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Research Article

Estimation of stress-strength probability in a multicomponent model based on geometric distribution

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Abstract

In this paper, the estimation of the stress-strength probability in a multicomponent model, in the case when all components follow the geometric distribution, is studied. This is the first time that multicomponent models with discrete probability distributions are considered. The MLE, UMVUE and Bayes point estimator, as well as asymptotic and bootstrap confidence intervals are presented. A simulation study is performed in order to compare the performance of various estimators. Finally, the methods are applied to real data examples from climatology and sport.

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1. Introduction

Stress-strength models have been popular for many years, mainly due to their applicability in engineering, meteorology, quality control, medicine, etc. The basic interpretation of the stress-strength probability is the reliability parameter of a system. In the simplest stress-strength model the system fails if the applied stress X is greater than strength Y, so the reliability parameter $R = P\{X \leq Y\}$ is a measure of system performance. In a broader interpretation, R can be viewed as a measure of difference between two populations. For example, in medicine, if Y represents the response of a treatment group, and X refers to a control group, R is a measure of the effect of the treatment.

Estimation of R is one of the main goals and it has been widely studied in statistical literature (see [17] for an excellent review of theory and applications in this area). The majority of papers deal with continuous probability distributions. However, there are some applications where stress and strength can have discrete distributions. For example, this is the case when the stress is the number of shocks the product undergoes and the strength is the number of shocks the product can withstand. The geometric distribution case was studied in [1, 18], the negative binomial distribution was considered in [9, 25], the Poisson distribution case was examined in [3], while the logarithmic distribution was considered

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in [19]. Recently, the case when the stress is from the geometric and the strength is from the Poisson distribution was investigated in [20], while the geometric-exponential model was studied in [10].

A system having more than one stress or strength component is called a multicomponent system. The most common is "s-out-of-k: G" system. In this model there is one stress X and k strength components $Y_1, Y_2, \dots, Y_k$. A system functions if at least s, $1 \leq s \leq k$, strength components are greater or equal than the stress component. In this setting, the reliability parameter is $R_{s,k}$, where $R_{s,k} = P\{\text{at least s of the } (Y_1, Y_2, \dots, Y_k) \geq X\}$. Parallel and series circuits are examples of extreme 1-out-of-k and k-out-of-k multicomponent systems. A four-engine airplane which can operate when at least two of its engines work is an example of 2-out-of-4 system.

The reliability in a multicomponent stress-strength model was initially studied in [4]. Recently, this topic has gained popularity and quite a few papers have been published. The estimation of $R_{s,k}$ for the generalized exponential distribution was examined in [22]. The cases of Burr-XII, Weibull and two parameter exponentiated Weibull distribution were studied in [14,23,24], respectively. The Kumaraswamy distribution was considered in [5,15,16]. General classes of inverse exponentiated and exponentiated inverted exponential distributions were investigated in [7,13], respectively. The power Lindley distribution was examined in [21], the Chen distribution was considered in [11], while the Topp-Leone case was investigated in [2].

All mentioned papers are devoted to multicomponent models with continuous probability distributions. In this paper, for the first time, we focus on the case when both stress and strength components follow discrete distributions. In particular, we consider the geometric distribution.

A motivation for such model can be found, e.g. in hydrology and climatology, when modelling durations of various phenomena, such as droughts, floods, warm and cold spells, etc. There, the number of time intervals (days, months, etc.) during which the process continuously remains above or below a reference level is frequently modelled with the geometric distribution. See, for instance, [12] and [6].

With increasing popularity of the climate change issue, it is of interest to estimate the probability that a duration of some phenomenon will not exceed a certain threshold, set as a quantile of its distribution in the past. For example, the probability that the duration of the next warm spell will not exceed 80th percentile of duration of warm spell in the past is $R_{1,5}$. We illustrate this in a real data example in Section 7.

If a random variable X has geometric $\mathcal{G}(p)$ distribution, then its probability mass and distribution functions are given by

$$P\{X = x; p\} = (1-p)^{x-1}p, \quad x \in \mathbb{N},$$
$$F_X(x) = 1 - (1-p)^x, \quad x \in \mathbb{N}.$$

Let the stress X and the strengths Y_i, $i \in \{1, 2, \dots, k\}$, be independent random variables such that X follows $\mathcal{G}(p_1)$ and each Y_i follows $\mathcal{G}(p_2)$ distribution. Then, the stress-strength probability is

$$\begin{aligned}
R_{s,k} &= P\{\text{at least s of the } (Y_1, Y_2, \dots, Y_k) \geq X\} \\
&= \sum_{i=s}^{k} \sum_{x=1}^{\infty} P\{X = x, \text{ exactly i of the } (Y_1, Y_2, \dots, Y_k) \geq x\} \\
&= \sum_{i=s}^{k} \sum_{x=1}^{\infty} (1-p_1)^{x-1} p_1 \binom{k}{i} \big(F_Y(x-1)\big)^{k-i} \big(1 - F_Y(x-1)\big)^i \\
&= \sum_{i=s}^{k} \sum_{x=1}^{\infty} (1-p_1)^{x-1} p_1 \binom{k}{i} \big((1-p_2)^{x-1}\big)^i \sum_{j=0}^{k-i} \binom{k-i}{j} (-1)^j \big((1-p_2)^{x-1}\big)^j
\end{aligned}$$

$$= \sum_{i=s}^{k}\sum_{j=0}^{k-i}(-1)^j\binom{k}{i}\binom{k-i}{j}p_1\sum_{x=1}^{\infty}\left((1-p_1)(1-p_2)^{i+j}\right)^{x-1}$$
$$= \sum_{i=s}^{k}\sum_{j=0}^{k-i}(-1)^j\binom{k}{i}\binom{k-i}{j}\frac{p_1}{1-(1-p_1)(1-p_2)^{i+j}}. \tag{1.1}$$

From this formula it is not easy to see how $R_{s,k}$ depends on p_1 and p_2, so this dependence, for some choices of s and k, is shown on Figure 1.

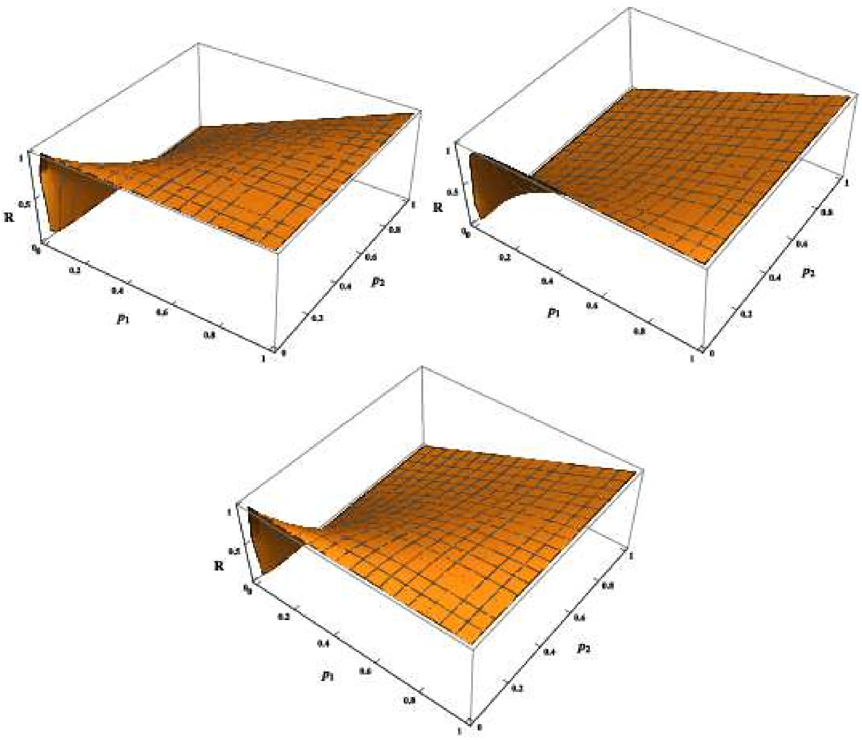

Figure 1. $R_{1,3}$ (top left), $R_{3,3}$ (top right), $R_{3,6}$ (bottom)

The rest of the paper is organized as follows. In section 2 the maximum likelihood estimator (MLE) of $R_{s,k}$ and its asymptotic distribution are derived. Based on it, in section 3, the asymptotic and bootstrap-p confidence intervals are constructed. The uniformly minimum variance unbiased estimator (UMVUE) of $R_{s,k}$ is obtained in section 4. Bayes estimator of $R_{s,k}$ with respect to square loss function is derived in section 5. In section 6 we perform a simulation study and compare the obtained estimators, while in section 7 we present two real data examples.

2. MLE of $R_{s,k}$ and its asymptotic distribution

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_m)$ be the samples from $\mathcal{G}(p_1)$ and $\mathcal{G}(p_2)$ distributions. The MLE's of p_1 and p_2 are

$$\widetilde{p_1} = \frac{1}{\overline{X}_n}, \quad \widetilde{p_2} = \frac{1}{\overline{Y}_m}. \tag{2.1}$$

Using the invariance property of MLE, from equation (1.1) we get the MLE of $R_{s,k}$

$$\widetilde{R}_{s,k} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}(-1)^j\binom{k}{i}\binom{k-i}{j}\frac{\widetilde{p}_1}{1-(1-\widetilde{p}_1)(1-\widetilde{p}_2)^{i+j}}. \tag{2.2}$$

In the following theorem we derive the asymptotic distribution of $\widetilde{R}_{s,k}$.

Theorem 2.1. *Let the ratio $\frac{n}{m}$ converge to a positive number λ when both n and m tend to infinity. Then*

$$\sqrt{n}(\widetilde{R}_{s,k} - R_{s,k}) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$$

when both n and m tend to infinity, where

$$\sigma^2 = p_1^2(1-p_1)\Big(\sum_{i=s}^{k}\sum_{j=0}^{k-i}(-1)^j\binom{k}{i}\binom{k-i}{j}\frac{1-(1-p_2)^{i+j}}{(1-(1-p_1)(1-p_2)^{i+j})^2}\Big)^2$$
$$+\lambda p_2^2(1-p_2)\Big(\sum_{i=s}^{k}\sum_{j=0}^{k-i}(-1)^{j+1}\binom{k}{i}\binom{k-i}{j}\frac{(i+j)p_1(1-p_1)(1-p_2)^{i+j-1}}{(1-(1-p_1)(1-p_2)^{i+j})^2}\Big)^2. \quad (2.3)$$

Proof. Since

$$-\frac{\partial^2 \ln P\{X=x\}}{\partial p_1^2} = \frac{x-1}{(1-p_1)^2} + \frac{1}{p_1^2}$$

and

$$E\Big(\frac{X-1}{(1-p_1)^2} + \frac{1}{p_1^2}\Big) = \frac{1}{p_1^2(1-p_1)},$$

from the asymptotic normality of maximum likelihood estimator (see [8]) it follows that

$$\sqrt{n}(\widetilde{p}_1 - p_1) \xrightarrow{d} \mathcal{N}(0, p_1^2(1-p_1))$$

when $n \to \infty$ and analogously

$$\sqrt{m}(\widetilde{p}_2 - p_2) \xrightarrow{d} \mathcal{N}(0, p_2^2(1-p_2))$$

when $m \to \infty$. Then

$$\sqrt{n}(\widetilde{p}_2 - p_2) = \sqrt{\frac{n}{m}}\sqrt{m}(\widetilde{p}_2 - p_2) \xrightarrow{d} \mathcal{N}(0, \lambda p_2^2(1-p_2))$$

when both n and m tend to infinity. From the independence of $\widetilde{p}_1$ and $\widetilde{p}_2$ we get

$$(\sqrt{n}(\widetilde{p}_1 - p_1), \sqrt{n}(\widetilde{p}_2 - p_2)) \xrightarrow{d} \mathcal{N}_2(\mathbf{0}, J(p_1, p_2))$$

when both n and m tend to infinity, where

$$J(p_1,p_2) = \begin{bmatrix} p_1^2(1-p_1) & 0 \\ 0 & \lambda p_2^2(1-p_2) \end{bmatrix}.$$

Since $R_{s,k} = R_{s,k}(p_1,p_2)$ is the transformation such that the matrix of partial derivatives

$$B = \begin{bmatrix} \frac{\partial R_{s,k}}{\partial p_1} & \frac{\partial R_{s,k}}{\partial p_2} \end{bmatrix},$$

where

$$\frac{\partial R_{s,k}}{\partial p_1} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}(-1)^j\binom{k}{i}\binom{k-i}{j}\frac{1-(1-p_2)^{i+j}}{(1-(1-p_1)(1-p_2)^{i+j})^2},$$
$$\frac{\partial R_{s,k}}{\partial p_2} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}(-1)^{j+1}\binom{k}{i}\binom{k-i}{j}\frac{(i+j)p_1(1-p_1)(1-p_2)^{i+j-1}}{(1-(1-p_1)(1-p_2)^{i+j})^2},$$

has continuous elements and does not vanish in the neighbourhood of (p_1,p_2), using the method from [8](Corollary 6.4.1.) we have

$$\sqrt{n}(\widetilde{R}_{s,k} - R_{s,k}) \xrightarrow{d} \mathcal{N}(0, BJB^T)$$

when both n and m tend to infinity. Inserting the values of B and J we get the statement of the theorem. □

3. Confidence intervals

3.1. Asymptotic confidence interval

Using theorem 2.1 the asymptotic confidence interval for $R_{s,k}$ can be constructed. From equation (2.3), using the invariance property, the MLE estimator of σ^2 is

$$\widetilde{\sigma}^2 = \widetilde{p}_1^2(1-\widetilde{p}_1)\Bigg(\sum_{i=s}^{k}\sum_{j=0}^{k-i}(-1)^j\binom{k}{i}\binom{k-i}{j}\frac{1-(1-\widetilde{p}_2)^{i+j}}{(1-(1-\widetilde{p}_1)(1-\widetilde{p}_2)^{i+j})^2}\Bigg)^2$$
$$+\lambda\widetilde{p}_2^2(1-\widetilde{p}_2)\Bigg(\sum_{i=s}^{k}\sum_{j=0}^{k-i}(-1)^{j+1}\binom{k}{i}\binom{k-i}{j}\frac{(i+j)\widetilde{p}_1(1-\widetilde{p}_1)(1-\widetilde{p}_2)^{i+j-1}}{(1-(1-\widetilde{p}_1)(1-\widetilde{p}_2)^{i+j})^2}\Bigg)^2.$$

Then, the asymptotic interval of confidence level $1-\alpha$ for $R_{s,k}$ is given by

$$I_{R_{s,k}}^{(ASYM)} = \left(\widetilde{R}_{s,k} - z_{1-\frac{\alpha}{2}}\frac{\widetilde{\sigma}}{\sqrt{n}}, \widetilde{R}_{s,k} + z_{1-\frac{\alpha}{2}}\frac{\widetilde{\sigma}}{\sqrt{n}}\right), \tag{3.1}$$

where z_γ is the γth quantile from standard normal distribution.

3.2. Bootstrap-p confidence interval

For small sample sizes asymptotic confidence intervals do not perform very well, so we propose a construction of the confidence interval based on parametric bootstrap-p method. The algorithm is illustrated below.

Step 1: From initial samples $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_m)$ calculate MLEs $\widetilde{p_1}$ and $\widetilde{p_2}$ using equations (2.1).

Step 2: Use those estimates to generate bootstrap sample $\mathbf{x_i^*}$ from geometric $\mathcal{G}(\widetilde{p}_1)$ distribution and bootstrap sample $\mathbf{y_i^*}$ from geometric $\mathcal{G}(\widetilde{p}_2)$ distribution. Based on these bootstrap samples compute estimates $\widetilde{p}_1^{*i}$ and $\widetilde{p}_2^{*i}$, using equations (2.1), and $\widetilde{R}_{s,k}^{*i}$ of $R_{s,k}$ using equation (2.2).

Step 3: Repeat step 2, N boot times.

Step 4: Let $\widetilde{R}_{s,k}^{*(\gamma)}$ be the γth empirical quantile of the $\widetilde{R}_{s,k}^{*i}$ values obtained in step 3, that is, the $N\gamma$th value in the ordered list of the N replications of $\widetilde{R}_{s,k}^{*i}$. If $N\gamma$ is not an integer, assuming $\gamma \leq 0.5$, the largest integer less or equal $(N+1)\gamma$ should be used. The bootstrap-p interval of confidence level $1-\alpha$ for $R_{s,k}$ is given by

$$I_{R_{s,k}}^{(BOOTp)} = \left(\widetilde{R}_{s,k}^{*(\frac{\alpha}{2})}, \widetilde{R}_{s,k}^{*(1-\frac{\alpha}{2})}\right).$$

4. UMVUE of $R_{s,k}$

In this section we find the UMVUE of $R_{s,k}$, denoted by $\widehat{R}_{s,k}$. The complete sufficient statistics for p_1 and p_2 are $T_X = \sum_{j=1}^{n} X_j$ and $T_Y = \sum_{j=1}^{m} Y_j$. The statistic T_X, as a sum of n independent identically distributed random variables with geometric distribution, has negative binomial distribution with parameters n and p_1, and the statistic T_Y, analogously, has negative binomial distribution with parameters m and p_2. Their probability mass functions are

$$P\{T_X = t_X\} = \binom{t_X-1}{n-1}p_1^n(1-p_1)^{t_X-n}, \quad t_X \geq n;$$
$$P\{T_Y = t_Y\} = \binom{t_Y-1}{m-1}p_2^m(1-p_2)^{t_Y-m}, \quad t_Y \geq m.$$

An unbiased estimator for $R_{s,k}$ is $I\{\text{at least s of the } (Y_1, Y_2, \ldots, Y_k) \geq X_1\}$. Because of this the sample size m must be greater or equal to k. Then

$$\begin{aligned}
&E(I\{\text{at least s of the } (Y_1, Y_2, \ldots, Y_k) \geq X_1\}|T_X = t_X, T_Y = t_Y)\\
=&P\{\text{at least s of the } (Y_1, Y_2, \ldots, Y_k) \geq X_1\}|T_X = t_X, T_Y = t_Y\}\\
=&\sum_{i=s}^{k} \frac{P\{\text{ exactly i of the } (Y_1, Y_2, \ldots, Y_k) \geq X_1, T_X = t_X, T_Y = t_Y\}}{P\{T_X = t_X, T_T = t_Y\}}\\
=&\frac{1}{P\{T_X = t_X\}P\{T_T = t_Y\}} \sum_{i=s}^{k} \sum_{x_1=1}^{t_X-n+1} \Bigg[P\{X_1 = x_1\}P\Big\{\sum_{j=2}^{n} X_j = t_X - x_1\Big\}\\
&\cdot P\{\text{ exactly i of the } (Y_1, Y_2, \ldots, Y_k) \geq x_1, T_Y = t_Y\}\Bigg]\\
=&\frac{1}{\binom{t_X-1}{n-1}p_1^n(1-p_1)^{t_X-n}\binom{t_Y-1}{m-1}p_2^m(1-p_2)^{t_Y-m}} \sum_{i=s}^{k} \sum_{x_1=1}^{t_X-n+1} \Bigg[(1-p_1)^{x_1-1}p_1\\
&\cdot \binom{t_X - x_1 - 1}{n-2} p_1^{n-1}(1-p_1)^{t_X-x_1-n+1}\\
&\cdot P\{\text{ exactly i of the } (y_1, y_2, \ldots, y_k) \geq x_1, \sum_{i=1}^{m} y_i = t_Y, Y_1 = y_1, Y_2 = y_2, \ldots, Y_m = y_m\}\Bigg]\\
=&\frac{1}{\binom{t_X-1}{n-1}\binom{t_Y-1}{m-1}p_2^m(1-p_2)^{t_Y-m}} \sum_{i=s}^{k} \sum_{x_1=1}^{t_X-n+1} \Bigg[\binom{t_X - x_1 - 1}{n-2}\binom{k}{i} A_i \prod_{j=1}^{m}\Big[(1-p_2)^{y_j-1}p_2\Big]\Bigg]\\
=&\frac{1}{\binom{t_X-1}{n-1}\binom{t_Y-1}{m-1}p_2^m(1-p_2)^{t_Y-m}} \sum_{i=s}^{k} \sum_{x_1=1}^{t_X-n+1} \Bigg[\binom{t_X - x_1 - 1}{n-2}\binom{k}{i} A_i (1-p_2)^{t_Y-m}p_2^m\Bigg]\\
=&\frac{\sum_{i=s}^{k}\sum_{x_1=1}^{t_X-n+1}\binom{t_X-x_1-1}{n-2}\binom{k}{i}A_i}{\binom{t_X-1}{n-1}\binom{t_Y-1}{m-1}},
\end{aligned} \tag{4.1}$$

where A_i is the number of solutions, on the set of natural numbers, of linear equation $y_1 + y_2 + \ldots + y_m = t_Y$ with constraints $y_1 \geq x_1, \ldots, y_i \geq x_1, y_{i+1} \leq x_1 - 1, \ldots, y_k \leq x_1 - 1$ (without loss of generality we can take the first i variables to be constrained by x_1). Denote $z_1 = y_1 - (x_1 - 1), \ldots, z_i = y_i - (x_1 - 1), z_{i+1} = y_{i+1}, \ldots, z_m = y_m$. Then, A_i is the number of solutions, on the set of natural numbers, of linear equation $z_1 + z_2 + \ldots + z_m = t_Y - i(x_1 - 1)$ with constraints $z_{i+1} \leq x_1 - 1, \ldots, z_k \leq x_1 - 1$. If we now denote with Ω the set of solutions of this linear equation without constrains, and with B_j the set of solutions of this linear equation with constraint $z_j \geq x_1$, where $j \in \{i+1, \ldots, k\}$, then $A_i = |\Omega \setminus (B_{i+1} \cup B_{i+2} \cdots \cup B_k)|$.

A linear equation of the form $w_1 + w_2 + \ldots + w_r = d$, where $d \geq r > 1$ and w_i, $i \in \{1, 2, \ldots, r\}$, are natural numbers, has $\binom{d-1}{r-1}$ solutions. The same linear equation $w_1 + w_2 + \ldots + w_r = d$, but with a constraint $w_j \geq a$, for some $j \in \{1, \ldots, r\}$, where a is a natural number, has the same number of solutions as the linear equation $w_1 + w_2 + \ldots + w'_j + \ldots + w_r = d - (a-1)$, where $w'_j = w_j - (a-1)$, and that number is $\binom{d-(a-1)-1}{r-1}$. Similarly, linear equation $w_1 + w_2 + \ldots + w_r = d$, with constraints $w_{j_1} \geq a, \ldots, w_{j_l} \geq a$, for some distinct $j_1, \ldots, j_l$ from $\{1, \ldots, r\}$, where a is a natural number, has $\binom{d-l(a-1)-1}{r-1}$ solutions.

Applying all this, we get that $|\Omega| = \binom{t_Y - i(x_1-1)-1}{m-1}$, $|B_j| = \binom{t_Y-(i+1)(x_1-1)-1}{m-1}$, where $j \in \{i+1, \ldots, k\}$, and, for $l > 1$, $|B_{j_1} \cap \cdots \cap B_{j_l}| = \binom{t_Y-(i+l)(x_1-1)-1}{m-1}$, where $j_1, \ldots, j_l$ are distinct numbers from $\{i+1,\dots,k\}$. Using the principle of inclusion and exclusion we get that

$$A_i=\begin{cases}\binom{t_Y-i(x_1-1)-1}{m-1}I\{t_Y-i(x_1-1)\geq m\}\\ -\sum_{l=1}^{k-i}(-1)^{l-1}\binom{k-i}{l}\binom{t_Y-(i+l)(x_1-1)-1}{m-1}I\{t_Y-(i+l)(x_1-1)\geq m\}, & s\leq i<k;\\ \binom{t_Y-k(x_1-1)-1}{m-1}I\{t_Y-k(x_1-1)\geq m\}, & i=k.\end{cases}$$

Applying Rao-Blackwell and Lehmann-Sheffé theorems to equation (4.1), it follows that the UMVUE of $R_{s,k}$ is

$$\begin{aligned}\widehat{R}_{s,k}=&\frac{1}{\binom{t_X-1}{n-1}\binom{t_Y-1}{m-1}}\sum_{x_1=1}^{t_X-n+1}\binom{t_X-x_1-1}{n-2}\Bigg[\sum_{i=s}^{k-1}\binom{k}{i}\Bigg[\binom{t_Y-i(x_1-1)-1}{m-1}\\ &\cdot I\{t_Y-(x_1-1)\geq m\}-\sum_{l=1}^{k-i}(-1)^{l-1}\binom{k-i}{l}\binom{t_Y-(i+l)(x_1-1)-1}{m-1}\\ &\cdot I\{t_Y-(i+l)(x_1-1)\geq m\}\Bigg]+\binom{t_Y-k(x_1-1)-1}{m-1}I\{t_Y-k(x_1-1)\geq m\}\Bigg].\end{aligned}\tag{4.2}$$

5. Bayes estimator of $R_{s,k}$

In this section we consider the Bayes estimator of $R_{s,k}$ with respect to square loss function. Let us suppose that p_1 and p_2 have conjugate prior distributions, beta $\mathcal{B}(a,b)$, $a,b\in\mathbb{N}$, and beta $\mathcal{B}(c,d)$, $c,d\in\mathbb{N}$, with the following joint density:

$$\pi(p_1,p_2)\propto p_1^{a-1}(1-p_1)^{b-1}p_2^{c-1}(1-p_2)^{d-1},\quad p_1\in(0,1),p_2\in(0,1).\tag{5.1}$$

Then, the joint posterior density given the sample $(\mathbf{x},\mathbf{y})$ is

$$\pi(p_1,p_2|\mathbf{x},\mathbf{y})=Wp_1^{a-1+n}(1-p_1)^{b-1+t_X-n}p_2^{c-1+m}(1-p_2)^{d-1+t_Y-m},$$

$p_1\in(0,1),p_2\in(0,1)$, where

$$\begin{aligned}W&=\left(\int_0^1\int_0^1 p_1^{a-1+n}(1-p_1)^{b-1+t_X-n}p_2^{c-1+m}(1-p_2)^{d-1+t_Y-m}dp_1\,dp_2\right)^{-1}\\ &=\frac{1}{B(a+n,b+t_X-n)B(c+m,d+t_Y-m)}\end{aligned}$$

and B is beta function.

Denote, for simplicity,

$$a^*=a+n,\quad b^*=b+t_X-n,\quad c^*=c+m,\quad d^*=d+t_Y-m.$$

The Bayes estimator $\check{R}_{s,k}$ of $R_{s,k}$ for the square loss function is the posterior mean of $R_{s,k}$, hence we obtain

$$\begin{aligned}\check{R}_{s,k}&=E(R_{s,k}|\mathbf{x},\mathbf{y})=\int_0^1\int_0^1 R_{s,k}(p_1,p_2)\pi(p_1,p_2|\mathbf{x},\mathbf{y})dp_1dp_2\\ &=\sum_{i=s}^{k}\sum_{j=0}^{k-i}\frac{(-1)^j\binom{k}{i}\binom{k-i}{j}}{B(a^*,b^*)B(c^*,d^*)}\int_0^1\int_0^1\frac{p_1^{a^*}(1-p_1)^{b^*-1}p_2^{c^*-1}(1-p_2)^{d^*-1}}{1-(1-p_1)(1-p_2)^{i+j}}dp_1dp_2\\ &=\sum_{i=s}^{k}\sum_{j=0}^{k-i}\frac{(-1)^j\binom{k}{i}\binom{k-i}{j}\sum_{l=0}^{\infty}\int_0^1 p_1^{a^*}(1-p_1)^{b^*+l-1}dp_1\int_0^1 p_2^{c^*-1}(1-p_2)^{d^*+l(i+j)-1}dp_2}{B(a^*,b^*)B(c^*,d^*)}\end{aligned}$$

$$= \sum_{i=s}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k}{i} \binom{k-i}{j} \sum_{l=0}^{\infty} \frac{B(a^*+1, b^*+l) B(c^*, d^*+l(i+j))}{B(a^*, b^*) B(c^*, d^*)}. \tag{5.2}$$

Next, we investigate the case when the prior is non-informative. We use Jeffreys non-informative prior, which is proportional to the square root of the determinant of the Fisher information matrix, and get

$$\pi(p_1, p_2) \propto p_1^{-1}(1-p_1)^{-\frac{1}{2}} p_2^{-1}(1-p_2)^{-\frac{1}{2}}, \quad p_1 \in (0,1), p_2 \in (0,1). \tag{5.3}$$

This prior has the form (5.1) with $a = 0$, $b = \frac{1}{2}$, $c = 0$, $d = \frac{1}{2}$, so the Bayes estimator of $R_{s,k}$ in this case is also given by equation (5.2), except that

$$a^* = n, \quad b^* = \frac{1}{2} + t_X - n, \quad c^* = m, \quad d^* = \frac{1}{2} + t_Y - m.$$

6. Simulation study

In this section some Monte Carlo simulations are performed to compare different estimators for $R_{s,k}$. We chose three multicomponent models: two extreme ones, 1-out-of-3 and 3-out-of-3, and "standard" one, 3-out-of-6. From Figure 1 it is visible that most interesting cases happen for smaller values of p_1 and p_2. Therefore, we consider $p_1 \in \{0.1, 0.2, 0.5\}$ and $p_2 \in \{0.1, 0.3, 0.7\}$. We study different sample sizes $(n, m) \in \{(10, 10), (10, 20), (20, 10), (20, 20), (100, 100)\}$.

For each model and each combination of n, m, p_1 and p_2 we generate one random sample from $\mathcal{G}(p_1)$ and one random sample from $\mathcal{G}(p_2)$ and calculate the MLE of $R_{s,k}$ using equation (2.2), and the UMVUE of $R_{s,k}$ using equation (4.2). We also calculate 95% asymptotic confidence interval using equation (3.1) and 95% bootstrap-p confidence interval using procedure described in subsection 3.2 with $N = 1000$ boot times. The Bayes estimates as well as 95% credible intervals are obtained from 5000 samples from two posterior distributions: one with Jeffreys prior given by expression (5.3) (noninformative case); and another with beta $\mathcal{B}(10p_1, 10(1-p_1))$ and beta $\mathcal{B}(10p_2, 10(1-p_2))$ priors (informative case – means of these prior distributions are equal to true parameter values). This procedure is repeated for 1000 times.

In Tables 1, 3 and 5 the averages of point estimates for $R_{1,3}$, $R_{3,3}$ and $R_{3,6}$, respectively, as well as their root-mean-square errors (denoted by Er), are presented. We can notice that in almost all cases the UMVUE has the value closest to R as expected due to its unbiasedness. Moreover, for smaller and larger values of $R_{s,k}$, it is the estimator with the smallest root-mean-square error. For intermediate values of $R_{s,k}$, the informative Bayes estimator is also competitive having the smallest root-mean-square error.

In Table 2, 4 and 6, for our three scenarios, the average lengths of asymptotic and bootstrap-p confidence intervals and both Bayes credible intervals are presented as well as the coverage percentages of these intervals (the percentage of intervals that contain true value of R). The noninformative Bayes credible interval has mostly the best coverage, with bootstrap-p slightly outperforming it for some higher values of $R_{s,k}$. The smallest length almost uniformly has the informative Bayes credible interval.

Table 1. Point estimates for $R_{1,3}$ and their root-mean-square errors

p_1	p_2	$R_{1,3}$	n	m	MLE $\widetilde{R}_{1,3}$	MLE $Er(\widetilde{R}_{1,3})$	UMVUE $\widehat{R}_{1,3}$	UMVUE $Er(\widehat{R}_{1,3})$	Bayes noninf. $\check{R}_{1,3}$	Bayes noninf. $Er(\check{R}_{1,3})$	Bayes inf. $\check{R}_{1,3}$	Bayes inf. $Er(\check{R}_{1,3})$
0.10	0.10	0.763	10	10	0.748	0.117	0.760	0.124	0.738	0.113	0.741	0.103
			10	20	0.762	0.096	0.765	0.102	0.747	0.095	0.748	0.088
			20	10	0.749	0.100	0.765	0.103	0.747	0.095	0.749	0.088
			20	20	0.758	0.084	0.764	0.087	0.752	0.082	0.753	0.079
			100	100	0.762	0.038	0.763	0.038	0.760	0.038	0.760	0.038
0.20	0.10	0.918	10	10	0.904	0.068	0.919	0.065	0.889	0.073	0.895	0.062
			10	20	0.911	0.055	0.919	0.054	0.895	0.061	0.899	0.052
			20	10	0.905	0.057	0.920	0.054	0.897	0.059	0.900	0.052
			20	20	0.909	0.048	0.917	0.046	0.901	0.050	0.903	0.046
			100	100	0.916	0.020	0.918	0.020	0.915	0.020	0.915	0.020
0.50	0.10	0.993	10	10	0.989	0.015	0.993	0.011	0.984	0.020	0.987	0.013
			10	20	0.990	0.010	0.993	0.008	0.985	0.015	0.989	0.009
			20	10	0.990	0.011	0.993	0.008	0.987	0.014	0.988	0.010
			20	20	0.991	0.007	0.993	0.006	0.989	0.009	0.990	0.007
			100	100	0.993	0.003	0.993	0.002	0.992	0.003	0.992	0.003
0.10	0.30	0.419	10	10	0.428	0.124	0.418	0.129	0.434	0.119	0.430	0.103
			10	20	0.440	0.110	0.425	0.111	0.438	0.105	0.435	0.094
			20	10	0.423	0.102	0.422	0.106	0.434	0.101	0.431	0.086
			20	20	0.425	0.092	0.419	0.094	0.428	0.090	0.426	0.083
			100	100	0.420	0.039	0.418	0.039	0.420	0.039	0.420	0.038
0.20	0.30	0.652	10	10	0.646	0.126	0.649	0.136	0.641	0.119	0.643	0.098
			10	20	0.654	0.117	0.649	0.124	0.642	0.112	0.644	0.096
			20	10	0.643	0.109	0.652	0.114	0.646	0.103	0.647	0.086
			20	20	0.650	0.092	0.651	0.096	0.647	0.089	0.647	0.081
			100	100	0.654	0.042	0.654	0.042	0.653	0.042	0.653	0.041
0.50	0.30	0.924	10	10	0.913	0.064	0.926	0.062	0.898	0.070	0.909	0.047
			10	20	0.918	0.055	0.926	0.055	0.902	0.061	0.912	0.043
			20	10	0.911	0.057	0.925	0.054	0.904	0.059	0.911	0.043
			20	20	0.917	0.045	0.924	0.044	0.909	0.048	0.913	0.039
			100	100	0.922	0.018	0.924	0.018	0.921	0.019	0.921	0.018
0.10	0.70	0.187	10	10	0.199	0.076	0.187	0.073	0.209	0.079	0.199	0.060
			10	20	0.201	0.066	0.187	0.062	0.204	0.067	0.199	0.056
			20	10	0.196	0.056	0.192	0.055	0.208	0.060	0.198	0.044
			20	20	0.193	0.048	0.187	0.047	0.198	0.050	0.194	0.042
			100	100	0.188	0.021	0.187	0.021	0.189	0.021	0.188	0.020
0.20	0.70	0.347	10	10	0.362	0.108	0.349	0.109	0.372	0.106	0.359	0.078
			10	20	0.368	0.103	0.351	0.101	0.369	0.099	0.361	0.079
			20	10	0.353	0.092	0.351	0.094	0.368	0.092	0.356	0.070
			20	20	0.357	0.078	0.351	0.078	0.362	0.077	0.356	0.065
			100	100	0.349	0.033	0.348	0.033	0.350	0.033	0.349	0.032
0.50	0.70	0.701	10	10	0.692	0.124	0.692	0.132	0.686	0.115	0.690	0.079
			10	20	0.707	0.113	0.701	0.119	0.693	0.107	0.696	0.076
			20	10	0.694	0.104	0.701	0.109	0.698	0.096	0.696	0.071
			20	20	0.703	0.086	0.704	0.089	0.699	0.083	0.699	0.066
			100	100	0.701	0.039	0.701	0.039	0.700	0.039	0.700	0.037

Table 2. Confidence intervals for $R_{1,3}$ and their coverages

p_1	p_2	$R_{1,3}$	n	m	asymptotic length	asymptotic cov	bootstrap-p length	bootstrap-p cov	Bayes noninf. length	Bayes noninf. cov	Bayes inf. length	Bayes inf. cov
0.10	0.10	0.763	10	10	0.435	88.7	0.416	94.3	0.419	94.4	0.404	96.0
			10	20	0.380	90.8	0.359	94.3	0.377	95.4	0.365	96.1
			20	10	0.382	92.6	0.376	95.4	0.364	95.7	0.353	96.5
			20	20	0.315	91.5	0.306	94.4	0.308	94.6	0.302	95.5
			100	100	0.145	93.6	0.144	94.1	0.144	94.7	0.143	94.7
0.20	0.10	0.918	10	10	0.246	88.4	0.257	95.0	0.262	95.3	0.240	96.5
			10	20	0.209	88.4	0.209	95.2	0.231	95.6	0.213	96.9
			20	10	0.214	91.8	0.231	95.1	0.215	96.5	0.203	97.2
			20	20	0.174	91.4	0.178	93.9	0.180	94.0	0.172	95.4
			100	100	0.076	94.4	0.076	95.1	0.077	94.9	0.076	95.1
0.50	0.10	0.993	10	10	0.042	81.4	0.058	93.8	0.062	93.9	0.044	97.4
			10	20	0.035	83.3	0.042	93.4	0.053	95.0	0.037	97.4
			20	10	0.034	86.6	0.049	94.8	0.043	94.5	0.036	96.6
			20	20	0.026	87.0	0.032	95.5	0.033	95.1	0.028	96.5
			100	100	0.010	92.7	0.010	94.9	0.011	95.2	0.010	95.4
0.10	0.30	0.419	10	10	0.462	90.3	0.449	94.5	0.450	95.1	0.420	97.0
			10	20	0.418	92.8	0.413	94.0	0.404	95.7	0.386	96.5
			20	10	0.397	91.3	0.385	93.7	0.396	94.4	0.365	96.7
			20	20	0.334	90.5	0.329	92.5	0.329	92.4	0.316	94.3
			100	100	0.152	93.6	0.151	94.0	0.152	93.8	0.150	94.4
0.20	0.30	0.652	10	10	0.486	90.7	0.458	95.1	0.462	95.4	0.428	97.1
			10	20	0.432	89.9	0.408	93.8	0.419	94.3	0.392	95.7
			20	10	0.420	92.4	0.404	95.6	0.402	96.0	0.373	97.7
			20	20	0.355	92.5	0.343	94.8	0.345	95.7	0.329	96.3
			100	100	0.163	94.0	0.161	94.4	0.162	94.7	0.160	95.0
0.50	0.30	0.924	10	10	0.231	85.8	0.240	94.8	0.250	95.5	0.203	98.9
			10	20	0.201	85.5	0.199	93.6	0.227	95.0	0.184	97.7
			20	10	0.200	89.4	0.215	93.4	0.203	94.2	0.177	96.7
			20	20	0.166	90.8	0.169	94.8	0.173	95.0	0.155	96.9
			100	100	0.073	94.7	0.073	94.9	0.074	95.0	0.072	95.2
0.10	0.70	0.187	10	10	0.265	90.6	0.279	93.3	0.278	94.0	0.237	96.9
			10	20	0.243	93.5	0.262	93.3	0.244	94.9	0.223	96.3
			20	10	0.220	93.0	0.223	95.2	0.238	95.8	0.196	97.4
			20	20	0.186	94.1	0.192	95.8	0.191	96.2	0.175	97.4
			100	100	0.082	93.7	0.083	94.9	0.083	95.1	0.081	95.1
0.20	0.70	0.347	10	10	0.409	92.3	0.410	94.0	0.407	94.7	0.350	97.6
			10	20	0.376	93.0	0.384	93.2	0.366	94.9	0.331	96.8
			20	10	0.337	91.0	0.331	92.8	0.345	94.3	0.293	97.0
			20	20	0.294	93.4	0.295	94.6	0.293	94.7	0.268	95.8
			100	100	0.132	95.4	0.132	95.7	0.132	96.0	0.129	96.5
0.50	0.70	0.701	10	10	0.458	88.1	0.435	93.3	0.439	94.6	0.372	98.6
			10	20	0.420	88.7	0.399	92.8	0.409	94.7	0.351	97.9
			20	10	0.381	89.3	0.366	93.4	0.364	94.8	0.318	98.4
			20	20	0.336	91.4	0.325	93.7	0.327	93.6	0.294	96.8
			100	100	0.155	94.6	0.154	94.9	0.154	95.1	0.150	95.7

Table 3. Point estimates for $R_{3,3}$ and their root-mean-square errors

					MLE		UMVUE		Bayes noninf.		Bayes inf.	
p_1	p_2	$R_{3,3}$	n	m	$\widetilde{R}_{3,3}$	$Er(\widetilde{R}_{3,3})$	$\widehat{R}_{3,3}$	$Er(\widehat{R}_{3,3})$	$\check{R}_{3,3}$	$Er(\check{R}_{3,3})$	$\check{R}_{3,3}$	$Er(\check{R}_{3,3})$
0.10	0.10	0.291	10	10	0.304	0.094	0.292	0.094	0.310	0.092	0.308	0.083
			10	20	0.306	0.087	0.292	0.086	0.306	0.085	0.304	0.077
			20	10	0.291	0.075	0.287	0.077	0.301	0.076	0.300	0.070
			20	20	0.297	0.062	0.291	0.063	0.301	0.062	0.300	0.059
			100	100	0.292	0.027	0.291	0.027	0.293	0.027	0.293	0.027
0.20	0.10	0.480	10	10	0.486	0.113	0.479	0.117	0.487	0.108	0.486	0.093
			10	20	0.496	0.099	0.484	0.101	0.490	0.095	0.489	0.082
			20	10	0.479	0.088	0.479	0.092	0.485	0.086	0.485	0.078
			20	20	0.483	0.076	0.479	0.078	0.483	0.074	0.483	0.069
			100	100	0.481	0.035	0.480	0.035	0.481	0.035	0.481	0.034
0.50	0.10	0.787	10	10	0.780	0.088	0.781	0.091	0.768	0.087	0.777	0.065
			10	20	0.793	0.080	0.790	0.083	0.778	0.080	0.784	0.057
			20	10	0.780	0.072	0.786	0.073	0.777	0.070	0.781	0.059
			20	20	0.783	0.059	0.784	0.059	0.777	0.059	0.781	0.049
			100	100	0.787	0.028	0.787	0.028	0.786	0.028	0.787	0.027
0.10	0.30	0.145	10	10	0.157	0.053	0.144	0.049	0.163	0.055	0.159	0.046
			10	20	0.158	0.053	0.145	0.049	0.160	0.053	0.157	0.046
			20	10	0.152	0.038	0.146	0.037	0.159	0.041	0.156	0.036
			20	20	0.152	0.036	0.146	0.034	0.155	0.037	0.154	0.034
			100	100	0.146	0.015	0.145	0.015	0.147	0.015	0.147	0.015
0.20	0.30	0.276	10	10	0.294	0.088	0.277	0.085	0.299	0.087	0.293	0.070
			10	20	0.294	0.083	0.276	0.080	0.294	0.081	0.290	0.066
			20	10	0.281	0.062	0.273	0.062	0.289	0.064	0.286	0.055
			20	20	0.286	0.061	0.277	0.060	0.289	0.061	0.287	0.055
			100	100	0.278	0.025	0.276	0.025	0.279	0.025	0.279	0.025
0.50	0.30	0.604	10	10	0.623	0.116	0.608	0.119	0.615	0.109	0.615	0.076
			10	20	0.617	0.114	0.600	0.117	0.604	0.108	0.608	0.075
			20	10	0.609	0.086	0.602	0.088	0.610	0.083	0.611	0.067
			20	20	0.608	0.082	0.600	0.083	0.604	0.079	0.607	0.065
			100	100	0.604	0.038	0.602	0.038	0.603	0.037	0.604	0.036
0.10	0.70	0.102	10	10	0.114	0.036	0.103	0.031	0.115	0.036	0.112	0.031
			10	20	0.115	0.040	0.105	0.035	0.115	0.040	0.113	0.034
			20	10	0.109	0.025	0.103	0.023	0.111	0.026	0.109	0.024
			20	20	0.107	0.023	0.102	0.022	0.107	0.023	0.107	0.022
			100	100	0.104	0.010	0.103	0.010	0.104	0.010	0.104	0.010
0.20	0.70	0.204	10	10	0.222	0.069	0.204	0.063	0.223	0.068	0.217	0.053
			10	20	0.218	0.064	0.200	0.059	0.217	0.063	0.214	0.050
			20	10	0.214	0.044	0.204	0.041	0.216	0.045	0.213	0.039
			20	20	0.215	0.045	0.206	0.042	0.216	0.045	0.214	0.040
			100	100	0.207	0.019	0.205	0.019	0.207	0.019	0.207	0.019
0.50	0.70	0.507	10	10	0.536	0.118	0.510	0.116	0.526	0.110	0.522	0.074
			10	20	0.536	0.118	0.510	0.116	0.524	0.109	0.522	0.074
			20	10	0.525	0.086	0.511	0.085	0.523	0.084	0.520	0.067
			20	20	0.523	0.082	0.509	0.081	0.518	0.079	0.518	0.064
			100	100	0.509	0.037	0.506	0.037	0.508	0.036	0.509	0.035

Table 4. Confidence intervals for $R_{3,3}$ and their coverages

p_1	p_2	$R_{3,3}$	n	m	asymptotic length	asymptotic cov	bootstrap-p length	bootstrap-p cov	Bayes noninf. length	Bayes noninf. cov	Bayes inf. length	Bayes inf. cov
			10	10	0.341	91.9	0.343	94.1	0.340	95.4	0.326	96.2
			10	20	0.310	92.1	0.317	92.7	0.304	93.7	0.292	94.3
0.10	0.10	0.291	20	10	0.280	91.1	0.275	93.5	0.287	94.6	0.277	95.3
			20	20	0.243	94.3	0.244	95.0	0.243	95.3	0.238	96.0
			100	100	0.109	95.1	0.109	94.8	0.109	94.9	0.109	95.0
			10	10	0.414	91.1	0.404	94.0	0.403	93.9	0.380	96.1
			10	20	0.382	92.9	0.376	94.0	0.371	95.6	0.348	96.9
0.20	0.10	0.480	20	10	0.346	93.3	0.336	95.2	0.342	95.6	0.328	96.5
			20	20	0.301	93.1	0.295	94.7	0.296	95.0	0.286	95.4
			100	100	0.137	95.3	0.136	95.2	0.136	95.5	0.135	95.9
			10	10	0.336	90.6	0.329	94.3	0.337	95.0	0.288	98.0
			10	20	0.305	89.5	0.297	91.0	0.313	95.1	0.264	98.1
0.50	0.10	0.787	20	10	0.269	92.3	0.268	94.4	0.267	94.9	0.245	96.5
			20	20	0.241	94.5	0.238	96.1	0.241	96.1	0.220	97.6
			100	100	0.108	94.0	0.107	94.4	0.108	95.1	0.106	94.9
			10	10	0.193	94.8	0.211	94.3	0.201	95.7	0.184	96.8
			10	20	0.184	95.5	0.204	92.3	0.184	95.2	0.173	96.4
0.10	0.30	0.145	20	10	0.148	93.8	0.153	95.2	0.160	96.1	0.145	96.8
			20	20	0.134	95.1	0.140	94.7	0.137	94.6	0.131	95.4
			100	100	0.058	95.2	0.059	94.7	0.059	94.6	0.058	94.7
			10	10	0.315	94.3	0.329	93.1	0.314	95.0	0.284	97.3
			10	20	0.300	95.2	0.318	92.1	0.294	95.9	0.270	98.2
0.20	0.30	0.276	20	10	0.238	93.1	0.239	95.2	0.246	95.3	0.226	96.9
			20	20	0.222	93.0	0.227	93.8	0.222	94.4	0.210	95.1
			100	100	0.098	95.2	0.098	95.0	0.098	95.4	0.097	95.9
			10	10	0.430	91.8	0.420	92.0	0.416	96.1	0.355	98.7
			10	20	0.417	91.7	0.410	92.5	0.405	93.9	0.344	97.8
0.50	0.30	0.604	20	10	0.329	92.8	0.321	94.2	0.324	95.2	0.292	97.1
			20	20	0.312	93.1	0.307	93.4	0.306	95.3	0.278	97.2
			100	100	0.142	93.3	0.141	93.2	0.141	93.9	0.138	94.5
			10	10	0.133	96.7	0.154	92.4	0.134	96.5	0.123	97.2
			10	20	0.133	94.2	0.154	90.7	0.132	94.3	0.124	96.0
0.10	0.70	0.102	20	10	0.091	96.3	0.099	91.8	0.094	94.6	0.088	95.6
			20	20	0.089	95.5	0.095	94.2	0.089	95.8	0.086	96.3
			100	100	0.039	96.1	0.039	94.5	0.039	95.0	0.038	95.3
			10	10	0.240	95.2	0.268	92.2	0.237	94.9	0.212	96.9
			10	20	0.236	95.6	0.263	93.8	0.230	95.6	0.209	96.8
0.20	0.70	0.204	20	10	0.167	95.2	0.177	93.3	0.169	94.8	0.157	96.0
			20	20	0.167	95.4	0.176	93.0	0.166	95.2	0.157	96.4
			100	100	0.072	94.1	0.073	93.5	0.072	94.0	0.071	94.2
			10	10	0.434	93.6	0.437	91.6	0.415	95.9	0.349	97.8
			10	20	0.433	94.1	0.437	90.4	0.414	96.7	0.349	99.7
0.50	0.70	0.507	20	10	0.311	93.0	0.312	91.1	0.305	94.4	0.274	96.1
			20	20	0.310	93.4	0.312	92.0	0.302	94.2	0.273	96.9
			100	100	0.139	93.8	0.138	92.3	0.138	94.0	0.135	94.4

Table 5. Point estimates for $R_{3,6}$ and their root-mean-square errors

p_1	p_2	$R_{3,6}$	n	m	MLE $\widetilde{R}_{3,6}$	MLE $Er(\widetilde{R}_{3,6})$	UMVUE $\widehat{R}_{3,6}$	UMVUE $Er(\widehat{R}_{3,6})$	Bayes noninf. $\check{R}_{3,6}$	Bayes noninf. $Er(\check{R}_{3,6})$	Bayes inf. $\check{R}_{3,6}$	Bayes inf. $Er(\check{R}_{3,6})$
			10	10	0.598	0.135	0.598	0.146	0.596	0.127	0.596	0.117
			10	20	0.599	0.121	0.590	0.129	0.590	0.115	0.590	0.107
0.10	0.10	0.593	20	10	0.585	0.116	0.592	0.122	0.592	0.110	0.592	0.102
			20	20	0.595	0.097	0.594	0.101	0.594	0.094	0.594	0.090
			100	100	0.593	0.044	0.593	0.044	0.593	0.044	0.593	0.043
			10	10	0.804	0.112	0.818	0.118	0.790	0.110	0.795	0.096
			10	20	0.819	0.091	0.825	0.096	0.801	0.091	0.805	0.080
0.20	0.10	0.822	20	10	0.803	0.099	0.820	0.101	0.798	0.095	0.802	0.086
			20	20	0.814	0.080	0.822	0.082	0.806	0.079	0.808	0.073
			100	100	0.819	0.034	0.821	0.034	0.817	0.034	0.818	0.034
			10	10	0.972	0.033	0.982	0.026	0.961	0.041	0.968	0.028
			10	20	0.978	0.022	0.984	0.019	0.968	0.030	0.973	0.020
0.50	0.10	0.982	20	10	0.974	0.028	0.982	0.023	0.968	0.032	0.971	0.025
			20	20	0.978	0.019	0.983	0.017	0.973	0.022	0.975	0.018
			100	100	0.982	0.007	0.983	0.006	0.981	0.007	0.981	0.007
			10	10	0.287	0.102	0.272	0.102	0.297	0.102	0.292	0.086
			10	20	0.292	0.088	0.275	0.085	0.294	0.086	0.291	0.076
0.10	0.30	0.276	20	10	0.279	0.078	0.274	0.078	0.292	0.080	0.288	0.068
			20	20	0.283	0.070	0.276	0.070	0.289	0.071	0.287	0.065
			100	100	0.276	0.030	0.275	0.030	0.278	0.030	0.277	0.030
			10	10	0.489	0.132	0.478	0.139	0.492	0.126	0.490	0.103
			10	20	0.499	0.118	0.484	0.122	0.494	0.113	0.492	0.095
0.20	0.30	0.484	20	10	0.486	0.106	0.485	0.111	0.496	0.104	0.493	0.088
			20	20	0.484	0.093	0.479	0.095	0.487	0.090	0.486	0.081
			100	100	0.483	0.041	0.482	0.041	0.484	0.041	0.484	0.040
			10	10	0.824	0.103	0.836	0.109	0.809	0.102	0.821	0.072
			10	20	0.828	0.093	0.832	0.099	0.809	0.096	0.821	0.070
0.50	0.30	0.839	20	10	0.820	0.089	0.834	0.090	0.814	0.086	0.822	0.067
			20	20	0.830	0.071	0.836	0.073	0.821	0.071	0.827	0.059
			100	100	0.838	0.032	0.839	0.033	0.836	0.032	0.837	0.031
			10	10	0.137	0.048	0.124	0.044	0.145	0.052	0.137	0.040
			10	20	0.135	0.048	0.123	0.044	0.139	0.049	0.134	0.041
0.10	0.70	0.124	20	10	0.130	0.035	0.123	0.034	0.138	0.039	0.131	0.030
			20	20	0.129	0.033	0.123	0.032	0.134	0.035	0.130	0.030
			100	100	0.125	0.013	0.123	0.013	0.126	0.013	0.125	0.013
			10	10	0.267	0.088	0.246	0.084	0.278	0.092	0.262	0.067
			10	20	0.267	0.085	0.248	0.079	0.271	0.085	0.261	0.066
0.20	0.70	0.242	20	10	0.258	0.065	0.246	0.065	0.270	0.071	0.258	0.054
			20	20	0.252	0.058	0.242	0.057	0.258	0.059	0.253	0.050
			100	100	0.245	0.025	0.243	0.024	0.246	0.025	0.246	0.024
			10	10	0.592	0.127	0.570	0.132	0.592	0.120	0.581	0.081
			10	20	0.597	0.126	0.575	0.128	0.589	0.118	0.582	0.080
0.50	0.70	0.565	20	10	0.574	0.094	0.561	0.100	0.584	0.092	0.574	0.070
			20	20	0.574	0.083	0.562	0.086	0.576	0.081	0.573	0.065
			100	100	0.567	0.040	0.565	0.040	0.568	0.040	0.568	0.038

Table 6. Confidence intervals for $R_{3,6}$ and their coverages

					asymptotic		bootstrap-p		Bayes noninf.		Bayes inf.	
p_1	p_2	$R_{3,6}$	n	m	length	cov	length	cov	length	cov	length	cov
			10	10	0.519	90.3	0.486	94.3	0.491	94.4	0.476	95.5
			10	20	0.463	90.7	0.438	94.1	0.444	94.6	0.430	95.9
0.10	0.10	0.593	20	10	0.455	92.1	0.433	94.9	0.435	96.1	0.423	96.3
			20	20	0.382	93.4	0.368	95.3	0.370	95.3	0.362	96.0
			100	100	0.176	94.6	0.174	95.1	0.175	95.1	0.174	95.1
			10	10	0.407	87.2	0.389	93.9	0.395	93.8	0.373	96.0
			10	20	0.352	87.4	0.331	93.4	0.356	95.6	0.334	97.3
0.20	0.10	0.822	20	10	0.356	90.3	0.352	94.2	0.339	94.4	0.325	95.2
			20	20	0.293	90.0	0.286	94.3	0.289	94.6	0.279	95.5
			100	100	0.135	94.1	0.134	94.9	0.135	94.8	0.134	95.1
			10	10	0.105	84.6	0.128	94.5	0.136	94.5	0.104	97.3
			10	20	0.079	82.6	0.090	92.8	0.110	94.9	0.082	97.4
0.50	0.10	0.982	20	10	0.084	84.5	0.107	93.3	0.097	93.4	0.084	95.1
			20	20	0.063	86.8	0.072	93.5	0.075	94.5	0.065	95.9
			100	100	0.026	92.2	0.027	94.5	0.027	95.1	0.026	95.5
			10	10	0.364	91.5	0.372	93.2	0.370	93.8	0.340	96.4
			10	20	0.332	94.0	0.346	95.0	0.327	95.5	0.311	96.8
0.10	0.30	0.276	20	10	0.304	92.1	0.303	94.7	0.319	95.3	0.290	96.9
			20	20	0.260	93.2	0.263	94.8	0.263	95.1	0.251	95.6
			100	100	0.116	94.2	0.116	94.1	0.116	94.2	0.115	94.8
			10	10	0.488	91.0	0.469	94.3	0.471	93.9	0.434	96.5
			10	20	0.445	90.9	0.434	92.9	0.428	94.2	0.400	96.4
0.20	0.30	0.484	20	10	0.418	91.7	0.402	95.0	0.412	95.0	0.379	96.5
			20	20	0.356	91.9	0.348	93.8	0.349	94.2	0.332	95.4
			100	100	0.163	94.7	0.162	94.8	0.162	94.9	0.160	95.4
			10	10	0.380	87.8	0.360	95.0	0.375	94.8	0.321	97.9
			10	20	0.343	88.1	0.322	92.5	0.349	93.2	0.297	96.5
0.50	0.30	0.839	20	10	0.329	90.0	0.322	94.6	0.315	95.6	0.283	98.1
			20	20	0.278	92.2	0.270	95.3	0.276	95.0	0.251	97.1
			100	100	0.126	94.3	0.125	94.3	0.126	94.9	0.123	95.0
			10	10	0.178	95.8	0.198	94.5	0.192	97.3	0.163	97.5
			10	20	0.165	94.8	0.186	93.1	0.168	95.4	0.154	96.5
0.10	0.70	0.124	20	10	0.134	93.4	0.139	95.4	0.150	95.8	0.125	97.5
			20	20	0.121	93.9	0.127	94.0	0.126	94.8	0.116	95.8
			100	100	0.053	95.2	0.053	95.7	0.053	95.8	0.053	95.7
			10	10	0.311	93.7	0.328	92.6	0.321	94.9	0.271	97.3
			10	20	0.293	94.0	0.314	91.7	0.292	94.3	0.259	97.3
0.20	0.70	0.242	20	10	0.240	94.3	0.241	93.8	0.258	94.7	0.217	96.8
			20	20	0.214	94.7	0.220	94.5	0.219	95.4	0.200	97.0
			100	100	0.095	94.9	0.095	94.7	0.096	95.2	0.094	95.6
			10	10	0.467	91.2	0.447	92.4	0.451	95.0	0.378	98.4
			10	20	0.446	89.9	0.434	90.0	0.429	94.4	0.367	97.8
0.50	0.70	0.565	20	10	0.367	92.7	0.348	94.7	0.365	95.2	0.315	97.4
			20	20	0.341	94.3	0.332	95.5	0.335	96.3	0.300	98.2
			100	100	0.155	95.0	0.154	95.1	0.155	95.4	0.151	96.2

7. Real data application

In this section we apply the presented methods to real datasets. The data of the first example represent durations of warm spells in summer months (June, July and August) in Budapest, Hungary. For this purpose we define a warm spell as number of days where daily maximum temperature is at least 30 degrees Celsius.

We are interested in probabilities that the duration of a warm spell in the 21st century (X) does not exceed the 60th and the 80th percentile of the duration of warm spell in the 20th century (Y). Those probabilities then can be modelled as $R_{2,5}$ and $R_{1,5}$ respectively.

Two populations, one containing summer months from the 21st century (2001 till 2018) and other from 20th century (1901 till 2000), are obtained from the database of the European Climate Assessment & Dataset project available at

https://www.ecad.eu/dailydata/predefinedseries.php.

Then, two simple random samples are drawn from each population. The samples, of sizes 20 (for the 21st century) and 100 (for the 20th century), are presented in Table 7.

Table 7. Warm spell duration

21st century									
duration	1	2	3	4	5	7	8	9	11
frequency	5	3	3	3	1	2	1	1	1

20st century									
duration	1	2	3	4	5	6	7	8	9
frequency	29	24	12	13	9	4	7	1	1

The point and interval estimates are presented in Table 8. Bayes estimates here are given for the noninformative case. From these results it follows that $R_{2,5} \approx 0.61$ and $R_{1,5} \approx 0.78$. Therefore, the probability that the duration of a warm spell in the 21st century will not exceed the 60th percentile of the duration of warm spell in the 20th century is estimated to be 0.61. In case of the 80th percentile the estimate of the corresponding probability is 0.78.

Table 8. Real data estimates for Budapest warm spells example

$\tilde{R}_{2,5}$	$\hat{R}_{2,5}$	$\check{R}_{2,5}$	asymptotic CI	bootstrap-p CI	Bayes CI
0.619	0.612	0.611	(0.462,0.776)	(0.478,0.789)	(0.452,0.758)

$\tilde{R}_{1,5}$	$\hat{R}_{1,5}$	$\check{R}_{1,5}$	asymptotic CI	bootstrap-p CI	Bayes CI
0.778	0.777	0.770	(0.643,0.914)	(0.637,0.898)	(0.618,0.888)

The data of the second example represent number of successive 2-point field goals made by a basketball player from the start of the game including his first miss. Anadolu Efes and Barcelona face each other in 2018/19 Euroleague playoffs. We choose two players with approximately equal regular season 2-point field goal percentage (Ante Tomić (62.11%) and Kevin Seraphin(62.50%)) from Barcelona and one player (Bryant Dunston(67.33%)) from Anadolu Efes. All three players play the center position. We are interested in estimating the probability that one (or both) of the Barcelona players will make great or equal number of consecutive field goals than B. Dunston. These probabilities can be utilized to set betting odds. This situation is an example of a multicomponent geometric model with two "strength" components, and underlying probabilities are $R_{1,2}$ and $R_{2,2}$.

Two simple random samples of the size ten were taken from the 2018/2019 Euroleague regular season games database available at https://www.euroleague.net. The first sample is taken from 29 regular season games played by B. Dunston and the second sample is taken from 30+24 regular season games played by A. Tomić and K. Seraphin. The samples, accompanied with the names of corresponding adversary teams, are presented in Table 9.

Table 9. Number of consecutive field goals including first miss

Bryant Dunston									
Khim. away	Bask. home	Arma. home	Olym. home	Barc. home	Real home	Arma. away	Olym. away	Zalg. away	Pana. away
1	2	3	2	1	1	4	1	4	2

Ante Tomić				Kevin Seraphin					
Pana. away	Gran. away	Macc. away	Baye. home	Bask. away	Arma. away	Zalg. away	Gran. away	CSKA home	Budu. away
1	3	1	4	3	1	1	2	4	1

Table 10. Real data estimates for Euroleague example

$\widetilde{R}_{1,2}$	$\widehat{R}_{1,2}$	$\check{R}_{1,2}$	asymptotic CI	bootstrap-p CI	Bayes CI
0.756	0.763	0.744	(0.541,0.972)	(0.523,0.932)	(0.505, 0.919)

$\widetilde{R}_{2,2}$	$\widehat{R}_{2,2}$	$\check{R}_{2,2}$	asymptotic CI	bootstrap-p CI	Bayes CI
0.556	0.536	0.552	(0.331,0.781)	(0.360,0.801)	(0.332, 0.759)

The point and interval estimates of $R_{1,2}$ and $R_{2,2}$ are given in Table 10. As in the previous example Bayes estimates are given for noninformative case. From these results it follows that $R_{1,2} \approx 0.75$ and $R_{2,2} \approx 0.54$. This means that the probability that at least one of the Barcelona centers will score at least as many consecutive field goals as Dunston is estimated to be 0.75, while the estimate of the probability of the corresponding event for both of them is 0.54.

8. Conclusion

In this paper we considered estimation of strength-stress probability in a multicomponent system, where both stress and strength come from the geometric distribution. This is the first study concerning such multicomponent system with discrete probability distributions.

We derived three point and three interval estimators and compared them in a simulation study. Based on both bias and root-mean-square error criteria, the UMVUE is shown to be most suitable among point estimators, with the informative Bayes estimator being also competitive for intermediate values of $R_{s,k}$. The Bayes credible intervals are mostly the best choices among interval estimators.

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