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Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 70, Number 2, Pages 612–621 (2021)
DOI: 10.31801/cfsuasmas.776651
ISSN 1303–5991 E-ISSN 2618–6470

Received by the editors: August 3, 2020; Accepted: March 3, 2021

THE EXISTENCE OF THE BOUNDED SOLUTIONS OF A SECOND ORDER NONHOMOGENEOUS NONLINEAR DIFFERENTIAL EQUATION

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Abstract. In this paper, we consider a second order nonlinear differential equation and establish two new theorems about the existence of the bounded solutions of a second order nonlinear differential equation. In these theorems, we use different Lyapunov functions with different conditions but we get the same result. In addition, two examples are given to support our results with some figures.

1. Introduction

For more than sixty years, a great deal of work has been done by various authors to investigate the autonomous and non-autonomous second order nonlinear ordinary differential equations (ODEs) ([1]- [5], [7]- [14], [16], [17], [19]) and references cited therein.

In investigating the qualitative properties of solutions for second order ODEs, the fixed point method, perturbation theory, variations of parameter formulas, etc. have been used to get information without solving the equations. Moreover, in some of these works, the authors have been studied the Lyapunov direct or second method by constructing different Lyapunov functions or using existing Lyapunov functions.

As far as we know, it should be noted in the relevant literature that so far, the second method of Lyapunov is the most effective tool for studying qualitative

2020 *Mathematics Subject Classification.* Primary 34C11; Secondary 34D05.
Keywords and phrases. Nonlinear differential equation, second order, boundedness, the Lyapunov second method.
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features of nonlinear higher order equations without getting solutions of the equations. This method needs the creation of an appropriate function or functionality that gives concrete results for the problem being studied.

In 1995, Meng [6] dealt with the ordinary linear differential equation of second order

$$x''(t) + p(t)x'(t) + [q_1(t) + q_2(t)]x(t) = f(t),$$

and in 2002, Yuangong and Fanwei [18] considered the second order time lag nonlinear differential equation

$$(r(t)x'(t))' + p(t)x'(t) + [q_1(t) + q_2(t)]x(t) = f(t, x(t)).$$

The authors got some interesting results on the boundedness and square integrability of solutions of the ODEs.

In 2019, Tunç and Mohammed [15] considered two different models for nonlinear of second order

$$x''(t) + p(t)g(x') + q_1(t)h(x) + q_2(t)x = f(t, x, x')$$

and

$$x''(t) + \Phi(t, x, x') + q_1(t)x + q_2(t)\theta(x) = q(t, x, x').$$

They investigate asymptotic boundedness of solutions of the ODEs as $t \to \infty$.

In this paper, motivated by the work of Tunc and Mohammed [15], we deal with the following second order nonlinear differential equation:

$$x'' + f(t, x, x') + q_1(t)\varphi(x) + q_2(t)\psi(x) = g(t, x, x'), \tag{1}$$

where $x \in \mathbb{R} = (-\infty, \infty)$, $t \in \mathbb{R}^+ = [0, \infty)$. $f \in C^1(\mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R})$, $q_1, q_2 \in C^1(\mathbb{R}^+, \mathbb{R})$, $\varphi, \psi \in C^1(\mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R})$ and $f(t, x, 0) = 0$, $\varphi(0) = 0$, $\psi(0) = 0$. Under the assumptions, the existence of the solutions of Eq. (1) is guaranteed. In addition, we assume that the functions f, φ, ψ and g fulfill the Lipschitz condition with respect to x and its derivative x'. So, the solutions of Eq. (1) are uniqueness.

Eq. (1) can be written as

$$\begin{aligned} x' &= y \\ y' &= -f(t, x, x') - q_1(t)\varphi(x) - q_2(t)\psi(x) + g(t, x, x'). \end{aligned} \tag{2}$$

Let

$$\varphi^*(x) = \begin{cases} x^{-1}\varphi(x), & x \neq 0 \\ \varphi'(0), & x = 0, \end{cases}$$

$$\psi^*(x) = \begin{cases} x^{-1}\psi(x), & x \neq 0 \\ \psi'(0), & x = 0 \end{cases}$$

and

$$f^*(t, x, y) = \begin{cases} y^{-1}f(t, x, y), & y \neq 0 \\ f'_y(t, x, 0), & y = 0. \end{cases}$$

2. Main Results

The following assumptions are needed to formulate our main results.

(A1) $f(t,x,0)=0,\ y^{-1}f(t,x,y)\geq f_0\geq 1$ for all $t\in\mathbb{R}^+,\ x\in\mathbb{R},\ y\in\mathbb{R}-\{0\}$.
(A2) $\varphi(0)=0,\ x^{-1}\varphi(x)\geq\varphi_0\geq 1$ for all $x\in\mathbb{R}-\{0\}$.
(A3) $\psi(0)=0,\ x^{-2}\psi^2(x)\leq 1$ for all $x\in\mathbb{R}-\{0\}$.
(A4) $\psi(0)=0,\ x^{-1}\psi(x)\geq\psi_0\geq 1$ for all $x\in\mathbb{R}-\{0\}$.
(A5) $q_1(t)>0,\ q_2(t)>0,\ q_1'(t)>0, \forall t\in\mathbb{R}^+$.
(A6) The functions $g_1(t),\Theta(t),h(t)$ are continuous such that

$$|g(t,x,y)|\leq|g_1(t)|, \forall t\in\mathbb{R}^+,\ \forall x,\ y\in\mathbb{R}, \tag{3}$$

$$\Theta(t)=\frac{1}{2}(q_1'(t)+2q_1(t)),\ \forall t\in\mathbb{R}^+,$$

$$\int_a^\infty\frac{q_2^2(s)}{h^2(s)\Theta(s)}ds<\infty,\ \int_a^\infty\frac{g_1^2(s)}{\Theta(s)}ds<\infty,$$

$$h^2(t)\geq 1,\ \forall t\in\mathbb{R}^+.$$

Theorem 1. *If the conditions* $(A1),(A2),(A3),(A5)$ *and* $(A6)$ *hold, any solution of Eq. (1) satisfies*

$$|x(t)|\leq O(1),\ \left|\frac{dx}{dt}\right|\leq O(\sqrt{q_1(t)}),\ t\to\infty.$$

Proof. We establish the following Lyapunov function because we use the Lyapunov second method

$$V(x,y)=2\int_0^x\varphi(\zeta)d\zeta+\frac{1}{q_1(t)}y^2. \tag{4}$$

From $(A1)$, $(A2)$, $(A5)$ and $(A6)$, we get $V(x,y)=0$ if and only if $x=0$ and $y=0$. From $(A2)$ and $q_1(t)>0$, we have

$$V(x,y)\geq x^2+\frac{1}{q_1(t)}y^2\geq 0.$$

Differentiating the Lyapunov function V in (4) along the solutions of the system (2) and using $(A1)$, we obtain

$$\begin{aligned}\frac{d}{dt}V&=-\frac{q_1'(t)}{q_1^2(t)}y^2-\frac{2}{q_1(t)}yf(t,x,y)-2\frac{q_2(t)}{q_1(t)}y\psi(x)+\frac{2}{q_1(t)}yg(t,x,y)\\&\leq-\frac{q_1'(t)}{q_1^2(t)}y^2-\frac{2}{q_1(t)}y^2-2\frac{q_2(t)}{q_1(t)}y\psi(x)+\frac{2}{q_1(t)}yg(t,x,y)\\&=-\frac{2}{q_1^2(t)}\left[\frac{1}{2}q_1'(t)+q_1(t)\right]y^2-2\frac{q_2(t)}{q_1(t)}y\psi(x)+\frac{2}{q_1(t)}yg(t,x,y).\end{aligned}$$

Since

$$\Theta(t)=\frac{1}{2}(q_1'(t)+2q_1(t)), \tag{5}$$

we have

$$\frac{d}{dt}V \leq -\frac{2\Theta(t)}{q_1^2(t)}y^2 - 2\frac{q_2(t)}{q_1(t)}y\psi(x) + \frac{2}{q_1(t)}yg(t,x,y).$$

We assume that $a > 0,\ b, x \in \mathbb{R}$. If we use the inequality

$$-ax^2 + bx \leq -\frac{a}{2}x^2 + \frac{b^2}{2a}, \tag{6}$$

to the terms

$$-\frac{2\Theta(t)}{q_1^2(t)}y^2 + \frac{2}{q_1(t)}yg(t,x,y),$$

and from $(A5),\ (A6)$, we get

$$\frac{d}{dt}V \leq -\frac{\Theta(t)}{q_1^2(t)}y^2 - 2\frac{q_2(t)}{q_1(t)}y\psi(x) + \frac{g_1^2(t)}{\Theta(t)}. \tag{7}$$

Let

$$W(x,y) = -\frac{\Theta(t)}{q_1^2(t)}y^2 - 2\frac{q_2(t)}{q_1(t)}y\psi(x).$$

Rearranging $W(x,y)$, we have

$$W(x,y) = -\frac{\Theta(t)}{q_1^2(t)}\left[h(t)y + \frac{q_1(t)q_2(t)}{h(t)\Theta(t)}\psi(x)\right]^2 + \frac{q_2^2(t)}{h^2(t)\Theta(t)}\psi^2(x) + \frac{\Theta(t)}{q_1^2(t)}\left(h^2(t)-1\right)y^2.$$

Since the first term of $W(x,y)$ is negative, it is clear that

$$W(x,y) \leq \frac{q_2^2(t)}{h^2(t)\Theta(t)}\psi^2(x) + \frac{\Theta(t)}{q_1^2(t)}(h^2(t)-1)y^2. \tag{8}$$

From (7) and (8)

$$\frac{d}{dt}V \leq \frac{q_2^2(t)}{h^2(t)\Theta(t)}\psi^2(x) + \frac{\Theta(t)}{q_1^2(t)}\left(h^2(t)-1\right)y^2 + \frac{g_1^2(t)}{\Theta(t)}. \tag{9}$$

We assume that

$$\frac{q_2^2(t)}{h^2(t)\Theta(t)} = \frac{\Theta(t)}{q_1(t)}\left(h^2(t)-1\right).$$

Hence

$$h^2(t) = \frac{\Theta^2(t) + \sqrt{\Theta^4(t) + 4q_1(t)q_2^2(t)\Theta^2(t)}}{2\Theta^2(t)}.$$

So, it can be seen that $h^2(t) \geq 1$ for $t \in \mathbb{R}^+$. Thus, we obtain

$$W(x,y) \leq \frac{q_2^2(t)}{h^2(t)\Theta(t)}\left[\psi^2(x) + \frac{1}{q_1(t)}y^2\right]. \tag{10}$$

From (9) and (10)

$$\frac{d}{dt}V \leq \frac{q_2^2(t)}{h^2(t)\Theta(t)}\left[\psi^2(x) + \frac{1}{q_1(t)}y^2\right] + \frac{g_1^2(t)}{\Theta(t)}. \tag{11}$$

Also, from $(A3)$, we know that

$$\psi^2(x) + \frac{1}{q_1(t)} y^2 \le x^2 + \frac{1}{q_1(t)} y^2 \le V(t).$$

And applying the inequality to (11), we can derive

$$\frac{d}{dt} V - \frac{q_2^2(t)}{h^2(t)\Theta(t)} V \le \frac{g_1^2(t)}{\Theta(t)}.$$

Multiplying the inequality by

$$\exp\left(-\int_{t_0}^{t} \frac{q_2^2(s)}{h^2(s)\Theta(s)} ds\right)$$

and integrating this inequality from t_0 to t, we get

$$V(t) \le V(t_0) \exp\left(\int_{t_0}^{t} \frac{q_2^2(s)}{h^2(s)\Theta(s)} ds\right) + \int_{t_0}^{t} \left[\frac{g_1^2(s)}{\Theta(s)} \exp\left(\int_{s}^{t} \frac{q_2^2(\eta)}{h^2(\eta)\Theta(\eta)} d\eta\right)\right] ds.$$

Hence we can take

$$V(t) \le V(t_0) \exp\left(\int_{t_0}^{\infty} \frac{q_2^2(s)}{h^2(s)\Theta(s)} ds\right) + \int_{t_0}^{\infty} \left[\frac{g_1^2(s)}{\Theta(s)} \exp\left(\int_{s}^{\infty} \frac{q_2^2(\eta)}{h^2(\eta)\Theta(\eta)} d\eta\right)\right] ds.$$

Because of $(A6)$, we can assume that

$$V(t_0) \exp\left(\int_{t_0}^{\infty} \frac{q_2^2(s)}{h^2(s)\Theta(s)} ds\right) + \int_{t_0}^{\infty} \left[\frac{g_1^2(s)}{\Theta(s)} \exp\left(\int_{s}^{\infty} \frac{q_3^2(\eta)}{h^2(\eta)\Theta(\eta)} d\eta\right)\right] ds = A,$$

where $A > 0,\ A \in \mathbb{R}$. So, we have

$$V(t) \le A$$

and

$$x^2 + \frac{1}{q_1(t)} y^2 \le V(t) \le A.$$

Therefore, we find

$$|x(t)| \le \sqrt{A}, \quad |y(t)| \le \sqrt{A q_1(t)}.$$

Hence

$$|x(t)| \le O(1), \quad |y(t)| \le O(\sqrt{q_1(t)}), \quad t \to \infty.$$

□

The result of the following theorem is the same as the result of Theorem 1 but we use different Lyapunov function and some different conditions in Theorem 2.

Theorem 2. *If the conditions* $(A1)$, $(A2)$, $(A4)$, $(A5)$ *and* $(A6)$ *hold, any solution of Eq. (1) satisfies*

$$|x(t)| \le O(1), \quad \left|\frac{dx}{dt}\right| \le O(\sqrt{q_1(t)}), \quad t \to \infty.$$

Proof. We determine the Lyapunov function as follows

$$V(x,y) = 2\int_0^x \left[\varphi(\zeta) + \frac{q_2(t)}{q_1(t)}\psi(\zeta)\right] d\zeta + \frac{1}{q_1(t)}y^2. \tag{12}$$

From $(A1)$, $(A2)$, $(A4)$, $(A5)$ and $(A6)$, we get $V(x,y) = 0$ if and only if $x = 0$ and $y = 0$. From $(A2)$, $(A4)$, $q_1(t) > 0$ and $q_2(t) > 0$, we have

$$V(x,y) \geq \left(1 + \frac{q_2(t)}{q_1(t)}\right) x^2 + \frac{1}{q_1(t)}y^2 \geq 0.$$

Differentiating the Lyapunov function V in (12) along the solutions of the system (2) and using $(A1)$, we find

$$\begin{aligned}\frac{d}{dt}V &= -\frac{2}{q_1(t)}yf(t,x,y) + \frac{2}{q_1(t)}yg(t,x,y) - \frac{q_1'(t)}{q_1^2(t)}y^2 \\ &\leq -\frac{2y^2}{q_1(t)} + \frac{2}{q_1(t)}yg(t,x,y) - \frac{q_1'(t)}{q_1^2(t)}y^2 \\ &= -\frac{2}{q_1^2(t)}\left[\frac{1}{2}q_1'(t) + q_1(t)\right]y^2 + \frac{2}{q_1(t)}yg(t,x,y).\end{aligned}$$

Defining $\Theta(t)$ as in (5), we have

$$\frac{d}{dt}V \leq -\frac{2\Theta(t)}{q_1^2(t)}y^2 + \frac{2}{q_1(t)}yg(t,x,y).$$

Let $a > 0,\ b, x \in \mathbb{R}$. From the inequality (6) and $(A6)$, we get

$$\frac{d}{dt}V \leq -\frac{\Theta(t)}{q_1^2(t)}y^2 + \frac{g_1^2(t)}{\Theta(t)}.$$

Since the first term of the inequality is negative, we can write

$$\frac{d}{dt}V \leq \frac{g_1^2(t)}{\Theta(t)}.$$

Integrating this inequality from t_0 to t, we get

$$V(t) \leq V(t_0) + \int_{t_0}^{t} \frac{g_1^2(s)}{\Theta(s)}ds.$$

Hence we can take

$$V(t) \leq V(t_0) + \int_{t_0}^{\infty} \frac{g_1^2(s)}{\Theta(s)}ds.$$

Because of $(A6)$, we can assume that

$$V(t_0) + \int_{t_0}^{\infty} \frac{g_1^2(s)}{\Theta(s)}ds = B,\ B > 0,\ B \in \mathbb{R}.$$

So, we have

$$V(t) \leq B.$$

From $(A2)$, $(A4)$ and $(A5)$, we know that

$$x^2 + \frac{1}{q_1(t)} y^2 \leq V(t) \leq B.$$

Therefore, we find

$$|x(t)| \leq \sqrt{B}, \quad |y(t)| \leq \sqrt{Bq_1(t)}.$$

Hence

$$|x(t)| \leq O(1),\ |y(t)| \leq O(\sqrt{q_1(t)}),\ t \to \infty.$$

□

Remark 3. *If it is taken $f(t,x,x') = p(t)g(x')$ and $\psi(x) = x$ in Eq. (1) or $\varphi(x) = x$ in Eq. (1), Theorem 1 or Theorem 2 in [15] is obtained, respectively.*

3. Examples

Example 4. *As a special case of Eq. (1), we consider the following second order nonlinear ODE*

$$x'' + 6x' + x'e^{-t-x^2} + 5e^{3t}(5 + sinx)x + 2e^{2t}(1 - e^{-x^2})x = \frac{cost}{e^{3t}(1 + 2e^{x^4})} \tag{13}$$

or

$$x' = y$$
$$y' = -6x' - x'e^{-t-x^2} - 5e^{3t}(5 + sinx)x - 2e^{2t}(1 - e^{-x^2})x + \frac{cost}{e^{3t}(1 + 2e^{x^4})}.$$

It is clear that the conditions $(A1)$, $(A2)$, $(A3)$, $(A5)$ and $(A6)$ are satisfied. So, from Theorem 1, all solutions of Eq. (13) satisfy

$$|x(t)| \leq O(1),\ \left|\frac{dx}{dt}\right| \leq O(\sqrt{5}e^{3t}),\ t \to \infty$$

as shown in Fig. 1 obtained by using the adaptive MATLAB solver ode45.

Example 5. *Taking $f(t,x,x') = 5x'e^t \sin^2 x$, $q_1(t) = 2e^{3t}$, $\varphi(x) = xe^{x^2}$, $q_2(t) = 5e^{4t}$, $\psi(x) = (3 + sinx)x$ and $g(t,x,x') = \dfrac{sinx'}{e^{6t}(2 + e^{x^2})}$ in Eq. (1), we get the following second order nonlinear ODE*

$$x'' + 5x'e^t \sin^2 x + 2e^{3t}xe^{x^2} + 5e^{4t}(3 + sinx)x = \frac{sinx'}{e^{6t}(2 + e^{x^2})} \tag{14}$$

or

$$x' = y$$
$$y' = -5x'e^t \sin^2 x - 2e^{3t}xe^{x^2} - 5e^{4t}(3 + sinx)x + \frac{sinx'}{e^{6t}(2 + e^{x^2})}.$$

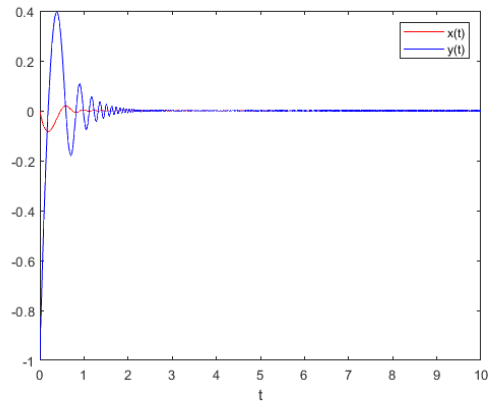

FIGURE 1. The solution of Eq. (13) with the initial conditions $x(0) = 0,\ y(0) = -1$ in $t \in [0, 10]$.

It is clear that the conditions $(A1)$, $(A2)$, $(A4)$, $(A5)$ *and* $(A6)$ *are satisfied. So, from Theorem 2, all solutions of Eq. (14) satisfy*

$$|x(t)| \leq O(1), \quad \left|\frac{dx}{dt}\right| \leq O(\sqrt{2}e^{-3t}), \quad t \to \infty$$

*as shown in Fig. 2 obtained by using the adaptive MATLAB solver ode*45*.*

4. Conclusion

We have presented a new second order nonlinear differential equation (1) to study the existence of the bounded solutions of the equation by using the Lyapunov direct or second method. Additionally, we give two examples to support our main results. Also, MATLAB has been used to draw two figures. Fig. 1 in first example shows the solution $(x(t), y(t))$ of Eq. (13) with the initial conditions $x(0) = 0,\ y(0) = -1$ in $t \in [0, 10]$. The solution is bounded since the conditions of Theorem 1 are satisfied. Fig. 2 exemplifies the solution $(x(t), y(t))$ of Eq. (14) with the initial conditions $x(0) = 1,\ y(0) = 0$ in $t \in [0, 7]$. The solution is bounded since the conditions of Theorem 2 are satisfied. Moreover, taking $f(t, x, x') = p(t)g(x')$ and $\psi(x) = x$ or $\varphi(x) = x$ in Eq. (1), Theorem 1 or Theorem 2 in [15] is gotten, respectively. So, Eq. (1) is a generalization of Eq. (6) and Eq. (7) in [15].

Declaration of Competing Interest The author has no competing interest to declare.

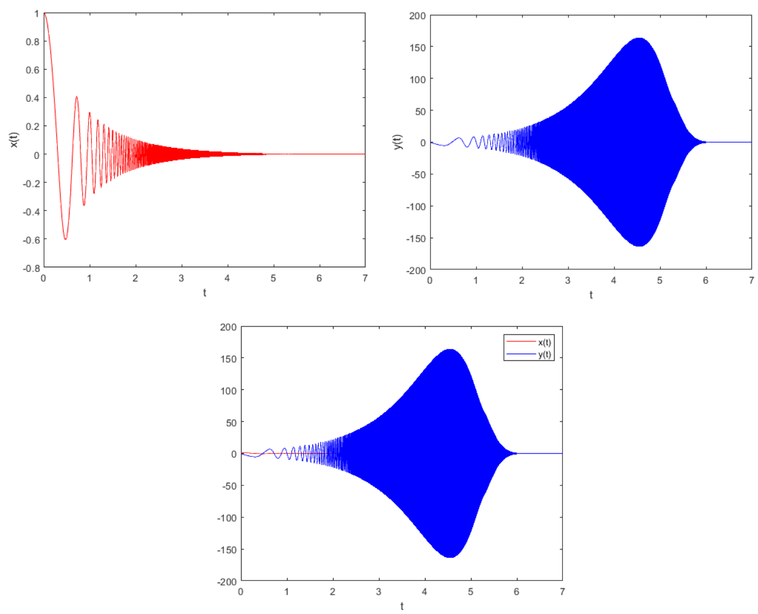

FIGURE 2. The solution of Eq. (14) with the initial conditions $x(0) = 1,\ y(0) = 0$ in $t \in [0, 7]$.

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