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On the Asymptotic Expansions for the Expected Value and Variance of the Reinsurance Surplus Process

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Abstract: In this paper, we consider the reinsurance surplus process. Depending on the type of the reinsurance we obtain formulas for the distribution functions and moments of claims in reinsurance surplus process, then using these moments we give the asymptotic results for the mathematical expectation and variance in each type of the reinsurance. Then we give numerical examples to compare the values of mathematical expectation and variance when there is no reinsurance and when the insurer effects reinsurance.

Keywords: Reinsurance, Surplus process, Asymptotic expansion, Mathematical expectation, variance, Renewal process, Renewal-reward process

1 Introduction

Reinsurance is one of the major risk and capital management tools available to primary insurance companies. Reinsurance is insurance for insurers. Insurers buy reinsurance for risks they cannot or do not wish to retain fully themselves. Reinsurers help the industry to provide protection for a wide range of risks, including the largest and most complex risks covered by the insurance system. Insurers also benefit from capital relief that reinsurance provides and from reinsurers' product development skills and risk expertise. As such, reinsurance is an indispensable part of the insurance system that makes insurance more secure and less expensive. This is ultimately for the benefit of policyholders who get more protection at a lower cost. That is why it is important to investigate surplus process and moments of this process when insurer effects reinsurance. We will call the insurer's surplus process as reinsurance surplus process when insurer effects reinsurance. Consider the insurance risk process

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \tag{1.1}$$

where $u = U(0) \geq 0$ is the initial capital of the insurance company, $c > 0$ is the premium rate, $\{X_i, i \geq 1\}$ denotes the sequence of independent and identically distributed (i.i.d.) non-negative successive claims, and $N(t)$ $(t \geq 0)$ denotes the number of claims up to time t, which is a counting process independent of $\{X_i, i \geq 1\}$.

If $N(t)$ is a renewal process, that is, the times $T_i, i \geq 1$, elapsed between successive claims are i.i.d., the model above is the renewal risk model, which introduced by Sparre Anderson (see, for example, [1] and [4]).

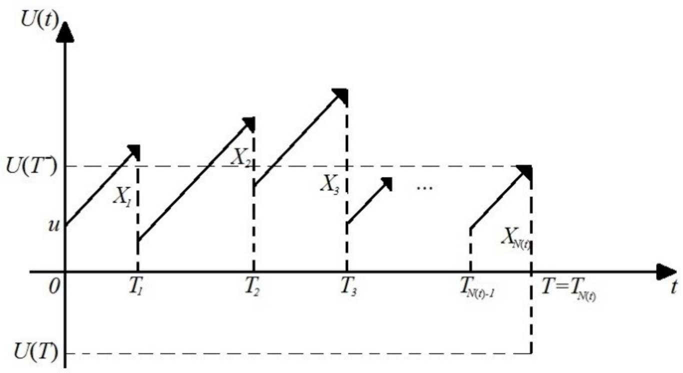

Fig. 1: A trajectory of the process $U(t)$.

If the insurer effects reinsurance by paying a reinsurance premium continuously at a constant rate, then this process becomes a net of reinsurance surplus process $U^*(t), t \geq 0$ given by

$$U^*(t) = u + c^*t - \sum_{i=1}^{N(t)} X_i^*, \tag{1.2}$$

where c^* denotes the insurer's premium income per unit time net of reinsurance, and X_i^* denotes the amount the insurer pays on the ith claim, net of reinsurance.

Basically there are some types of reinsurance contracts: proportional reinsurance, excess of loss reinsurance and excess stop loss reinsurance. If the insurer effects reinsurance, then the amount of claim paid by insurer is given by function h in each type of reinsurance, so, if the amount of claim is x, then the insurer pays the amount of $h(x)$ (see, for example, [2]-[7]-[9]): $0 \leq h(x) \leq x$. Consequently, in (1.2) we can write $h(X_i)$ instead of $X_i^* : X_i^* = h(X_i)$. We will denote by F and F_h the distribution functions of X and $X^* = h(X)$, respectively.

If the insurer effects proportional reinsurance, then the insurer pays some proportion α of each claim. In this case $h(x) = \alpha x, 0 < \alpha \leq 1$.

If the insurer effects excess of loss reinsurance with retention level M, then the reinsurance company pays claims that exceed the level M. In this case, $h(x) = \min\{x, M\}, M > 0$.

The reinsurance company can apply some upper bound L to insure itself against big losses, so, the maximum amount which can be paid by reinsurance company equals to L. In this case, the insurer effects excess stop loss reinsurance with retention level M and upper bound L, then

$$h(x) = x - \min\{\max\{x - M; 0\}; L\} = \max\{\min\{x; M\}; x - L\}.$$

To distinguish distribution function of claims for each type of reinsurance, we will denote them by $F_{h,\alpha}$, $F_{h,M}$ and $F_{h,M,L}$, when the insurer effects proportional reinsurance, excess of loss reinsurance with retention level M and excess stop loss reinsurance with retention level M and upper bound L, respectively.

Denote

$$\lambda_k = E\{X^k\} = k \int_0^\infty x^{k-1} \overline{F}(x) dx,$$

$$\lambda_k^* = E\{(X^*)^k\} = E\{h^k(X)\} = k \int_0^\infty x^{k-1} \overline{F}_h(x) dx,$$

$$G_k(t) = k \int_0^t x^{k-1} \overline{F}(x) dx,$$

where $k = 1; 2$.

We will denote the moments of claims by $\lambda_{k,\alpha}^*$, $\lambda_{k,M}^*$ and $\lambda_{k,M,L}^*$, when the insurer effects proportional reinsurance, excess of loss reinsurance with retention level M and the excess stop loss reinsurance with retention level M and upper bound L, respectively.

The following proposition gives us the tail of distribution, first and second moments of claims for each type of reinsurance when insurer effects reinsurance.

Proposition 1.1. *1) If the insurer effects proportional reinsurance with proportion α, then*

$$\overline{F}_{h,\alpha}(x) = \overline{F}(x/\alpha), \tag{1.3}$$

$$\lambda_{1,\alpha}^* = \alpha \lambda_1, \tag{1.4}$$

$$\lambda_{2,\alpha}^* = \alpha^2 \lambda_2. \tag{1.5}$$

2) If the insurer effects excess of loss reinsurance with retention level M, then

$$\overline{F}_{h,M}(x) = \begin{cases} \overline{F}(x), & x < M \\ 0, & x \geq M \end{cases}, \tag{1.6}$$

$$\lambda_{1,M}^* = G_1(M), \tag{1.7}$$

$$\lambda_{2,M}^* = G_2(M). \tag{1.8}$$

3) If the insurer effects excess stop loss reinsurance with retention level M and upper bound L, then

$$\overline{F}_{h,M,L}(x) = P\{h(X) > x\} = \begin{cases} \overline{F}(x), & x < M \\ \overline{F}(x + L), & x \geq M \end{cases}, \tag{1.9}$$

$$\lambda_{1,M,L}^* = \lambda_1 + G_1(M) - G_1(M + L), \tag{1.10}$$

$$\lambda_{2,M,L}^* = \lambda_2 + G_2(M) - G_2(M + L) - 2L\lambda_1 + 2LG_1(M + L). \tag{1.11}$$

Proof: 1) If the insurer effects proportional reinsurance with proportion α, then

$$\overline{F}_{h,\alpha}(x) = P\{h(X) > x\} = P\{\alpha X > x\} = P\{X > x/\alpha\} = \overline{F}(x/\alpha),$$

$$\lambda^*_{1,\alpha} = \int_0^\infty \overline{F}_{h,\alpha}(x)dx = \int_0^\infty \overline{F}(x/\alpha)dx = \alpha \int_0^\infty \overline{F}(x)dx = \alpha\lambda_1,$$

$$\lambda^*_{2,\alpha} = 2\int_0^\infty x\overline{F}_{h,\alpha}(x)dx = 2\int_0^\infty x\overline{F}(x/\alpha)dx = \alpha^2 \cdot 2\int_0^\infty x\overline{F}(x)dx = \alpha^2\lambda_2.$$

2) If the insurer effects excess of loss reinsurance with retention level M, then

$$\overline{F}_{h,M}(x) = P\{h(X) > x\}$$

$$= P\{h(X) > x, X > M\} + P\{h(X) > x, X \le M\}$$

$$= P\{M > x, X > M\} + P\{X > x, X \le M\}$$

$$= \begin{cases} \overline{F}(M), & x < M \\ 0, & x \ge M \end{cases} + \begin{cases} F(M) - F(x), & x < M \\ 0, & x \ge M \end{cases} = \begin{cases} \overline{F}(x), & x < M \\ 0, & x \ge M \end{cases},$$

$$\lambda^*_{1,M} = \int_0^\infty \overline{F}_{h,M}(x)dx = \int_0^M \overline{F}(x)dx = G_1(M),$$

$$\lambda^*_{2,M} = 2\int_0^\infty x\overline{F}_{h,M}(x)dx = 2\int_0^M x\overline{F}(x)dx = G_2(M).$$

3) If the insurer effects excess stop loss reinsurance with retention level M and upper bound L, then

$$\overline{F}_{h,M,L}(x) = P\{h(X) > x\}$$

$$= P\{h(X) > x, X > M+L\} + P\{h(X) > x, M < X \le M+L\} + P\{h(X) > x, X \le M\}$$

$$= P\{X - L > x, X > M+L\} + P\{M > x, M < X \le M+L\} + P\{X > x, X \le M\}$$

$$= \begin{cases} \overline{F}(M+L), & x < M \\ \overline{F}(x+L), & x \ge M \end{cases} + \begin{cases} F(M+L) - F(M), & x < M \\ 0, & x \ge M \end{cases} + \begin{cases} F(M) - F(x), & x < M \\ 0, & x \ge M \end{cases}$$

$$= \begin{cases} \overline{F}(x), & x < M \\ \overline{F}(x+L), & x \ge M \end{cases},$$

$$\lambda^*_{1,M,L} = \int_0^\infty \overline{F}_{h,M,L}(x)dx = \int_0^M \overline{F}(x)dx + \int_M^\infty \overline{F}(x+L)dx$$

$$= G_1(M) + \int_{M+L}^\infty \overline{F}(x)dx = \lambda_1 + G_1(M) - G_1(M+L),$$

$$\lambda^*_{2,M,L} = 2\int_0^\infty x\overline{F}_{h,M,L}(x)dx = 2\int_0^M x\overline{F}(x)dx + 2\int_M^\infty x\overline{F}(x+L)dx$$

$$= G_2(M) + 2\int_{M+L}^\infty x\overline{F}(x)dx - 2L\int_{M+L}^\infty \overline{F}(x)dx$$

$$= \lambda_2 + G_2(M) - G_2(M+L) - 2L\lambda_1 + 2LG_1(M+L).$$

This completes the proof of Proposition 1.1. □

It is not difficult to see that, if we take $\alpha = 1$ in proportional reinsurance or $M = \infty$ in excess of loss reinsurance and excess stop loss reinsurance, we obtain insurance without reinsurance. Also, if we take $L = \infty$ in excess stop loss reinsurance, we obtain excess of loss reinsurance. It can be also seen mathematically from the following relations:

1) Proportional reinsurance with proportion $\alpha = 1$:

$$\overline{F}_{h,\alpha=1}(x) = \overline{F}(x/1) = \overline{F}(x),$$

$$\lambda^*_{1,\alpha=1} = 1 \cdot \lambda_1 = \lambda_1,$$

$$\lambda^*_{2,\alpha=1} = 1^2 \cdot \lambda_2 = \lambda_2.$$

2) Excess of loss reinsurance with retention level $M = \infty$:

$$\overline{F}_{h,M=\infty}(x) = \begin{cases} \overline{F}(x), & x < \infty \\ 0, & x = \infty \end{cases} = \overline{F}(x),$$

$$\lambda^*_{1,M=\infty} = \lim_{M\to\infty} G_1(M) = \lambda_1,$$

$$\lambda^*_{2,M=\infty} = \lim_{M\to\infty} G_2(M) = \lambda_2.$$

3) Excess stop loss reinsurance with retention level $M = \infty$ and upper bound L:

$$\overline{F}_{h,M=\infty,L}(x) = \begin{cases} \overline{F}(x), & x < \infty \\ \overline{F}(x+L), & x = \infty \end{cases} = \begin{cases} \overline{F}(x), & x < \infty \\ 0, & x = \infty \end{cases} = \overline{F}(x),$$

$$\lambda^*_{1,M=\infty,L} = \lim_{M\to\infty} (\lambda_1 + G_1(M) - G_1(M+L)) = \lambda_1 + \lambda_1 - \lambda_1 = \lambda_1.$$

Since λ_1 exists, then $\int_t^\infty \overline{F}(x)dx \to 0$ as $t \to \infty$. So,

$$\lambda^*_{2,M=\infty,L} = \lim_{M\to\infty} (\lambda_2 + G_2(M) - G_2(M+L) - 2L\lambda_1 + 2LG_1(M+L)) =$$

$$= \lambda_2 + \lambda_2 - \lambda_2 - 2L \lim_{M\to\infty} \int_{M+L}^\infty \overline{F}(x)dx = \lambda_2.$$

4) Excess stop loss reinsurance with retention level M and upper bound $L = \infty$:

$$\overline{F}_{h,M,L=\infty}(x) = \lim_{L\to\infty} \begin{cases} \overline{F}(x), & x < M \\ \overline{F}(x+L), & x \geq M \end{cases} = \begin{cases} \overline{F}(x), & x < M \\ 0, & x \geq M \end{cases} = \overline{F}_{h,M}(x),$$

$$\lambda^*_{1,M,L=\infty} = \lim_{L\to\infty} (\lambda_1 + G_1(M) - G_1(M+L)) = \lambda_1 + G_1(M) - \lambda_1 = G_1(M) = \lambda^*_{1,M}.$$

Since λ_2 exists, then $0 \leq t \int_t^\infty \overline{F}(x)dx \leq \int_t^\infty x\overline{F}(x)dx \to 0$ as $t \to \infty$. So,

$$\lambda^*_{2,M,L=\infty} = \lim_{L\to\infty} (\lambda_2 + G_2(M) - G_2(M+L) - 2L\lambda_1 + 2LG_1(M+L)) =$$

$$= \lambda_2 + G_2(M) - \lambda_2 - 2 \lim_{L\to\infty} \left(L \int_{M+L}^\infty \overline{F}(x)dx \right) = G_2(M) = \lambda^*_{2,M}.$$

2 Asymptotics for the expected value of the reinsurance surplus process

In this section we derive asymptotics for the expected value of $U^*(t)$. For this, let us introduce the following definition from the literature [10]:

Definition 2.1. *We call F a non-lattice distribution function if and only if $f(\theta) \neq 1$ for $\theta \neq 0$, and as a special case, we call F strongly non-lattice if*

$$\liminf_{|\theta|\to\infty} \mid 1 - f(\theta) \mid > 0,$$

where $f(\theta)$ is the characteristic function of F defined by

$$f(\theta) = \int_{-\infty}^\infty e^{i\theta x} dF(x).$$

The following theorem describes asymptotic expansions for the mathematical expectation of the reinsurance surplus process in each type of the reinsurance.

Theorem 2.1. *If distribution function of T is a strongly non-lattice and $\lambda^*_1 = E\{X^*\}$, $\mu_{k+2} = E\{T^{k+2}\}$, $k \geq 0$ exist, then the following asymptotic expansions as $t \to \infty$ can be written for the mathematical expectation of the reinsurance surplus process depending on the type of reinsurance:*

1) for the proportional reinsurance with proportion α:

$$E\{U^*(t)\} = \left(c^* - \frac{\alpha\lambda_1}{\mu_1}\right)t + u - \frac{\alpha\lambda_1(\mu_2 - 2\mu_1^2)}{2\mu_1^2} + o(t^{-k}), \tag{2.1}$$

2) for the excess of loss reinsurance with retention level M:

$$E\{U^*(t)\} = \left(c^* - \frac{G_1(M)}{\mu_1}\right)t + u - \frac{G_1(M)(\mu_2 - 2\mu_1^2)}{2\mu_1^2} + o(t^{-k}), \tag{2.2}$$

3) for the excess stop loss reinsurance with retention level M and upper bound L:

$$E\{U^*(t)\} = \left(c^* - \frac{\lambda_1 + G_1(M) - G_1(M+L)}{\mu_1}\right)t + u - \frac{(\lambda_1 + G_1(M) - G_1(M+L))(\mu_2 - 2\mu_1^2)}{2\mu_1^2} + o(t^{-k}). \tag{2.3}$$

Proof: From (1.2) we can write the following:

$$E\{U^*(t)\} = u + c^*t - E\left\{\sum_{i=1}^{N(t)} X_i^*\right\}, \tag{2.4}$$

Since X^* and T are independent, using Wald's identity (see, for example, [5]) we can write

$$E\left\{\sum_{i=1}^{N(t)} X_i^*\right\} = E\{N(t)\}E\{X^*\} = \lambda_1^* H(t) \tag{2.5}$$

where $H(t) = E\{N(t)\}$ is a renewal function. Under the conditions of this theorem a second-order approximation for the renewal function has been derived in [10]:

$$H(t) = \frac{t}{\mu_1} + \frac{\mu_2 - 2\mu_1^2}{2\mu_1^2} + o(t^{-k}), \quad t \to \infty. \tag{2.6}$$

Taking into account (2.6) in (2.5) we obtain

$$E\left\{\sum_{i=1}^{N(t)} X_i^*\right\} = \frac{\lambda_1^*}{\mu_1}t + \frac{\lambda_1^*(\mu_2 - 2\mu_1^2)}{2\mu_1^2} + o(t^{-k}), \quad t \to \infty. \tag{2.7}$$

Then taking into account (2.7) in (2.4) we obtain

$$E\{U^*(t)\} = \left(c^* - \frac{\lambda_1^*}{\mu_1}\right)t + u - \frac{\lambda_1^*(\mu_2 - 2\mu_1^2)}{2\mu_1^2} + o(t^{-k}), \quad t \to \infty. \tag{2.8}$$

For the proportional reinsurance with proportion α, taking into account (1.4) in (2.8) we can obtain (2.1), for the excess of loss reinsurance with retention level M, taking into account (1.7) in (2.8) we can obtain (2.2) and for the excess stop loss reinsurance with retention level M and upper bound L, taking into account (1.10) in (2.8) we can obtain (2.3).

This completes the proof of Theorem 2.1. □

3 Asymptotics for the variance of the reinsurance surplus process

In this section we derive asymptotics for the variance of $U^*(t)$. For this, let us introduce the following definition from the literature ([6]):

Definition 3.1. *A distribution function F is said to belong to the class ϑ if some convolution of F has an absolutely continuous component.*

The following theorem describes asymptotic expansions for the variance of the reinsurance surplus process in each type of the reinsurance.

Theorem 3.1. *If distribution function of T belongs to the class ϑ and $\lambda_2^* = E\{(X^*)^2\}$, $\mu_{k+3} = E\{T^{k+3}\}$, $k \geq 0$ exist, then the following asymptotic expansions as $t \to \infty$ can be written for the variance of the reinsurance surplus process depending on the type of reinsurance:*

1) for the proportional reinsurance with proportion α:

$$Var\{U^*(t)\} = \alpha^2\bigg(\frac{\lambda_1^2\mu_2}{\mu_1^3} + \frac{\lambda_2 - 2\lambda_1^2}{\mu_1}\bigg)t + \alpha^2\bigg(\frac{5\lambda_1^2\mu_2^2}{4\mu_1^4} - \frac{2\lambda_1^2\mu_3}{3\mu_1^3} + \frac{(\lambda_2 - 2\lambda_1^2)\mu_2}{2\mu_1^2} - \lambda_2 + \lambda_1^2\bigg) + o(t^{-k}), \tag{3.1}$$

2) for the excess of loss reinsurance with retention level M:

$$\begin{aligned} Var\{U^*(t)\} = \bigg(\frac{(G_1(M))^2\mu_2}{\mu_1^3} + \frac{G_2(M) - 2(G_1(M))^2}{\mu_1}\bigg)t + \\ + \frac{5(G_1(M))^2\mu_2^2}{4\mu_1^4} - \frac{2(G_1(M))^2\mu_3}{3\mu_1^3} + \frac{(G_2(M) - 2(G_1(M))^2)\mu_2}{2\mu_1^2} - G_2(M) + (G_1(M))^2 + o(t^{-k}), \end{aligned} \tag{3.2}$$

3) for the excess stop loss reinsurance with retention level M and upper bound L:

$$\begin{aligned} Var\{U^*(t)\} = \bigg(\frac{(\lambda_1 + G_1(M) - G_1(M+L))^2\mu_2}{\mu_1^3} \\ + \frac{\lambda_2 + G_2(M) - G_2(M+L) - 2L\lambda_1 + 2LG_1(M+L) - 2(\lambda_1 + G_1(M) - G_1(M+L))^2}{\mu_1}\bigg)t \\ + \frac{5(\lambda_1 + G_1(M) - G_1(M+L))^2\mu_2^2}{4\mu_1^4} - \frac{2(\lambda_1 + G_1(M) - G_1(M+L))^2\mu_3}{3\mu_1^3} \\ + \frac{(\lambda_2 + G_2(M) - G_2(M+L) - 2L\lambda_1 + 2LG_1(M+L) - 2(\lambda_1 + G_1(M) - G_1(M+L))^2)\mu_2}{2\mu_1^2} \\ -(\lambda_2 + G_2(M) - G_2(M+L) - 2L\lambda_1 + 2LG_1(M+L)) + (\lambda_1 + G_1(M) - G_1(M+L))^2 + o(t^{-k}), \end{aligned} \tag{3.3}$$

Proof: From (1.2) we can write the following:

$$Var\{U^*(t)\} = Var\bigg\{\sum_{i=1}^{N(t)} X_i^*\bigg\}. \tag{3.4}$$

Using (3.4) and Wald's second identity (see, for example, [5]) we get

$$Var\{U^*(t)\} = E\{N(t)\}Var\{X^*\} + (E\{X^*\})^2 Var\{N(t)\}. \tag{3.5}$$

For $Var\{N(t)\}$ we can write (see, [3], Corollary 2.3)

$$Var\{N(t)\} = \bigg(\frac{\mu_2}{\mu_1^3} - \frac{1}{\mu_1}\bigg)t + \frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{\mu_2}{2\mu_1^2} + o(t^{-k}), \quad t \to \infty. \tag{3.6}$$

Since $Var\{X^*\} = \lambda_2^* - (\lambda_1^*)^2$, $E\{X^*\} = \lambda_1^*$, then taking into account (2.6) and (3.6) in (3.5), we can write

$$\begin{aligned} Var\{U^*(t)\} = \bigg(\frac{t}{\mu_1} + \frac{\mu_2 - 2\mu_1^2}{2\mu_1^2} + o(t^{-k})\bigg)(\lambda_2^* - (\lambda_1^*)^2) \\ +(\lambda_1^*)^2\bigg(\bigg(\frac{\mu_2}{\mu_1^3} - \frac{1}{\mu_1}\bigg)t + \frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{\mu_2}{2\mu_1^2} + o(t^{-k})\bigg) \\ = \bigg(\frac{\lambda_2^* - (\lambda_1^*)^2}{\mu_1} + (\lambda_1^*)^2\bigg(\frac{\mu_2}{\mu_1^3} - \frac{1}{\mu_1}\bigg)\bigg)t + (\lambda_2^* - (\lambda_1^*)^2)\bigg(\frac{\mu_2}{2\mu_1^2} - 1\bigg) \\ +(\lambda_1^*)^2\bigg(\frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{\mu_2}{2\mu_1^2}\bigg) + o(t^{-k}) = \bigg(\frac{(\lambda_1^*)^2\mu_2}{\mu_1^3} + \frac{\lambda_2^* - 2(\lambda_1^*)^2}{\mu_1}\bigg)t \\ + \frac{5(\lambda_1^*)^2\mu_2^2}{4\mu_1^4} - \frac{2(\lambda_1^*)^2\mu_3}{3\mu_1^3} + \frac{(\lambda_2^* - 2(\lambda_1^*)^2)\mu_2}{2\mu_1^2} - \lambda_2^* + (\lambda_1^*)^2 + o(t^{-k}), \quad t \to \infty. \end{aligned} \tag{3.7}$$

For the proportional reinsurance with proportion α, taking into account (1.4) and (1.5) in (3.7) we can obtain (3.1), for the excess of loss reinsurance with retention level M, taking into account (1.7) and (1.8) in (3.7) we can obtain (3.2) and for the excess stop loss reinsurance with retention level M and upper bound L, taking into account (1.10) and (1.11) in (3.7) we can obtain (3.3).

This completes the proof of Theorem 3.1. □

4 Numerical examples

In this section we give numerical examples to compare insurance surplus process and reinsurance surplus procces. For this, we take $u = 100$ as the initial capital of insurance company, $c = 0,6$ as the premium rate. Also, we assume that the distribution of T is an Erlang distribution with shape parameter $n = 2$ and scale parameter $\beta = 1$, and the distribution of X is an exponential distribution with scale parameter $\beta = 1$. In this case, $\mu_1 = 2$, $\mu_2 = 6$, $\mu_3 = 24$ and $\lambda_1 = 1$, $\lambda_2 = 2$. It is not difficult to see that safety loading condition holds:

$$\rho = \frac{c\mu_1}{\lambda_1} - 1 = 0,2 > 0.$$

In the following table the expected value and variance for different values of time are calcluated when the insurer does not effect reinsurance:

μ_1	μ_2	μ_3	λ_1	λ_2	c	$t = 10$		$t = 100$		$t = 1000$	
						Exp. value	**Variance**	**Exp. value**	**Variance**	**Exp. value**	**Variance**
2	6	24	1	2	$0,6$	$101,25$	$7,3125$	$110,25$	$74,8125$	$200,25$	$749,8125$

Table 4.1: No reinsurance

We let ρ to be a constant for every surplus process. Thus we can calculate c^* from the formula of ρ for each type of reinsurance surplus process:

$$c^* = \frac{\lambda_1^*(\rho + 1)}{\mu_1}.$$

In the following table the expected value and variance for different values of time are calcluated when the insurer effects proportional reinsurance with different values of α:

α	λ_1^*	λ_2^*	c^*	$t = 10$		$t = 100$		$t = 1000$	
				Exp. value	**Variance**	**Exp. value**	**Variance**	**Exp. value**	**Variance**
$0,5$	$0,5$	$0,5$	$0,3$	$100,625$	$1,8281$	$105,125$	$18,7031$	$150,125$	$187,4531$
$0,65$	$0,65$	$0,845$	$0,39$	$100,8125$	$3,0895$	$106,6625$	$31,6083$	$165,1625$	$316,7958$
$0,87$	$0,87$	$1,5138$	$0,522$	$101,0875$	$5,5348$	$108,9175$	$56,6256$	$187,2175$	$567,5331$

Table 4.2: Proportional reinsurance

In the following table the expected value and variance for different values of time are calcluated when the insurer effects excess of loss reinsurance with different values of M:

M	λ_1^*	λ_2^*	c^*	$t = 10$		$t = 100$		$t = 1000$	
				Exp. value	**Variance**	**Exp. value**	**Variance**	**Exp. value**	**Variance**
1	$0,6321$	$0,5285$	$0,3793$	$100,7902$	$1,6362$	$106,4792$	$16,4275$	$163,3701$	$164,3398$
2	$0,8647$	$1,188$	$0,5188$	$101,0808$	$4,0075$	$108,8628$	$40,6449$	$186,6826$	$407,0195$
3	$0,9502$	$1,6017$	$0,5701$	$101,1878$	$5,633$	$109,7397$	$57,3942$	$195,2588$	$575,0073$

Table 4.3: Excess of loss reinsurance

In the following table the expected value and variance for different values of time are calcluated when the insurer effects excess stop loss reinsurance with different values of M and L:

M	L	λ_1^*	λ_2^*	c^*	$t = 10$		$t = 100$		$t = 1000$	
					Exp. value	**Variance**	**Exp. value**	**Variance**	**Exp. value**	**Variance**
1	1	$0,7675$	$1,0698$	$0,4605$	$100,9593$	$3,7932$	$107,8664$	$38,683$	$176,9374$	$387,5812$
1	2	$0,6819$	$0,7276$	$0,4091$	$100,8524$	$2,4391$	$106,9896$	$24,72$	$168,3612$	$247,5292$
2	1	$0,9145$	$1,4867$	$0,5487$	$101,1431$	$5,2326$	$109,3731$	$53,3196$	$191,6738$	$534,1895$
2	2	$0,883$	$1,2979$	$0,5298$	$101,1037$	$4,4594$	$109,0505$	$45,3219$	$188,5188$	$453,9467$

Table 4.4: Excess stop loss reinsurance

It can be easily seen from the tables that when the insurer effects reinsurance there does not happen big decrease in expected value in comparison with no reinsurance. But there happens a distinguishable decrease in variance which can be accepted as a good result.

5 Conclusion

In this paper we consider insurance surplus process when insurer effects reinsurance and derive asymptotic results for the mathematical expectation and variance of this process. Depending on the type of the reinsurance we first give formulas for the distribution functions and moments of claims in reinsurance surplus process, then using these moments we give the asymptotic results for the mathematical expectation and variance in each type of the reinsurance.

Finally, we give numerical examples to compare the values of mathematical expectation and variance when there is no reinsurance and when the insurer effects reinsurance.

6 References

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