n\rangle|r_1=r_2=\ldots r_m=e\rangle$ and if r is a word $r^1(r_2)^{-1}$, then we are *write* $r_1=r_2$ as a replacement to $r=e$.
Remark 2.3. There is no unique presentation of group (there is many presentations of group G. See the following example:

Example 2.4. It is clear $Z_5=\langle a|a^5=e\rangle$ because:

$$\langle a, b, c|ab=c, cb=a, bca=b\rangle$$
$$=\langle a, b. c|ab=c, cbc=a, ca=e\rangle$$
$$=\langle a, b, c|ab=c, cbc=a, c=a^{-1}\rangle$$
$$=\langle a, b|ab=a^{-1}, a^{-1}ba^{-1}=a\rangle$$
$$=\langle a, b|b=a^{-2}, a^{-1}ba^{-1}=a$$
$$=\langle a|a^{-1}a^{-2}a^{-1}=a\rangle$$

$=\langle a|a^5=e\rangle$.

Theorem 2.5. Let $G=\langle S|R\rangle$ and $H=\langle S|R_1\rangle$ such that $R\subseteq R_1$ then H is isomorphic of quotient group.

Proof. Suppose that F=F(S) and N, N_1 are two normal subgroups of F which are generated by two sets R and R_1 and, respectively. Since $R\subseteq R_1$, then $N\leq N_1$.Since $N\lhd F$ then $N\lhd N_1$. Since $G\cong F/N$ and $H\cong (F/N)$ then by (The Third Theorem of Isomorphism), we get $H\cong F/N\cong (F/N)/(N_1/N)\cong G/N_1$. We can get presentation of group G by other presentation of other group G_1 especially G finite group.

Lemma 2.6. If $G=\langle S|R\rangle$ and $H=\langle S|R_1\rangle$ such that R subset of R_1, then H isomorphic quotient of free group G.

Proof. Suppose that F=$F(S)$ and let N, N_1 be two normal subgroups of F and generated by R and R_1. Since R subset of R_1, then N subgroup of N_1. Also, since $N\trianglelefteq F$, then $N\trianglelefteq N_1$. We have $G\cong(F/N)$ and $H\cong(F/N)$, therefore from The Third Theorem of Isomorphism we get

$$H=F/N_1\cong(F/N)/(N_1/N)\cong G/N_1$$

Corollary 2.7. Let $\langle S,R\rangle$ be finite group and let H be a group satisfying $|Q|\leq|H|$. Let $T\subseteq H$ be a set of generators such that $Q:S\rightarrow T$ and $(S_2)^{a2},\ldots,(S_n)^{an}$, implies to $Q(S_2)^{a1},Q(S_2)^{a2},\ldots, Q(S_n)^{an}=e$, *then* $G\cong H$.

Proof. Note that $G\cong\langle T|Q(R)\rangle$ *and* $H=\langle T|Q(R)\rangle\cup R_1\rangle$. From Lemma 2.6, we obtain $H\cong G/N$ such that $N\trianglelefteq G$. Thus $|G|=|H||N|$. Since $|G|\leq|H|$, then $|N|=1$ and hence $|N|=\{e\}$. Thus $H\cong G$.

Corollary 2.8. Let $G\langle a,b|a^2=e, b^3=e, (ab)^2=e\rangle$. Then $G\cong S_3$.

Proof. Suppose that $N=\langle b\rangle$. Since $(ab)^2=e$, then $a^{-1}b^{-1}a=b\in N$, *also* $b^{-1}bb=b\in N$. Thus $N\trianglelefteq G$. Now $G/N=G/\langle b\rangle\cong\langle a, b|a^2=e, b^3=e, (ab)^2=e, b=-e\rangle=\langle a|a^2=e\rangle\cong Z_2$. *Since* $|N|=3$, *then* $|G|\leq 6$. *But* $S_3=\langle(1\ 2), (1\ 2\ 3)\ni(1\ 2)\circ(1\ 2\ 3)^2=(1)$ *and* $|S_3|=6$. *Thus* $G\cong S_3$.

Corollary 2.9. Let $\langle a, b|a^n=e, b^m=e, ab=ba\rangle$ then $G\cong Z_n\times Z_m$.

Proof. Since ab=ba then G is commutative group. Suppose that $N=\langle a\rangle$. Hence $N\trianglelefteq G$ and $G/N=\langle a,b|a^n=e, b^m=e,ab,ba,a=e\rangle=\langle b|b^m=e\rangle\cong Z_m$. Since $|N|\leq n$, then $|G|\leq nm$. Now $Z_n\times Z_m=\langle[1], [0], [1]\rangle$ such that $m([0], [1])=([0],[0])n$, $([1], [0])=([0], [0])$ and $([1], [0])+([0], [0])=([0], [1])+([1],[0])$. Thus $G\cong Z_n\times Z_m$.

Corollary 2.10. Let $G=\langle A, B\rangle\leq GL(2, R)\ni A=\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}$, $B=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$. Then $G\cong D_4$.

Proof. Note that $A^4=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $B^2=\begin{bmatrix}-1 & 0\\ 0 & 0\end{bmatrix}$ and $BA=A^3B=\begin{bmatrix}-1 & 0\\ 0 & 0\end{bmatrix}$. Thus $G=\langle A,B,A^4=I, B^2=I,BA=A^3B\rangle$. Suppose that $N=\langle A\rangle$. Since $A^3=A^{-1}$ and $BA=A^3B=A^{-1}B$, then $BAB^{-1}=A^{-1}\in N$. Also, $AAA^{-1}=A\in N$ and so $N\trianglelefteq G$ and hence $G/N=\langle A,B|A^4=I,B^2=I, BA=A^3B,$

$A=I\rangle=\langle B|B^2=I\rangle=Z_2$. Since $|N|\leq 4$, *then* $|G|\leq 8$. *But* $D_4=\langle a,\ b|a^4=e,\ b^2=e,\ ba=a^3b\rangle$ and $|D_4|=8$. Thus $G\cong D_4$.

Corollary 2.11. Let $G=\langle a,\ b|a^2=b^3=e,\ (ab)^n=(ab^{-1}ab)^k\rangle$. Then

1. $G=\langle ab,\ ab^{-1}ab\rangle$
2. $\langle (ab)^n\rangle\leq G$.

Proof.

1. Suppose that $H=\langle ab,\ ab^{-1}ab\rangle\leq G$, then $ab^{-1}=ab^{-1}ab(ab)\in H$, and $a=abb\text{-}1\in H$, then $H=G$.
2. Suppose that $M=\langle(ab)^n\rangle\leq G$. To prove $M\leq G$ we need by (1) prove that $[(ab)^n,\ ab]=e$ and, $[ab]=(ab)^{-n}(ab)^n(ab)=(ab)^{-(n+1)}(ab)^{n+1}=e$. Hence $(ab)^n=(ab^{-1}ab)^k=(a^{-1}b^{-1}ab)^k\in G$. Thus, $M\leq G$.

Theorem 2.12. Let $G=\langle a,b\rangle$ such that $x^3=e$, for all x in G. Then

1. $[a,b]\in Z(G)$.
2. G is a finite group.

Proof.

1. To prove $[a,\ b]\in Z(G)$, we should prove that $[a,b]$ *commutator* with a,b. Now

$$
\begin{aligned}
a^{-1}[a,\ b] &= a\text{-}1(a^{-1}b^{-1}ab)\\
&= a(b^{-1}a)ba\\
&= a(a^{-1}ba^{-1}b)ba\\
&= ba^{-1}b^{-1}a\\
&= b(baba)a\\
&= b^{-1}aba^{-1}\\
&= a^{-1}ba^{-1}bba^{-1}\\
&= a^{-1}ba^{-1}b^{-1}a^{-1}\\
&= a^{-1}ba^{-1}b^{-1}a^{-1}\\
&= a^{-1}ba^{-1}b^{-1}a^{-1}\\
&= a^{-1}ba^{-1}bba^{-1}\\
&= a^{-1}ba^{-1}b^{-1}a^{-1}\\
&= a^{-1}ba^{-1}b\text{-}1a^{-1}\\
&= a^{-1}bbabaa^{-1}\\
&= a^{-1}b^{-1}ab\\
&= [a,\ b]
\end{aligned}
$$

Thus $[a,b]a=a[a,b]$ and by similarly $[a,b]b=b[a,b]$ and *so* $[a,\ b]\in Z(G)$.

2. Since [a,b] $\in Z(G)$, then $G/Z(G)$ commutative group. Therefore, any commutative group generated by a finite set and satisfying $x^n=e$ such that $n\in Z^+$. Then $G/Z(G)$ is a finite. So, $G/Z(G)$, $Z(G)$ are finite groups. Thus $G$is a finite group.

3 Rank of Free Group

In this subsection, we study another application of free group, namely, the rank of $abelian$ group. We can use a generalized linear algebra, therefore we will discuss the factors of integers Z. Let G be an $abelian$ group. Then the set of $ndistintelements$ $\{g_1,\dots,g_n\}$ subset of G is called $Z\ lineraly$ independent and we have $\sum r_i g_i=0,\ i=1,\dots,n$. If and only if $r_1=r_2=\dots=r_n=0$.

Definition 3.1 Let G be an $abelian$ group. Then the rank of G is the size of the largest set of Z-$lineraly$ independent elements in G.

Remark 3.2 Let G be an abelian group. Then G is called free of rank n if G has a set of n linearly independent generators. Moreover, the set of generators is called basis for G.

Theorem 3.3 Let G be a group. Then G is free rank n if and only if $G\cong Z^n$.

Proof. Let $g_1,\dots,g_n$ be a basis for G , and let $\alpha:Z^n\rightarrow G$ such that $\alpha(r_1,\dots r_n)=\sum r_i g_i,\ i=1,\dots,n$. Since x is surjective homomorphism, therefore we need to prove that $\alpha$is injective. Let $(r_1,\dots,r_n)$ belong to Ker(α), then by definition $\sum r_i g_i=0$, i=1,…,n. Now since g_i are linearly independent, then $r_i=0$ for all i and so $(r_1,\dots,r_n)=(0,\dots,0)$. Thus Ker(x)={0} and this means α is injective and then is isomorphism.

Corollary 3.4 Let G be an abelian group. If G is n-generated, then rank $(G)\leq n$.

Proof. Suppose that $x:Z^n\rightarrow G$ be an onto all g_i are linearly independent. Since α is onto, then there exists $h_1,\dots,h_k$ belong to Z^n andα $(h_i)=g_i$. Suppose that$\sum r_i h_i=0$. Therefore by α we get: $0=\alpha(\sum r_i h_i)=\sum r_i\alpha(h_i)=\sum r_i g_i$, i=1,…k. Since g_i is a linearly independent, then, $r_1=r_2=r_3=0$ and this implies that $h_i=0$. Also, linearly independent, but $k\leq n$ and so rank $(G)\leq n$.

Example 3.5 If $F_n$is F-group has rank n, then F_n contain sub-F-group $F_{k,}$ for all $1\leq k\leq n$. If $S_n=\{S_1,\dots,S_n\}$ and if $S_k=\{S_1,\dots,S_k\}$, we get $F_k=F(S_k)\leq F(S_n)=F_n$.

Remark 3.6 We call S is a free basis of F of and the order of S is a rank of free group F_n . If |S|=n, then F is a free group has rank n and denoted by F_n.

Corollary 3.7. Let G_1 and G_2 be an abelian group with finite rank. Then $\text{rank}(G_1\times G_2)\geq$ rank (G_1)+rank(G_2)

Proof. Let $rank(G_1)=k$ and $rank(G_2)=h$. Suppose $g_1,\dots,g_n$ belong to G_1 such that are linearly independent. Also suppose $g_1,\dots,g_n$ be a linearly independent. Therefore we can claim linearly $(g_1,0),\dots,(g_k,0),\ (0,\ s_1),\dots,(0,\ s_n)$ are linearly independent in $G_1\times G_2$. Now suppose that

$$\sum r_i(g_i+0)\sum b_j(0,\ h_i)=(0,\ 0),\ j=1,\dots h.$$

Therefore, by linear independence we have $r_1=r_2=\ldots=r_k=0$ *and* $b_1=b_2=\ldots b_n=0$, as desired.

Theorem 3.8 Let n be a positive integer number and let F_{n+1} be *F-group* has rank *(n+1)*. Then there exists image homomorphism G of group F_{n+1} $G\cong F_n$.

Proof. Suppose that $F_{n+1}=F(S)$ such that $|S|=n+1$. Let $S=S_1\cup\{\alpha\}$ such that $|S_1|=n$. It is clear $G=G(S_1)\leq F(S_1)$ and $\varphi:S\rightarrow G$ which define by $\varphi(x)=\begin{cases}x, x\neq a\\ e, x=a\end{cases}$

Can be extended into onto homomorphism φ:F(S) → G, thus F_{n+1} $G\cong F_n$.

Theorem 3.9 Let be a free group with rank *n*. Then F_n contains subgroup with index m for each m $\in Z^+$.

Proof. Suppose that G⟨a⟩ is a finite cyclic group with order m and let F_n = F(S) such that S={$s_1,\ldots s_n$} then there exists onto homo$\varphi:F_n\rightarrow G$ and satisfying φ(Si)=a, for all $1\leq i\leq n$. Then by (First Theorem of Isomorphic) we get $F_n/\ker\varphi\cong G$. Thus $Ker\varphi\leq F_n$ with index *n*.

Let G be a free abelian group with *Z-basis X* In this case we can say the rank of G is the cardinality of the basis X. The next example shows some properties of Z-basis:

Example 3.10 We know that Z is free abelian group of Z has cardinality one.

Remark 3.11

1. Only cyclic generators of Z are *{1, -1}* and this mean *Z* has only two basis {1} and *{-1}*.
2. Note that {2, 3} is a generating set of *Z* and not contain a *Z-basis* of *Z* as a subset.
3. Note that {2} is a *Z-independent* subset of *Z* and cannot be extended to *Z-basis* of *Z*.

4. Inverse of Free Semigroup

In this subsection, we will study some results of inverse of free semigroup, but before that we should discuss a concept of inverse of semigroup. A study of inverse elements in semi-group very important in this section.

Definition 4.1 An element y belong to *S* is called inverse element of x in *S*, if $x=xy$ and $y=yxy$.

Remark 4.2 The inverse element of x in *S* need not be unique.

Lemma 4.3 Any regular element of x in a set *S* has an inverse element.

Proof. Let s be an element in *S* such that s is regular, therefore:

For some *s* in *S*, $t=tst$. Now $sts=sts.t.sts$ and so *sts* is also regular element. Now $t=t.sts.t=t.sts.t$ and sts is an inverse element of *t*.

Now we transition from inverse element to inverse semigroup *S*.

Definition 4.4 Let *S* be a semigroup. Then *S* is called an inverse semigroup, if for all s belong to S has a unique inverse element s^{-1} such that $s=sss^{-1}$ *and* $s^{-1}=s^{-1}ss^{-1}$.

Example 4.5 Let *S* be a group. Then it is an inverse semigroup and the inverse of an element *s* is s^{-1}.

Theorem 4.6 *Let S=[X].* If generator s in *X* has unique inverse element. Then *S* is an inverse semigroup $(s_1s_2\ldots s_n)^{-1} = (sn)^{-1},\ldots,s^{n-1})^{-1}$, s_i *in X.*

Proof. Suppose that x, y ∈ S and have unique inverse elements s-1 and t^{-1}, respectively. Therefore $st.t^{-1}s^{-1}.st=s1tt^{-1}.s-1s.t=ss^{-1}s.tt^{-1}.t=st$.

Corollary 4.7 For all x in S such that S is inverse semigroup, then $s=(s^{-1})^{-1}$.

Remark 4.8 For a semigroup S, we have the following statements are true:

1. Let *S* be a semigroup. An element a∈ *S* is said to be regular if there exists s∈*S* such that *a=asa*. The elements is an inverse of a if *a=asa* and *s=sas*.

2. If *s* is an inverse of *a*, then the elements ax and *xa* are idempotents in *S*. i.e. as=sas and sa=sasa.

3. We say that the semigroup *S* is inverse if a unary operation $s_1 \rightarrow s^{-1}$ is definitions with the properties; $\forall s, t \in S(s^{-1})=s, ss^{-1}s=ss^{-1}tt^{-1}=tt^{-1}ss^{-1}$.

4. Let s be nonempty set and $X^{-1}=\{s^{-1}:s\in X\}$ be a set in one-one correspondence with *X* and disjoint from it.

5. Let $Y=X\cup X^{-1}$and consider Y^+ the free semigroup on *Y*. We define the inverse for the elements of Y^+ by the result $(s^{-1})^{-1}$, $s\in X$. Also $(t_1,\ldots,t_n)^{-1}=(t_n)^{-1},\ldots,(t_n)^{-1}$, $t_1,\ldots,t_n\in Y$, and for any $w\in Y^+$ $(w^{-1})^{-1}=w$.

6. Let T be the congruence generated by the set $T=\{(w^{-1}w, w):w \text{ in } Y^+\}\cup\{ww^{-1}zz^{-1}, zz^{-1}ww\text{-}1):w, z \text{ in } Y^+\}$, Y^+/T is a semigroup under the multiplication $(Wt)(Zt)=(wz)T$, *w, z in* Y^+. The map:$X\rightarrow(Y^+/T)$, $s\rightarrow sT$, is obviously well-defined and is the mapping that we associate to Y^+/T to prove that this inverse semigroup is in fact the free inverse semigroup.

7. Let Sbe any inverse semigroup and φany map from X into S. We can extend φ to Y by defining $s^{-1}\varphi=(s\varphi)^{-1}s\in X$, where $(s\ \varphi)^{-1}$is the inverse of $s\varphi \in S$. Since Y^T is the free semigroup on Y, we can define a semigroup $\varphi_1:Y^T\rightarrow S$ by the rule $(t_1t_2\ldots t_n)\ \varphi_1 = t_1\varphi t_2\varphi$, $t_1, t_2, \ldots, tn \in Y$.

Corollary 4.9 There is no free inverse semigroup is $f.g$ as a semigroup.

Proof. Let $X\neq\emptyset$ be a set and FI_X is defined by the S-group presentation $\langle Y/R\rangle$, where *Y* and *R* are finite. If *X* is infinite. Let $s\in X$ not occurring in the relations of R, then the relation $s=s^{-1}$does not hold in FI_X, this is a contradiction, so *X* must be finite. We may express each element in Y as a product of elements in X, so *Y=X*. Let use the finite set of relations $\{s_i=s_j: s_i, s_j \in X, i \neq j\}$, we are identifying all the elements of X as a unique element, so we obviously obtain the free inverse semigroup, but this semigroup is not finitely generated and so FI_X is not finitely generated.

Remark 4.11 Let S be an inverse semigroup. Then we have the following is true in general:

$s \leq t \Leftrightarrow \exists\, e: t=te \Leftrightarrow ss^{-1}=ts^{-1} \Leftrightarrow s=st^{-1}s= \Leftrightarrow ss^{-1}=st^{-1} \Leftrightarrow ss^{-1}=t^{-1}s \Leftrightarrow t^{-1}s \Leftrightarrow s^{-1}t \Leftrightarrow s=ss^{-1}t.$

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