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The Deformation of an (α, β)-Metric

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(Dedicated to the memory of **Prof. Dr. Aurel BEJANCU (1946 - 2020)**)

ABSTRACT

In this paper, we will continue our investigation on the new recently introduced (α,β)-metric $F=\beta+\frac{a\alpha^2+\beta^2}{\alpha}$ in [12]; where α is a Riemannian metric; β is a 1-form, and $a\in\left(\frac{1}{4},+\infty\right)$ is a real positive scalar. We will investigate the deformation of this metric, and we will investigate its properties.

Keywords: *Finsler metric, Finsler (α,β)-metric, deformation of an (α,β)-metric.*
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1. Introduction

Finsler metric is a generalization of the Riemannian metric, which depends on the position and direction. The (α,β) metric is a fruitful branch of Finsler geometry which is introduced by M. Matsumoto [7]. It can be expressed in the form $F=\alpha\phi(s),\ s=\frac{\beta}{\alpha}$, where α is a Riemannian metric, β is a 1-form and ϕ is a smooth positive function on the domain. (α,β)-metrics are very significant since they are computable.

In [12], the authors introduced a new class of (α,β)-metric as $F=\beta+\frac{a\alpha^2+\beta^2}{\alpha}$. In this paper, we consider a perturbation of this metric, given by

$$F_\epsilon(\alpha,\beta)=\frac{\beta^2+\alpha^2(a+1)}{\alpha}+\epsilon\beta \tag{1.1}$$

where ϵ is a real parameter with $|\epsilon|<2\sqrt{a+1}$, and we show that F_ϵ is also a (α,β)-metric. Some interesting results on the theory of Finsler spaces and also of Lagrange spaces were obtained in the last years underlying the importance for the study of this spaces. In this respect, please see ([2], [3], [4], [5], [9], [11], [13], [20])

The notion of dually flat metrics arises from information geometry on Riemannian manifolds [1]. In [14], Z. Shen extends the notion of locally dually flat metrics to Finsler geometry. It plays a significant role in considering the flat Finsler information structure. A Finsler metric on an n-dimensional manifold M is called locally dually flat if at every point there is a coordinate system in which the spray coefficients are in the following form

$$G^i=-\frac{1}{2}g^{ij}H_{y^j}, \tag{1.2}$$

where $H=H(x,y)$ is a local scalar function on the tangent bundle TM of M.

In this paper, we want to extend the result of [12] to F_ϵ. In fact, we obtain the followings:

Theorem 1.1. *Let $F_\epsilon=\frac{\beta^2+\alpha^2(a+1)}{\alpha}+\epsilon\beta$ be the (α,β)-metric on an n-dimensional manifold M^n, $(n\geq 3)$, where $\alpha=\sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i\neq 0$ is a 1-form on M. Then F_ϵ is a locally dually flat if and only if α,β and $\phi=\phi(s)$, satisfy:*

1. $s_{l0}=\frac{1}{3}(\beta\theta_l-\theta b_l)$,

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2. $r_{00} = \frac{2}{3}\theta\beta + \left[\theta + \frac{2}{3}(b^2\theta - \theta_l b^l)\right]\alpha^2 + \frac{1}{3}(3k_2 - 2 - 3k_3 b^2)\theta\beta^2$,
3. $G^l_\alpha = \frac{1}{3}\left[2\theta + (3k_1 - 2)\theta\beta\right] y^l + \frac{1}{3}(\theta^l - \tau b^l)\alpha^2 + \frac{1}{2}k_3\tau\beta^2 b^l$,
4. $\tau\left[s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi')\right] = 0$,

where k_1, k_2, and k_3 are defined by (2.6), (2.7), and (2.8) in Section 2, $\tau = \tau(x)$ is a scalar function, $\theta = \theta_i(x)y^i$ is a 1-form on M, $\theta^l = a^{lm}\theta_m$.

Projectively flat metrics on $\mathcal{U} \subseteq \mathbb{R}^n$, where $\mathcal{U}$ is a convex subset of $\mathbb{R}^n$ are the solutions to the Hilbert's Fourth Problem. By Beltrami's theorem, a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. Locally projectively flat Finsler metrics are much more complicated.

In this paper, we find the necessary condition for F_ϵ to be projectively flat.

Theorem 1.2. *Let $F_\epsilon = \frac{\beta^2 + \alpha^2(a+1)}{\alpha} + \epsilon\beta$ be the (α, β)-metric on an n-dimensional manifold M^n, $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i \neq 0$ is a 1-form on M. Then F_ϵ is locally projectively flat if and only if there is a scalar function $\tau = \tau(x)$ such that*

$$b_{i|j} = \frac{\tau}{2(a+1)}\{(a + 1 + b^2)a_{ij} - 3(b_i)(b_j)\}, G^i_\alpha = \xi y^i \frac{2\tau}{a+1}\alpha^2 b^i.$$

2. Preliminaries

Let M be an n-dimensional, real, differentiable manifold and $\pi : TM \longrightarrow M$ be the tangent bundle of M.

Definition 2.1. Let $F : TM - \{0\} \longrightarrow \mathbb{R}$, be a Finsler metric. The Finsler space $F^n = (M, F(x, y))$, is endowed with an (α, β)-metric, if the fundamental function F, can be written as follows: $F(x, y) = \tilde{F}(\alpha(x, y), \beta(x, y))$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ with a_{ij} a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form field on TM.

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) F is C^∞ on TM_0;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;
(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y + su + tv)\right]|_{s,t=0}, u, v \in T_xM.$$

The following notion can be found in [8]:

Definition 2.2. A Lagrange space is a pair $L^n = (M, L(x, y))$ formed by a smooth real, n-dimensional manifold M and a regular differentiable Lagrangian $L(x, y)$, for which the d-tensor field g_{ij} has a constant signature over the manifold $\widetilde{TM}$.

As we know from [18] and [10], Finsler spaces endowed with (α, β)-metrics were applied successfully to the study of gravitational magnetic fields. Other important results from [8] are presented as follows:

Let $F^n = (M, F(x, y))$ be a Finsler space. It has an (α, β)-metric if the fundamental function can be expressed in the following form: $F(x, y) = \breve{F}(\alpha(x, y), \beta(x, y))$, where $\breve{F}$ is a differentiable function of two variables with: $\alpha^2(x, y) = a_{ij}(x)y^i y^j$; $\beta(x, y) = b_i(x)y^i$. In fact, $a_{ij}(x)dx^i dx^j$ is a Riemannian metric on the base manifold M and $b_i(x)dx^i$ is the electromagnetic 1-form on M. As we know from [8], if we denote by $L^n = (M, L)$ a Lagrange space; the fundamental tensor $g_{ij}(x, y)$ of L^n is: $g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}$ and this tensor can be written as follows for (α, β)-Lagrangians:

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_{-1}(b_i\mathcal{Y}_j + b_j\mathcal{Y}_i) + \rho_{-2}\mathcal{Y}_i\mathcal{Y}_j$$

where $b_i = \frac{\partial\beta}{\partial y^i}$, $\mathcal{Y}_i = a_{ij}y^j = \alpha\frac{\partial\alpha}{\partial y^i}$ and $\rho, \rho_0, \rho_{-1}, \rho_{-2}$ are invariants of the space L^n and given by (see [8]):

$$\rho = \frac{1}{2\alpha}L_\alpha, \qquad \rho_0 = \frac{1}{2}L_{\beta\beta},$$
$$\rho_{-1} = \frac{1}{2\alpha}L_{\alpha\beta}, \quad \rho_{-2} = \frac{1}{2\alpha^2}\left(L_{\alpha\alpha} - \frac{1}{\alpha}L_\alpha\right). \tag{2.1}$$

where $L_\alpha = \frac{\partial L}{\partial \alpha}$, $L_\beta = \frac{\partial L}{\partial \beta}$, $L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2}$, $L_{\beta\beta} = \frac{\partial^2 L}{\partial \beta^2}$, and $L_{\alpha\beta} = \frac{\partial^2 L}{\partial \alpha \partial \beta}$.

Shimada and Sabău in [17], have proved that the system of covectors $\{b_i, \mathcal{Y}_i\}$ is independent. The following formulas hold (see [8]):

$$
\begin{aligned}
& y_i = \frac{1}{2}\frac{\partial L}{\partial y^i} = \rho_1 b_i + \rho \mathcal{Y}_i, && \rho_1 = \frac{1}{2} L_\beta, \\
& \frac{\partial \rho_1}{\partial y^i} = \rho_0 b_i + \rho_{-1}\mathcal{Y}_i, && \frac{\partial \rho}{\partial y^i} = \rho_{-1} b_i + \rho_{-2}\mathcal{Y}_i \\
& \frac{\partial \rho_0}{\partial y^i} = r_{-1} b_i + r_{-2}\mathcal{Y}_i, && \frac{\partial \rho_{-1}}{\partial y^i} = r_{-2} b_i + r_{-3}\mathcal{Y}_i \\
& \frac{\partial \rho_{-2}}{\partial y^i} = r_{-3} b_i + r_{-4}\mathcal{Y}_i
\end{aligned}
\tag{2.2}
$$

where

$$
\begin{aligned}
& r_{-1} = \frac{1}{2} L_{\beta\beta\beta}, \quad r_{-2} = \frac{1}{2\alpha} L_{\beta\beta\beta}, \\
& r_{-3} = \frac{1}{2\alpha^2}\left(L_{\alpha\alpha\beta} - \frac{1}{\alpha} L_{\alpha\beta}\right), \quad r_{-4} = \frac{1}{2\alpha^3}\left(L_{\alpha\alpha\alpha} - \frac{3}{\alpha} L_{\alpha\alpha} + \frac{3}{\alpha^2} L_\alpha\right).
\end{aligned}
$$

By [6], we have the following:

Remark 2.1. The function $F = \alpha\phi(s)$ is a Finsler function if and only if three conditions are satisfied:

1. $\phi(s) > 0$,
2. $\phi(s) - s\phi'(s) > 0$,
3. $[\phi(s) - s\phi'(s)] + (b^2 - s^2)\phi''(s) > 0$.

Now, let us consider the (α, β)-metric $F = \beta + \frac{a\alpha^2 + \beta^2}{\alpha}$ where α is a Riemannian metric; β is a 1-form and $a \in \left(\frac{1}{4}, +\infty\right)$ is a real positive scalar. We will consider a perturbation of this metric, given by

$$
F_\epsilon(\alpha, \beta) = \frac{\beta^2 + \alpha^2(a+1)}{\alpha} + \epsilon\beta \tag{2.3}
$$

where ϵ is a real parameter with $|\epsilon| < 2\sqrt{a+1}$.

Let us first observe the following:

Remark 2.2. The metric F_ϵ is also an (α, β)-metric since it can be rewritten in the following form:

$$
F_\epsilon(\alpha, \beta) = \alpha\phi(s)
$$

where

$$
\phi(s) = s^2 + \epsilon s + a + 1 \tag{2.4}
$$

Also, it is easy to compute

$$
\phi'(s) = 2s + \epsilon, \quad \phi''(s) = 2. \tag{2.5}
$$

Let

$$
k_1 = \Pi(0), \qquad k_2 = \frac{\Pi'(0)}{Q(0)}, \tag{2.6}
$$

$$
k_3 = \frac{1}{6Q(0)^2}\left[3Q''(0)\Pi'(0) - 6\Pi(0)^2 - Q(0)\Pi'''(0)\right], \tag{2.7}
$$

$$
Q = \frac{\phi'}{\phi - s\phi'}, \qquad \Pi = \frac{\phi'^2 + \phi\phi''}{\phi(\phi - s\phi')}. \tag{2.8}
$$

An important result obtained in [19], is the following:

Theorem 2.1. *([19]) Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be an (α, β)-metric on an n-dimensional manifold $M^n, (n \geq 3)$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i \neq 0$ is a 1-form on M. Suppose that $\phi'(s) \neq 0$, $\phi'(0) \neq 0$. Then F is a locally dually flat on M if and only if α, β and $\phi = \phi(s)$, satisfy:*

1. $s_{lo} = \frac{1}{3}(\beta\theta_l - \theta b_l)$,
2. $r_{00} = \frac{2}{3}\theta\beta + \left[\theta + \frac{2}{3}(b^2\theta - \theta_l b^l)\right]\alpha^2 + \frac{1}{3}(3k_2 - 2 - 3k_3 b^2)\theta\beta^2$,
3. $G^l_\alpha = \frac{1}{3}\left[2\theta + (3k_1 - 2)\theta\beta\right] y^l + \frac{1}{3}(\theta^l - \tau b^l)\alpha^2 + \frac{1}{2}k_3\tau\beta^2 b^l$,
4. $\tau\left[s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi')\right] = 0$,

where k_1, k_2, and k_3 are defined by (2.6), (2.7), and (2.8), $\tau = \tau(x)$ is a scalar function, $\theta = \theta_i(x)y^i$ is a 1-form on M, $\theta^l = a^{lm}\theta_m$,

By [16], for a projectively flat (α, β)-metric, we have:

Theorem 2.2. *[16] For an (α, β)-metric, if ϕ satisfies*

$$\phi(s) - s\phi'(s) = (p + rs^2)\phi''(s) \tag{2.9}$$

where $p \neq 0, r$ are constants, then it satisfies

$$\{1 + (m_1 + m_2 s^2)s^2 + m_3 s^2\}\phi''(s) = (m_1 + m_2 s^2)\{\phi(s) - s\phi'(s)\}, \tag{2.10}$$

with $m_1 = \frac{1}{p}, m_2 = 0$, and $m_3 = \frac{(r-1)}{p}$. Then $F = \alpha\phi(s)$ is projectively flat if and only if there is a scalar function $\tau = \tau(x)$ such that

$$b_{i|j} = \frac{2\tau}{p}\{(p + b^2)a_{ij} + (r-1)b_i b_j\}, G^i_\alpha = \xi y^i \frac{\tau}{p}\alpha^2 b^i$$

3. Main Results

Using Remark 2.2, we can give now the following theorem:

Theorem 3.1. *The function $F_\epsilon = \alpha\phi(s)$ is a Finsler function as defined in (1.1) if and only if $|s| < \min\{\sqrt{a+1}, \sqrt{\frac{2b^2+a+1}{3}}\}$.*

Proof. First, note that due to $\|\epsilon\| < 2\sqrt{a+1}$, we have $\phi(s) > 0$. It is easy to see that two conditions $\phi(s) - s\phi'(s) > 0$ and $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$ are equivalent to $-s^2 + a + 1 > 0$ and $3s^2 < 2b^2 + a + 1$, respectively. From which, we get the result. □

Proof of Theorem 1.1:
By using Theorem 2.1 and Maple program we have

$$k_1 = \frac{\epsilon^2 + 2 + 2a}{(1+a)^2}, \qquad k_2 = \frac{4 + 4a^2}{(1+a)^2},$$

$$k_3 = -\frac{1}{3\epsilon^2(1+a)^3}\left[12a^3 + (12\epsilon^2 + 6\epsilon + 36)a^2 + (\epsilon^4 + 4\epsilon^3 + 20\epsilon + 34)a + \epsilon^5 + 6\epsilon^4 - 4\epsilon^3 + 6\epsilon + 12\right]$$

$$Q(s) = \frac{2s + \epsilon}{-s^2 + a + 1}, \qquad \Pi(s) = \frac{-\epsilon^3 - 6\epsilon^2 s - 6\epsilon s^2 - 2a\epsilon + 4\epsilon + 12s}{(s^2 + s\epsilon + a + 1)(-s^2 + a + 1)}$$

By substituting the above equations, we obtain Theorem 1.1. □

Now we will compute the following quantities:

$$\Theta(s) = \frac{1}{2}\frac{\phi(s)\phi'(s) - s(\phi(s)\phi''(s) + (\phi'(s))^2)}{\phi(s)(\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s))} \tag{3.1}$$

$$\Psi(s) = \frac{1}{2}\frac{\phi''(s)}{\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s)} \tag{3.2}$$

By computing these quantities for F_ϵ, one obtains:

$$\Theta(s) = \frac{1}{2}\frac{-4s^3 - 3s^2\epsilon + a\epsilon + \epsilon}{(s^2 + s\epsilon + a + 1)(-3s^2 + a + 1 + 2b^2)}, \tag{3.3}$$

and

$$\Psi(s) = \left(-3s^2 + a + 1 + 2\,b^2\right)^{-1}$$

Suppose that $F = \alpha\phi(s),\ s = \frac{\beta}{\alpha}$ be an (α,β)-metric, satisfying $\|\beta\|_\alpha < b_0$, $\forall x \in M$. Let $\nabla\beta = b_{i|j}dx^i \otimes dx^j$ be the covariant derivative of β with respect to α By [15], the relation between the geodesic coefficients G^i of F and the geodesic coefficients G^i_α of α is given by:

$$G^i = G^i_\alpha + \alpha Q(s)s^i_0 + \{-2Q(s)\alpha s_0 + r_{00}\}\left\{\Psi(s)b^i + \Theta(s)\alpha^{-1}y^i\right\}. \tag{3.4}$$

By using (3.4), we have the followings:

Theorem 3.2. *The relation between the geodesic coefficients G^i of the metric F_ϵ and the geodesic coefficients G^i_α of α is given by:*

$$\begin{aligned} G^i &= G^i_\alpha + \alpha\left(\frac{2s+\epsilon}{-s^2+a+1}\right)s^i_0 \\ &+ \left\{-2\left(\frac{2s+\epsilon}{-s^2+a+1}\right)\alpha s_0 + r_{00}\right\}\left\{\left(\left(-3s^2+a+1+2b^2\right)^{-1}\right)b^i + \Theta(s)\alpha^{-1}y^i\right\} \end{aligned} \tag{3.5}$$

where $\Theta(s)$ is given by (3.3).

Proof. The proof can be obtained directly using the above results. □

Now, we consider a Finsler space endowed with the fundamental function $L = F^2_\epsilon = \frac{(\beta^2+\alpha\beta\epsilon+\alpha^2(a+1))^2}{\alpha^2}$. To determine the fundamental tensor $g_{ij}(x,y)$, of a Finsler space F_ϵ, we compute the following invariants:

$$\begin{aligned} \rho &= \frac{1}{2\alpha}L_\alpha = \frac{a^2\alpha^4 + a\alpha^3\beta\epsilon + 2a\alpha^4 + \alpha^3\beta\epsilon - \alpha\beta^3\epsilon + \alpha^4 - \beta^4}{\alpha^4}, \\ \rho_0 &= \frac{1}{2}L_{\beta\beta} = \frac{6\beta^2 + 6\epsilon\alpha\beta + 2\alpha^2 + \alpha^2\epsilon^2 + 2\alpha^2 a}{\alpha^2}, \\ \rho_{-1} &= \frac{1}{2\alpha}L_{\alpha\beta} = \frac{a\alpha^3\epsilon + \alpha^3\epsilon - 3\alpha\beta^2 e - 4\beta^3}{\alpha^4}, \\ \rho_{-2} &= \frac{1}{2\alpha^2}\left(L_{\alpha\alpha} - \frac{1}{\alpha}L_\alpha\right) = -\frac{\beta(a\alpha^3\epsilon + \alpha^3\epsilon - 3\alpha\beta^2\epsilon - 4\beta^3)}{\alpha^6}, \\ \rho_1 &= \frac{1}{2}L_\beta = \frac{a\alpha^3\epsilon + \alpha^2\beta\epsilon^2 + 2a\alpha^2\beta + \alpha^3\epsilon + 3\alpha\beta^2\epsilon + 2\alpha^2\beta + 2\beta^3}{\alpha^2}, \end{aligned}$$

where $L_\alpha = \frac{\partial L}{\partial\alpha}, L_\beta = \frac{\partial L}{\partial\beta}, L_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha^2}, L_{\beta\beta} = \frac{\partial^2 L}{\beta^2}$, and $L_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\partial\beta}$.

Using all invariants $\rho, \rho_{-1}, \rho_{-2}, \rho_1, \rho_0$, we can now compute the fundamental tensor for the perturbed metric.

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_{-1}(b_i p_j + b + jp_i) + \rho_{-2}p_i p_j,$$

where $p_i = \alpha\frac{\partial\alpha}{\partial y^i} = a_{ij}y^j$.

Proof of Theorem 1.2:

Using (2.4) and (2.5), we have

$$(\phi - s\phi') = a + 1 - s^2 \tag{3.6}$$

By comparing (3.6) and (2.9), one concludes that $p = \frac{a+1}{2}$ and $r = -\frac{1}{2}$. Then by Theorem 2.2, we have

$$m_1 = \frac{2}{a+1}, \quad m_2 = 0, \quad m_3 = -\frac{3}{a+1},$$

which concludes that (2.10) is correct. Hence F_ϵ is projectively flat if and only if

$$b_{i|j} = \frac{\tau}{2(a+1)}\{(a+1+b^2)a_{ij} - 3b_i b_j\}, \quad G^i_\alpha = \xi y^i\frac{2\tau}{a+1}\alpha^2 b^i,$$

which complete the proof. □

4. Conclusion

In this paper, we have continued the investigations on the new introduced (α, β)-metric $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ where α is a Riemannian metric, β is a 1-form, and $a \in \left(\frac{1}{4}, +\infty\right)$ is a real positive scalar. We investigate the perturbation of this metric.

Remark 4.1. In this paper, we used the Maple 13 program for computations.

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